Filomat 36:12 (2022), 4225–4239
https://doi.org/10.2298/FIL2212225B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On the Mixed Multifractal Densities and Regularities with Respect to Gauges

Anouar Ben Mabrouk[a,b,c], Mohamed Menceur[d], Bilel Selmi[e]

[a] *Laboratory of Algebra, Number Theory and Nonlinear Analysis, LR18ES15, Department of Mathematics, Faculty of Sciences, University of Monastir, Avenue of the Environment, 5019 Monastir, Tunisia.*
[b] *Department of Mathematics, Higher Institute of Applied Mathematics and Computer Sciences, University of Kairouan, Street of Assad Ibn Alfourat, 3100 Kairouan, Tunisia.*
[c] *Department of Mathematics, Faculty of Sciences, University of Tabuk, King Faisal road, 47512 Tabuk, Saudi Arabia.*
[d] *Department of Algebra and Number Theory, Faculty of Mathematics, University of Sciences and Technology Houari Boumediene, BP 32 EL Alia 16111 Bab Ezzouar, Algiers, Algeria.*
[e] *Laboratory of Analysis, Probability and Fractals, LR18ES17, Department of Mathematics, Faculty of Sciences, University of Monastir, Avenue of the Environment, 5019 Monastir, Tunisia.*

Abstract. In this paper, we are concerned with some density estimations of vector-valued measures in the framework of the so-called mixed multifractal analysis. We precisely consider some Borel probability measures that are no longer Gibbs and introduce some mixed multifractal generalizations of densities in a framework of relative mixed multifractal analysis. Results on multifractal regularities are developed in the new framework.

1. Introduction

A first step in the mixed multifractal analysis has been developed by the same author for one very restrictive class of measures known as the self-affine measures [27] dealing precisely with Rényi dimensions for finitely many self-affine measures. Next, motivated by this study, a mixed multifractal analysis has been developed in [5, 28] for vector-valued measures in some more general contexts. By assuming a restrictive hypothesis looking like Gibbs-type measures and by proving a general mixed large deviation formalism a mixed multifractal formalism has been proved. In [5] and [6] a mixed multifractal analysis inspired from the one for measures has been developed in the functional case. By concentring a vector-valued Gibbs-like measure on the singularities set of finitely and simultaneously many functions, a mixed multifractal formalism for functions has been developed. General results for almost all functions have been proved and a mixed multifractal formalism have been proved for self-similar quasi self-similar functions as well as their superpositions (which are not self-similar neither quasi self-similar). For more details and backgrounds on multifractal analysis as well as the mixed generalizations the readers may be referred also to the following essential references [5, 27, 38–46].

2020 *Mathematics Subject Classification*. Primary: 28A78, 28A80; Secondary: 28A20, 28A75, 49Q15
Keywords. Mutual multifractal analysis, Density Theorem, Moran sets.
Received: 12 April 2021; Accepted: 09 August 2022
Communicated by Dragan S. Djordjević
Email addresses: anouar.benmabrouk@fsm.rnu.tn (Anouar Ben Mabrouk), m.m.m@live.fr (Mohamed Menceur), bilel.selmi@fsm.rnu.tn, bilel.selmi@isetgb.rnu.tn (Bilel Selmi)

In this paper, we are concerned with the multifractal analysis of measures in a mixed case (which can already be adapted to single cases) where the hypothesis of the existence of Gibbs-like and/or doubling measures supported by the singularities sets is relaxed. We aim to consider some cases of simultaneous behaviors of measures where the local Hölder behavior is controlled by a special and suitable function that allows the extra-hypothesis of Gibbs-like measures not to be necessary.

In the present paper, we are precisely focusing on some density estimations of vector-valued measures in a general framework of mixed multifractal analysis for measures that have been developed recently in pure mathematics to extend the multifractal analysis. However, in the applied point of view, mixed multifractal analysis has been developed independently especially in physics and statistics ([18, 23]). We propose to extend the existing cases of mixed multifractal analysis of measures to the more general context that will englobe all the existing cases as special ones by introducing a mixed density notion where the hypothesis of being Gibbs for the measures is no longer assumed. For backgrounds, we may refer to [1, 2, 11–13, 21–23, 36, 37].

The problem studied here has been the subject of several studies. In [2], some multifractal densities in the framework of single multifractal analysis have been investigated. Motivated by [2], an extending study has been developed in [4] where the hypotheses on the applied measure have been revised. The theory was next applied to a case of quasi Ahlfors regular vector-valued measures introduced in [21]. Next, in [22] a more general mixed (and thus single) multifractal analysis for vector-valued measures has been developed extending all the existing cases ([3, 12, 25, 42]) and where the hypothesis of being Gibbs-like is no longer necessary for the applied measures. Instead, a gauge control function φ is included in the definition of the Hausdorff and packing measures to best control the local behavior of the vector-valued measure μ. In the present paper we aim to consider the last context of mixed multifractal analysis developed in [22] and to develop some mixed multifractal densities estimations extending the results of [2, 4, 9, 12, 14, 15, 20, 25–27, 29, 30, 32–34, 36, 37].

Finally, we recall that this work has been reformulated with more generalities, extensions and all the necessary details in the form of a chapter. Eventual discussions of the usefulness of the theoretical results as well as concrete applications have been developed in [7]. This will respond to readers' suggestions and wishes. See also [10] for more applications.

The next section is devoted to the presentation of the general settings of the present study. Section 3 is devoted to the main results. In section 4, results on the regularities of the φ-mixed multifractal generalizations of Hausdorff and packing measures are developed leading to the decomposition theorem of Besicovitch's type. Section 5 is concerned with an application of the previous results in which a necessary condition for a strong regularity with the φ-mixed multifractal generalizations of Hausdorff and packing measures has been established. Section 6 is an appendix in which some useful well-known theorems have been recalled.

2. The φ-mixed Hausdorff and packing measures and dimensions

In this section, we review briefly the φ-mixed multifractal generalizations of the Hausdorff and packing measures and dimensions already developed in [22] and [35], and thus we introduce the general settings and context for our study to be developed next.

Let $k \in \mathbb{N}$ be a fixed integer and $\mu = (\mu_1, \mu_2, \dots, \mu_k)$ be a vector-valued measure composed of Borel probability measures on $\mathbb{R}^d$ with common support equal to $\mathrm{supp}(\mu)$. Let also $\varphi : \mathbb{R}_+ \to \mathbb{R}$ be such that

$$\varphi \text{ is non-decreasing and } \varphi(r) < 0 \text{ for } r \text{ small enough.} \tag{2.1}$$

For $x \in \mathbb{R}^d$ and $r > 0$ we denote $B(x, r)$ the ball of radius r and center x and

$$\mu(B(x,r)) \equiv \Big(\mu_1(B(x,r)), \dots, \mu_k(B(x,r))\Big)$$

and for $\mathbf{q} = (q_1, q_2, \dots, q_k) \in \mathbb{R}^k$, we write

$$(\mu(B(x,r)))^{\mathbf{q}} \equiv (\mu_1(B(x,r)))^{q_1} \dots (\mu_k(B(x,r)))^{q_k}.$$

For $E \subseteq \mathbb{R}^d$ nonempty, $\epsilon > 0$ and $t \in \mathbb{R}$ consider the quantity

$$\overline{\mathscr{H}}^{\mathbf{q},t}_{\mu,\varphi,\epsilon}(E) = \inf\left\{ \sum_i (\mu(B(x_i, r_i)))^{\mathbf{q}} e^{t\varphi(r_i)} \right\},$$

where the inf is taken over the set of all centered ϵ-coverings of E, and for the empty set, $\overline{\mathscr{H}}^{\mathbf{q},t}_{\mu,\epsilon}(\emptyset) = 0$. Consider next

$$\overline{\mathscr{H}}^{\mathbf{q},t}_{\mu,\varphi}(E) = \lim_{\epsilon \downarrow 0} \overline{\mathscr{H}}^{\mathbf{q},t}_{\mu,\varphi,\epsilon}(E).$$

The φ-mixed multifractal generalization of the Hausdorff measure is

$$\mathscr{H}^{\mathbf{q},t}_{\mu,\varphi}(E) = \sup_{F \subseteq E} \overline{\mathscr{H}}^{\mathbf{q},t}_{\mu,\varphi}(F).$$

Similarly, we introduce the φ-mixed multifractal generalization of the packing measure as follows. Let

$$\overline{\mathscr{P}}^{\mathbf{q},t}_{\mu,\varphi,\epsilon}(E) = \sup\left\{ \sum_i (\mu(B(x_i, r_i)))^{\mathbf{q}} e^{t\varphi(r_i)} \right\}$$

where the sup is taken over the set of all centered ϵ-packings of E. For the empty set, we set as usual $\overline{\mathscr{P}}^{\mathbf{q},t}_{\mu,\varphi,\epsilon}(\emptyset) = 0$. Next, we denote

$$\overline{\mathscr{P}}^{\mathbf{q},t}_{\mu,\varphi}(E) = \lim_{\epsilon \downarrow 0} \overline{\mathscr{P}}^{\mathbf{q},t}_{\mu,\varphi,\epsilon}(E) = \inf_{\epsilon > 0} \overline{\mathscr{P}}^{\mathbf{q},t}_{\mu,\varphi,\epsilon}(E)$$

and finally we obtain the φ-mixed multifractal generalization of the packing measure as

$$\mathscr{P}^{\mathbf{q},t}_{\mu,\varphi}(E) = \inf_{E \subseteq \cup_i E_i} \sum_i \overline{\mathscr{P}}^{\mathbf{q},t}_{\mu,\varphi}(E_i).$$

The following theorem resumes some properties of the φ-mixed multifractal measure $\mathscr{H}^{\mathbf{q},t}_{\mu,\varphi}$ introduced above.

Theorem 2.1. *$\mathscr{H}^{\mathbf{q},t}_{\mu,\varphi}$ and $\mathscr{P}^{\mathbf{q},t}_{\mu,\varphi}$ are outer metric measures on $\mathbb{R}^d$.*

It holds as for the case of the multifractal analysis of a single measure that the measures $\mathscr{H}^{\mathbf{q},t}_{\mu,\varphi}$, $\mathscr{P}^{\mathbf{q},t}_{\mu,\varphi}$ and the pre-measure $\overline{\mathscr{P}}^{\mathbf{q},t}_{\mu,\varphi}$ assign a dimension to every set $E \subseteq \mathbb{R}^d$. More precisely, the following result hold.

Proposition 2.2. *Given a subset $E \subseteq \mathbb{R}^d$,*

1. *There exists a unique number $dim^{\mathbf{q}}_{\mu,\varphi}(E) \in [-\infty, +\infty]$ such that*

$$\mathscr{H}^{\mathbf{q},t}_{\mu,\varphi}(E) = \begin{cases} +\infty & \text{for} \quad t < dim^{\mathbf{q}}_{\mu,\varphi}(E) \\ 0 & \text{si} \quad t > dim^{\mathbf{q}}_{\mu,\varphi}(E). \end{cases}$$

2. *There exists a unique number $Dim^{\mathbf{q}}_{\mu,\varphi}(E) \in [-\infty, +\infty]$ such that*

$$\mathscr{P}^{\mathbf{q},t}_{\mu,\varphi}(E) = \begin{cases} +\infty & \text{for} \quad t < Dim^{\mathbf{q}}_{\mu,\varphi}(E) \\ 0 & \text{for} \quad t > Dim^{\mathbf{q}}_{\mu,\varphi}(E). \end{cases}$$

3. *There exists a unique number $\Delta^{\mathbf{q}}_{\mu,\varphi}(E) \in [-\infty, +\infty]$ such that*

$$\overline{\mathscr{P}}^{\mathbf{q},t}_{\mu,\varphi}(E) = \begin{cases} +\infty & \text{for} \quad t < \Delta^{\mathbf{q}}_{\mu,\varphi}(E) \\ 0 & \text{for} \quad t > \Delta^{\mathbf{q}}_{\mu,\varphi}(E). \end{cases}$$

Definition 2.3. *For all set $E \subseteq \mathbb{R}^d$, we have*

- $dim^{\boldsymbol{q}}_{\mu,\varphi}(E)$ *is called the φ-mixed multifractal generalization of the Hausdorff dimension of the set E.*
- $Dim^{\boldsymbol{q}}_{\mu,\varphi}(E)$ *is called the φ-mixed multifractal generalization of the packing dimension of the set E.*
- $\Delta^{\boldsymbol{q}}_{\mu,\varphi}(E)$ *is called the φ-mixed multifractal generalization of the logarithmic index of the set E.*

When $E = \operatorname{supp}(\mu)$, we denote respectively

$$b_{\mu,\varphi}(q) = dim^{\boldsymbol{q}}_{\mu,\varphi}(E),\ \ B_{\mu,\varphi}(q) = Dim^{\boldsymbol{q}}_{\mu,\varphi}(E)\ \ and\ \ \Delta_{\mu,\varphi}(q) = \Delta^{\boldsymbol{q}}_{\mu,\varphi}(E).$$

Next, we aim to study the characteristics of the mixed multifractal generalizations of dimensions.

Proposition 2.4.

a. $b^{\boldsymbol{q}}_{\mu,\varphi}(.)$ *and* $B^{\boldsymbol{q}}_{\mu,\varphi}(.)$ *are non decreasing with respect to the inclusion property in $\mathbb{R}^d$.*

b. $b^{\boldsymbol{q}}_{\mu,\varphi}(.)$ *and* $B^{\boldsymbol{q}}_{\mu,\varphi}(.)$ *are σ-stable.*

c. *The functions $\boldsymbol{q} \longmapsto B_{\mu,\varphi}(\boldsymbol{q})$ and $\boldsymbol{q} \longmapsto \Lambda_{\mu,\varphi}(\boldsymbol{q})$ are convex.*

d. *For $i = 1, 2, ..., k$, the functions $q_i \longmapsto b_{\mu,\varphi}(\boldsymbol{q})$, $q_i \longmapsto B_{\mu,\varphi}(\boldsymbol{q})$ and $q_i \longmapsto \Lambda_{\mu,\varphi}(\boldsymbol{q})$ are non increasing.*

e. $b_{\mu,\varphi}(\boldsymbol{q}) \leq B_{\mu,\varphi}(\boldsymbol{q}) \leq \Lambda_{\mu,\varphi}(\boldsymbol{q})$.

More details about such measures, dimensions, the associated multifractal formalism for non necessary Gibbs measures and also on applications and links to existing cases may be found in [22] and [35]. For example, for $k = 1$, $\varphi = \log$, $\mathbf{q} = 0$, we come back to the classical definitions of the Hausdorff and packing measures and dimensions in their original forms.

3. The φ-mixed multifractal densities

In this section we propose to develop our main results by introducing a mixed type of multifractal density for vector valued non necessary Gibbs measures relatively to the φ-mixed multifractal analysis developed in [22], [35]. Consider a vector valued measure $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$ composed of probability measures on $\mathbb{R}^d$. For $x \in \operatorname{supp}(\mu)$, we define the upper and lower $(\mathbf{q}, t)$-densities of a probability measure ν with respect to μ and φ by

$$\overline{d}^{\mathbf{q},t}_{\mu,\varphi}(x,\nu) = \limsup_{r\to 0} \frac{\nu\Big(B(x,r)\Big)}{\mu\Big(B(x,r)\Big)^{\mathbf{q}} e^{t\varphi(r)}} \quad \text{and} \quad \underline{d}^{\mathbf{q},t}_{\mu,\varphi}(x,\nu) = \liminf_{r\to 0} \frac{\nu\Big(B(x,r)\Big)}{\mu\Big(B(x,r)\Big)^{\mathbf{q}} e^{t\varphi(r)}}.$$

For $a > 1$ and $1 \leq j \leq k$, we write

$$P^j_a(\mu) = \limsup_{r\searrow 0} \left(\sup_{x\in \operatorname{supp}(\mu_j)} \frac{\mu_j\Big(B(x,ar)\Big)}{\mu_j\Big(B(x,r)\Big)} \right).$$

We will now say that the measure μ satisfies the doubling condition if there exists $a > 1$ such that $P^j_a(\mu) < \infty$ for all $1 \leq j \leq k$. It is easily seen that the exact value of the parameter a is unimportant: $P^j_a(\mu) < \infty$, for some $a > 1$ if and only if $P^j_a(\mu) < \infty$, for all $a > 1$. Also, we will write $\mathscr{P}(\mathbb{R}^d)$ for the family of Borel probability measures on $\mathbb{R}^d$ and $\mathscr{P}_D(\mathbb{R}^d)$ for the family of Borel probability measures on $\mathbb{R}^d$ which satisfy the doubling condition. Our first main result which extends the results of [2, 4, 12, 14–16, 26, 27, 32, 34] is stated as follows.

Theorem 3.1. *Let E be a Borel subset of* $\operatorname{supp}(\mu)$.

1. *If $\mathscr{H}^{q,t}_{\mu,\varphi}(E) < \infty$, then there exists a constant $\xi > 0$ such that*

$$\frac{1}{\xi}\mathscr{H}^{q,t}_{\mu,\varphi}(E)\inf_{x\in E}\overline{d}^{q,t}_{\mu,\varphi}(x,\nu) \le \nu(E) \le \mathscr{H}^{q,t}_{\mu,\varphi}(E)\sup_{x\in E}\overline{d}^{q,t}_{\mu,\varphi}(x,\nu), \tag{3.1}$$

2. *Let φ be a doubling function and $\mu \in \mathscr{P}_D(\mathbb{R}^d)$. If $\mathscr{H}^{q,t}_{\mu,\varphi}(E) < \infty$, then*

$$\mathscr{H}^{q,t}_{\mu,\varphi}(E)\inf_{x\in E}\overline{d}^{q,t}_{\mu,\varphi}(x,\nu) \le \nu(E) \le \mathscr{H}^{q,t}_{\mu,\varphi}(E)\sup_{x\in E}\overline{d}^{q,t}_{\mu,\varphi}(x,\nu). \tag{3.2}$$

3. *If $\mathscr{P}^{q,t}_{\mu,\varphi}(E) < \infty$, then*

$$\mathscr{P}^{q,t}_{\mu,\varphi}(E)\inf_{x\in E}\underline{d}^{q,t}_{\mu,\varphi}(x,\nu) \le \nu(E) \le \mathscr{P}^{q,t}_{\mu,\varphi}(E)\sup_{x\in E}\underline{d}^{q,t}_{\mu,\varphi}(x,\nu). \tag{3.3}$$

Proof.

1. Denote $m = \inf_{x\in E}\overline{d}^{q,t}_{\mu,\varphi}(x,\nu)$. Without loss of generality we may assume that $m > 0$. Let $\varepsilon,\eta > 0$, and $F \subset E$ be closed set such that $\eta < m$. Let finally $H \subset F$ and denote for $\delta > 0$,

$$B_\delta(F) = \Big\{x \in \mathbb{R}^d;\ dist(F,x)\ \le \delta\Big\}.$$

As $B_F(\delta) \searrow F$ whenever $\delta \searrow 0$, there exists δ_0 for which

$$\nu\Big(B_F(\delta)\Big) \le \nu(F) + \frac{\epsilon}{2\xi}, \quad \forall\ 0 < \delta < \delta_0.$$

Let next $\delta < \delta_0$ be such that

$$\overline{\mathscr{H}}^{q,t}_{\mu,\varphi}(H) - \frac{\epsilon}{2(m-\eta)} \le \overline{\mathscr{H}}^{q,t}_{\mu,\varphi,\delta}(H).$$

and denote

$$\mathfrak{T}_\delta = \Big\{B(x,r), x \in H, 0 < r < \delta\ \text{ and }\ \nu\Big(B(x,r)\Big) \ge (m-\eta)\mu\Big(B(x,r)\Big)^{q} e^{t\varphi(r)}\Big\}.$$

By applying Theorem 6.1, there exists a ξ countable or finite set $(\mathfrak{T}^i_\delta)_{1\le i\le\xi}$, with $\mathfrak{T}^i_\delta = \Big(B(x_{ij},r_{ij})\Big)_j$, such that, for each i, $\mathfrak{T}^i_\delta$ consists of pairwise disjoint sets and $H \subset \underset{i}{\cup}\underset{B\in\mathfrak{T}^i_\delta}{\cup} B$. We then obtain

$$\begin{aligned}
\overline{\mathscr{H}}^{q,t}_{\mu,\varphi}(H)(m-\eta) &\le \overline{\mathscr{H}}^{q,t}_{\mu,\varphi,\delta}(H)(m-\eta) + \frac{\epsilon}{2}\\
&\le (m-\eta)\sum_{i=1}^{\xi}\sum_j \mu\Big(B(x_{ij},r_{ij})\Big)^{q} e^{t\varphi(r_{ij})} + \frac{\epsilon}{2}\\
&\le \sum_{i=1}^{\xi}\nu\Big(\underset{j}{\cup} B(x_{ij},r_{ij})\Big) + \frac{\epsilon}{2} \le \xi\nu\Big(B_F(\delta)\Big) + \frac{\epsilon}{2}\\
&\le \nu(F) + \epsilon \le \nu(E) + \epsilon.
\end{aligned}$$

We now prove the second inequality of (3.1). Denote $M = \sup_{x\in E}\overline{d}^{q,t}_{\mu,\varphi}(x,\nu)$, $\varepsilon,\eta > 0$, $s \in \mathbb{N}$ and consider the set

$$E_s = \Big\{x \in E, \nu\Big(B(x,r)\Big) \le (M+\eta)\mu\Big(B(x,r)\Big)^{q} e^{t\varphi(r)} \text{ and } 0 < r < \frac{1}{s}\Big\}.$$

There exists a $(1/s)$-centered covering $\Big(B(x_i, r_i)\Big)_i$ of E_s, such that

$$\sum_i \mu\Big(B(x_i, r_i)\Big)^{\mathbf{q}} e^{t\varphi(r_i)} \leq \overline{\mathscr{H}}^{\mathbf{q},t}_{\mu,\varphi,\frac{1}{s}}(E_s) + \frac{\varepsilon}{M+\eta}.$$

Consequently,

$$\begin{aligned} \nu(E_s) &\leq \sum_i \nu\Big(B(x_i, r_i)\Big) \\ &\leq (M+\eta)\sum_i \mu\Big(B(x_i, r_i)\Big)^{\mathbf{q}} e^{t\varphi(r_i)} \\ &\leq (M+\eta)\overline{\mathscr{H}}^{\mathbf{q},t}_{\mu,\varphi,\frac{1}{s}}(E_s) + \varepsilon. \end{aligned}$$

As $E_s \searrow E$ whenever $s \nearrow \infty$, we get

$$\nu(E) \leq (M+\eta)\overline{\mathscr{H}}^{\mathbf{q},t}_{\mu,\varphi}(E) + \varepsilon.$$

2. Let $m = \inf_{x\in E} \overline{d}^{\mathbf{q},t}_{\mu,\varphi}(x,\nu)$. Without loss of the generality we may assume that $m > 0$. Consider for $s \in \mathbb{N}^*$ the set

$$E_s = \left\{x \in E, \frac{\mu_j\Big(B(x, 5r)\Big)}{\mu_j\Big(B(x, r)\Big)} < s,\ \forall\, 1 \leq j \leq k \text{ and } \frac{\varphi(5r)}{\varphi(r)} < s,\ 0 < r < \frac{1}{s}\right\}.$$

We claim as previously, that for any $\epsilon > 0, 0 < \eta < m$, $s \in \mathbb{N}^*$, for all closed subset $F \subset E_s$ and for any $H \subset F$, we have

$$\overline{\mathscr{H}}^{\mathbf{q},t}_{\mu,\varphi}(H)(m-\eta) \leq \nu(E) + \epsilon, \tag{3.4}$$

which in turns yields that

$$\mathscr{H}^{\mathbf{q},t}_{\mu,\varphi}(F)(m-\eta) \leq \nu(E) + \epsilon$$

and

$$\mathscr{H}^{\mathbf{q},t}_{\mu,\varphi}(E_s)(m-\eta) \leq \nu(E) + \epsilon.$$

Next, as $E_s \nearrow E$ when $s \nearrow \infty$ we get

$$\mathscr{H}^{\mathbf{q},t}_{\mu,\varphi}(E)(m-\eta) \leq \nu(E) + \epsilon.$$

It remains to prove the inequality (3.4). To do this, we consider for $\delta > 0$ the set

$$\mathfrak{T}_\delta = \Big\{B(x, r), x \in H, 5r < \delta \text{ and } \nu\Big(B(x, r)\Big) \geq (m-\eta)\mu\Big(B(x, r)\Big)^{\mathbf{q}} e^{t\varphi(r)}\Big\}.$$

Next let $\Big(B(x_i, r_i)\Big)_i$ be the countable disjoint sub-family of $\mathfrak{T}_\delta$ defined in Vitali's Theorem 6.4, such that

$$H \setminus \bigcup_{i=1}^{k} B(x_i, r_i) \subset \bigcup_{i\geq k} B(x_i, 5r_i),\ \forall\, k \geq 1. \tag{3.5}$$

From the definition of E_s, there exists a constant $c = c(\mathbf{q}, t, s) > 0$ such that

$$\begin{aligned} \sum_i \mu\Big(B(x_i, 5r_i)\Big)^{\mathbf{q}} e^{t\varphi(5r_i)} &\leq c\sum_i \mu\Big(B(x_i, r_i)\Big)^{\mathbf{q}} e^{t\varphi(r_i)} \\ &\leq c\,(m-\eta)^{-1}\sum_i \nu\Big(B(x_i, r_i)\Big) \\ &\leq c\,(m-\eta)^{-1}\nu\Big(\bigcup_i B(x_i, r_i)\Big) < \infty. \end{aligned}$$

We deduce that there exists an integer N, such that

$$\sum_{i>N}\mu\Big(B(x_i,5r_i)\Big)^{\mathbf{q}}e^{t\varphi(5r_i)}\leq\frac{\epsilon}{3}(m-\eta)^{-1}.$$

Moreover, since $\overline{\mathscr{H}}^{\mathbf{q},t}_{\mu,\varphi}(H)<\infty$, we may choose $\delta<\delta_0$, with

$$\overline{\mathscr{H}}^{\mathbf{q},t}_{\mu,\varphi}(H)-\frac{\epsilon}{3(m-\eta)}\leq\overline{\mathscr{H}}^{\mathbf{q},t}_{\mu,\varphi,\delta}(H). \tag{3.6}$$

It follows from (3.5) and (3.6) that

$$\begin{array}{rcl}\overline{\mathscr{H}}^{\mathbf{q},t}_{\mu,\varphi}(H)(m-\eta) & \leq & \overline{\mathscr{H}}^{\mathbf{q},t}_{\mu,\varphi,\delta}(H)(m-\eta)+\dfrac{\epsilon}{3}\\ & \leq & \displaystyle\sum_{i>N}\mu\Big(B(x_i,5r_i)\Big)^{\mathbf{q}}e^{t\varphi(5r_i)}+\sum_{i\leq N}\mu\Big(B(x_i,r_i)\Big)^{\mathbf{q}}e^{t\varphi(r_i)}+\frac{\epsilon}{3}\\ & \leq & \displaystyle\sum_{i}\nu\Big(B(x_i,r_i)\Big)+\frac{2\epsilon}{3}.\end{array}$$

Recall now that when $\delta\searrow 0$, there holds that $B_F(\delta)\searrow F$. Consequently, there exists $\delta_0>0$ for which

$$\nu\Big(B_F(\delta)\Big)\leq\nu(F)+\frac{\epsilon}{3},\ \forall\,0<\delta<\delta_0.$$

This implies that

$$\overline{\mathscr{H}}^{\mathbf{q},t}_{\mu,\varphi}(H)(m-\eta)\leq\nu\Big(B_F(\delta)\Big)+\frac{2\epsilon}{3}\leq\nu(E)+\epsilon.$$

3. We start by proving the right inequality of (3.3). Denote $a=\sup_{x\in E}\underline{d}^{\mathbf{q},t}_{\mu,\varphi}(x,\nu)$ and let $F\subset E$. It suffices to prove that

$$\nu(F)\leq a\overline{\mathscr{P}}^{\mathbf{q},t}_{\mu,\varphi}(F),\qquad\forall F\subset E. \tag{3.7}$$

Indeed, whenever (3.7) holds, we consider a covering $(E_i)_i$ of E and obtain

$$\nu(E)=\nu\Big(\cup_i\,(E\cap E_i)\Big)\leq\sum_i\nu(E\cap E_i)\leq a\sum_i\overline{\mathscr{P}}^{\mathbf{q},t}_{\mu,\varphi}(E\cap E_i)\leq a\sum_i\overline{\mathscr{P}}^{\mathbf{q},t}_{\mu,\varphi}(E_i).$$

Taking the inf over all the coverings $(E_i)_i$, the result follows immediately. We now proceed by proving (3.7). Let $F\subset E,\epsilon,\eta,\delta>0$ be such that

$$\overline{\mathscr{P}}^{\mathbf{q},t}_{\mu,\varphi,\delta}(F)\leq\overline{\mathscr{P}}^{\mathbf{q},t}_{\mu,\varphi}(F)+\frac{\epsilon}{a+\eta}.$$

Consider next the set

$$\mathfrak{T}_\delta=\Big\{B(x,r),x\in F,r<\delta\ \text{ and }\ \nu\Big(B(x,r)\Big)\leq(a+\eta)\mu\Big(B(x,r)\Big)^{\mathbf{q}}e^{t\varphi(r)}\Big\}.$$

By Theorem 6.3, there exists a δ-packing $\Big(B(x_i,r_i)\Big)_i\subset\mathfrak{T}_\delta$ of F satisfying

$$\nu\Big(F\setminus\cup_iB(x_i,r_i)\Big)=0.$$

Moreover, we have

$$\begin{array}{rcl}\nu(F) & = & \nu\Big(\cup_i\,(F\cap B(x_i,r_i))\Big)\leq\displaystyle\sum_i\nu(B(x_i,r_i))\leq(a+\eta)\sum_i\mu\Big(B(x_i,r_i)\Big)^{\mathbf{q}}e^{t\varphi(r_i)}\\ & \leq & (a+\eta)\overline{\mathscr{P}}^{\mathbf{q},t}_{\mu,\varphi,\delta}(F)\leq(a+\eta)\overline{\mathscr{P}}^{\mathbf{q},t}_{\mu,\varphi}(F)+\epsilon.\end{array}$$

Letting $\epsilon \to 0$, we obtain

$$\nu(F) \le (a+\eta)\overline{\mathscr{P}}^{\mathbf{q},t}_{\mu,\varphi}(F).$$

Next, making $\eta \to 0$ equation (3.7) follows immediately.

We now proceed by proving the left inequality of (3.3). Denote $m = \inf_{x\in E} \underline{d}^{\mathbf{q},t}_{\mu,\varphi}(x,\nu)$ and assume with out loss of the generality that $m > 0$. We just need to prove that

$$\mathscr{P}^{\mathbf{q},t}_{\mu,\varphi}(E)(m-\eta) \le \nu(E) + \varepsilon, \qquad \forall\, \varepsilon > 0, \quad \forall\, 0 < \eta < m.$$

Fix $\varepsilon > 0$ and $0 < \eta < m$. It is sufficient to prove that, for any closed subset F of E,

$$\mathscr{P}^{\mathbf{q},t}_{\mu,\varphi}(F)(m-\eta) \le \nu(E) + \varepsilon.$$

Recall here again that, if $\delta \searrow 0$, then $B_F(\delta) \searrow F$. So, there exists δ_0 satisfying

$$\nu\Big(B_F(\delta)\Big) \le \nu(F) + \epsilon, \qquad \forall\, 0 < \delta < \delta_0.$$

For $s \in \mathbb{N}$, consider the set

$$F_s = \Big\{ x \in F, \nu\Big(B(x,r)\Big) \ge (m-\eta)\mu\Big(B(x,r)\Big)^{\mathbf{q}} e^{t\varphi(r)}, \ \text{ for } \ 0 < r < \frac{1}{s} \Big\}.$$

Fix $s \in \mathbb{N}$ and $0 < \delta < \sup\{\frac{1}{s}, \delta_0\}$. Let $\Big(B(x_i, r_i)\Big)_i$ be a centered δ-packing of F_s. Then,

$$(m-\eta)\sum_i \mu(B(x_i,r_i))^{\mathbf{q}} e^{t\varphi(r_i)} \le \sum_i \nu\Big(B(x_i,r_i)\Big) = \nu\Big(\cup_i B(x_i,r_i)\Big) \le \nu\Big(B_F(\delta)\Big) \le \nu(F) + \epsilon \le \nu(E) + \epsilon.$$

Hence,

$$\mathscr{P}^{\mathbf{q},t}_{\mu,\varphi}(F_s)(m-\eta) \le \overline{\mathscr{P}}^{\mathbf{q},t}_{\mu,\varphi}(F_s)(m-\eta) \le \overline{\mathscr{P}}^{\mathbf{q},t}_{\mu,\varphi,\delta}(F_s)(m-\eta) \le \nu(E) + \varepsilon.$$

As $F_s \nearrow F$ when $s \nearrow \infty$, we obtain

$$\mathscr{P}^{\mathbf{q},t}_{\mu,\varphi}(F)(m-\eta) \le \nu(E) + \varepsilon.$$

□

In the following part we propose to link the previous estimations of the φ-mixed multifractal densities to the exact computation of the both φ-mixed generaliations of Hausdorff and packing measures and dimensions. We will show precisely that these densities permit in some special cases to compute the φ-mixed multifractal generalizations of both Hausdorff and packing dimensions of sets characterized by the existence of some suitable measure(s) supported on them. This problem has been the object of several papers such as [2, 4, 12, 32], ...

Let $E \subset \mathbb{R}^d$ be a Borel subset and denote by $\mathscr{H}^{\mathbf{q},s}_{\mu,\varphi} \llcorner E$ (resp. $\mathscr{P}^{\mathbf{q},t}_{\mu,\varphi} \llcorner E$) the s-dimensional centered Hausdorff measure $\mathscr{H}^{\mathbf{q},s}_{\mu,\varphi}$ (resp. t-dimensional centered packing measure $\mathscr{P}^{\mathbf{q},t}_{\mu,\varphi}$) restricted to E (if the Hausdorff or packing measure of E is zero, then the restriction measure is in fact a zero measure).

For $\nu = \mathscr{H}^{\mathbf{q},t}_{\mu,\varphi} \llcorner_E$, we define

$$\overline{D}^{\mathbf{q},t}_{\mu,\varphi}(x,E) = \overline{d}^{\mathbf{q},t}_{\mu,\varphi}(x,\nu) \qquad \text{and} \qquad \underline{D}^{\mathbf{q},t}_{\mu,\varphi}(x,E) = \underline{d}^{\mathbf{q},t}_{\mu,\varphi}(x,\nu).$$

Similarly, for $\nu = \mathscr{P}^{\mathbf{q},t}_{\mu,\varphi} \llcorner_E$, we define

$$\overline{\Delta}^{\mathbf{q},t}_{\mu,\varphi}(x,E) = \overline{d}^{q,t}_{\mu,\varphi}(x,\nu) \qquad \text{and} \qquad \underline{\Delta}^{\mathbf{q},t}_{\mu,\varphi}(x,E) = \underline{d}^{\mathbf{q},t}_{\mu,\varphi}(x,\nu).$$

Whenever $\overline{D}^{\mathbf{q},t}_{\mu,\varphi}(x,E) = \underline{D}^{\mathbf{q},t}_{\mu,\varphi}(x,E)$ $\Big($ resp. $\overline{\Delta}^{\mathbf{q},t}_{\mu,\varphi}(x,\nu) = \underline{\Delta}^{\mathbf{q},t}_{\mu,\varphi}(x,\nu)$ $\Big)$, we write $D^{\mathbf{q},t}_{\mu,\varphi}(x,E)$ $\Big($ resp. $\Delta^{\mathbf{q},t}_{\mu,\varphi}(x,E)$ $\Big)$ for the common value.

As a result of Theorem 3.1 we obtain the estimations

$$\inf_{x\in E}\overline{D}^{\mathbf{q},t}_{\mu,\varphi}(x,E) \le 1 \le \sup_{x\in E}\overline{D}^{\mathbf{q},t}_{\mu,\varphi}(x,E) \quad \text{and} \quad \inf_{x\in E}\underline{\Delta}^{q,t}_{\mu,\varphi}(x,E) \le 1 \le \sup_{x\in E}\underline{\Delta}^{\mathbf{q},t}_{\mu,\varphi}(x,E).$$

An interesting question is then to study the case of equality for the last inequalities. The equality means in some sense that the measure ν plays the role of the Gibbs measure constructed on the singularities set of the vector valued measure μ. Here, it means that the vector valued measure μ when controled by the gauge function φ permits the construction of a Borel probability measure ν on the φ-mixed singularities sets which permit in turns to prove the validity of an associated variant of the multifractal formalism conjectured in [22]. So, consider the sets

$$\underline{K} = \Big\{x\in E,\ \underline{D}^{\mathbf{q},t}_{\mu,\varphi}(x,E) = 1\Big\},\ \overline{K} = \Big\{x\in E,\ \overline{D}^{\mathbf{q},t}_{\mu,\varphi}(x,E) = 1\Big\},$$

$$\underline{T} = \Big\{x\in E,\ \underline{\Delta}^{\mathbf{q},t}_{\mu,\varphi}(x,E) = 1\Big\},\ \overline{T} = \Big\{x\in E,\ \overline{\Delta}^{\mathbf{q},t}_{\mu,\varphi}(x,E) = 1\Big\},$$

$$K = \underline{K}\cap\overline{K} \quad \text{and} \quad T = \underline{T}\cap\overline{T}.$$

The following result provides a description of these sets by means of their φ-mixed multifractal generalizations of Hausdorff and packing dimensions.

Theorem 3.2. *Let E be a Borel subset of* $\operatorname{supp}(\mu)$.

1. *Let φ be a doubling function and $\mu\in\mathscr{P}_D(\mathbb{R}^d)$. If $\mathscr{H}^{\mathbf{q},t}_{\mu,\varphi}(E)<\infty$, then* $\dim^{q}_{\mu,\varphi}(\overline{K}) = t$.
2. *If $\mathscr{P}^{\mathbf{q},t}_{\mu,\varphi}(E)<\infty$, then* $\operatorname{Dim}^{q}_{\mu,\varphi}(\underline{T}) = t$.
3. *Let φ be a doubling function and $\mu\in\mathscr{P}_D(\mathbb{R}^d)$. If $\mathscr{P}^{\mathbf{q},t}_{\mu,\varphi}(E)<\infty$, then the following assertions are equivalent*
   (a) $\mathscr{H}^{\mathbf{q},t}_{\mu,\varphi} = \mathscr{P}^{\mathbf{q},t}_{\mu,\varphi}$.
   (b) $\overline{D}^{q,t}_{\mu,\varphi}(x,E) = 1 = \underline{D}^{q,t}_{\mu,\varphi}(x,E)$ *for* $\mathscr{P}^{\mathbf{q},t}_{\mu,\varphi}$ *-a.a.* $x\in E$.
   (c) $\overline{\Delta}^{q,t}_{\mu,\varphi}(x,E) = 1 = \underline{\Delta}^{q,t}_{\mu,\varphi}(x,E)$ *for* $\mathscr{P}^{\mathbf{q},t}_{\mu,\varphi}$ *-a.a.* $x\in E$.
4. *If $\mathscr{H}^{\mathbf{q},t}_{\mu,\varphi} = \mathscr{P}^{\mathbf{q},t}_{\mu,\varphi}<\infty$, then*

$$\dim^{q}_{\mu,\varphi}(\overline{K}) = \dim^{q}_{\mu,\varphi}(K) = \dim^{q}_{\mu,\varphi}(\overline{T}) = \dim^{q}_{\mu,\varphi}(T) = t.$$

Proof. 1. We show firstly that

$$\overline{D}^{\mathbf{q},t}_{\mu,\varphi}(x,E) \le 1, \quad \text{for} \quad \mathscr{H}^{\mathbf{q},t}_{\mu,\varphi}\text{-a.a. } x\in E. \tag{3.8}$$

Consider the sets

$$F = \Big\{x\in E,\ \overline{D}^{\mathbf{q},t}_{\mu,\varphi}(x,E) > 1\Big\}$$

and

$$F_m = \Big\{x\in E,\ \overline{D}^{\mathbf{q},t}_{\mu,\varphi}(x,E) > 1+\frac{1}{m}\Big\},\ m\in\mathbb{N}.$$

It follows from (3.2) that

$$\Big(1+\frac{1}{m}\Big)\mathscr{H}^{\mathbf{q},t}_{\mu,\varphi}(F_m) \le \mathscr{H}^{\mathbf{q},t}_{\mu,\varphi}(F_m).$$

Consequently, $\mathscr{H}^{\mathbf{q},t}_{\mu,\varphi}(F_m) = 0$. Since $F = \bigcup_m F_m$, we obtain (3.8).

We now prove the inequality

$$1 \le \overline{D}^{\mathbf{q},t}_{\mu,\varphi}(x,E) \quad \text{for} \quad \mathscr{H}^{\mathbf{q},t}_{\mu,\varphi}\text{-a.a. } x\in E. \tag{3.9}$$

We consider as previously the sets

$$G = \Big\{x \in E,\ \overline{D}^{\mathbf{q},t}_{\mu,\varphi}(x,E) < 1\Big\}$$

and

$$G_m = \Big\{x \in E,\ \overline{D}^{\mathbf{q},t}_{\mu,\varphi}(x,E) \le 1 - \frac{1}{m}\Big\},\ m \in \mathbb{N}.$$

Applying (3.2) we obtain

$$\Big(1 - \frac{1}{m}\Big)\mathscr{H}^{\mathbf{q},t}_{\mu,\varphi}(G_m) \ge \mathscr{H}^{\mathbf{q},t}_{\mu,\varphi}(G_m).$$

Then, $\mathscr{H}^{\mathbf{q},t}_{\mu,\varphi}(G_m) = 0$. Since $G = \underset{m}{\cup} G_m$, we obtain (3.9). Finally, (3.8) and (3.9) lead to the desired result.

2. The proof is similar to assertion 1.

3. $\underline{(a) \Rightarrow (b).}$ It follows from assertion 1 that for $\mu \in \mathscr{P}_D(E)$ and $\mathscr{H}^{\mathbf{q},t}_{\mu,\varphi}(E) < \infty$. Thus

$$\overline{D}^{\mathbf{q},t}_{\mu,\varphi}(x,E) = 1 \quad \text{for} \quad \mathscr{H}^{\mathbf{q},t}_{\mu,\varphi}\text{-a.a. } x \in E. \tag{3.10}$$

Next, observe that

$$\mathscr{H}^{\mathbf{q},t}_{\mu,\varphi}(F) = \mathscr{P}^{\mathbf{q},t}_{\mu,\varphi}(F), \quad \text{for any } \ F \subset E. \tag{3.11}$$

Thanks to (3.10) and (3.11), we get

$$\overline{D}^{\mathbf{q},t}_{\mu,\varphi}(x,E) = 1 \quad \text{for} \quad \mathscr{P}^{\mathbf{q},t}_{\mu,\varphi}\text{-a.a. } x \in E. \tag{3.12}$$

Now, we consider the sets

$$F = \Big\{x \in E, \underline{D}^{\mathbf{q},t}_{\mu,\varphi}(x,E) < 1\Big\},$$

and

$$F_m = \Big\{x \in E, \underline{D}^{\mathbf{q},t}_{\mu,\varphi}(x,E) < 1 - \frac{1}{m}\Big\},\ m \in \mathbb{N}.$$

From Theorem 3.1 and (3.12), we obtain

$$\mathscr{P}^{\mathbf{q},t}_{\mu,\varphi}(F_m) = \mathscr{H}^{\mathbf{q},t}_{\mu,\varphi}(F_m) \le \Big(1 - \frac{1}{m}\Big)\mathscr{P}^{\mathbf{q},t}_{\mu,\varphi}(F_m).$$

This implies that $\mathscr{P}^{\mathbf{q},t}_{\mu,\varphi}(F_m) = 0$. As $F = \underset{m}{\cup} F_m$, we obtain $\mathscr{P}^{\mathbf{q},t}_{\mu,\varphi}(F) = 0$, i.e.

$$\underline{D}^{\mathbf{q},t}_{\mu,\varphi}(x,E) \ge 1 \quad \text{for} \quad \mathscr{P}^{\mathbf{q},t}_{\mu,\varphi}\text{-a.a. } x \in E. \tag{3.13}$$

Finally, (3.12) and (3.13) lead to (b).
$\underline{(b) \Rightarrow (a).}$ Consider the set

$$F = \Big\{x \in E, D^{\mathbf{q},t}_{\mu,\varphi}(x,E) = 1\Big\}.$$

It follows from Theorem 3.1 and (b) that

$$\mathscr{P}^{\mathbf{q},t}_{\mu,\varphi}(E) = \mathscr{P}^{\mathbf{q},t}_{\mu,\varphi}(F) \le \mathscr{H}^{\mathbf{q},t}_{\mu,\varphi}(F) \le \mathscr{H}^{\mathbf{q},t}_{\mu,\varphi}(E) \le \mathscr{P}^{\mathbf{q},t}_{\mu,\varphi}(E),$$

which yields that $\mathscr{H}^{\mathbf{q},t}_{\mu,\varphi}(E) = \mathscr{P}^{\mathbf{q},t}_{\mu,\varphi}(E)$.
$\underline{(b) \Leftrightarrow (c).}$ It may be checked by following similar techniques as in the proof of $\big((a) \Leftrightarrow (b)\big)$.

4. It is an immediate consequence of 3.

□

Remark 3.3. *This result is important as it consists of a first information leading to the computation of the multifractal spectrum due to the introduced densities. Indeed, related to the original form of the multifractal spectrum, the starting point is to establish an estimation of the form*

$$\nu(B(x,r)) \sim (\mu(B(x,r))^q e^{(t\pm\varepsilon)\varphi(r)}, \quad r \to 0.$$

For example, when φ is the classical logarithm ($\varphi(r) = \log r$), we obtain an estimation of the form

$$\nu(B(x,r)) \sim (\mu(B(x,r))^q (2r)^{t\pm\varepsilon}, \quad r \to 0$$

which in the case of a Hölderian (Gibbs) measur μ means that the densities considered above are all equals 1 and thus permits to compute the multifractal spectrum (evaluated as the Hausdorff dimension of the level sets of the densities) by means of a Legendre transform of a convex function issued from the multifractal generalized dimensions $b^{\mathbf{q}}_{\mu,\varphi}$, $B^{\mathbf{q}}_{\mu,\varphi}$ and $\Delta^{\mathbf{q}}_{\mu,\varphi}$.

4. Regularities of φ-mixed Hausdorff and packing measures

In this section, we will prove a decomposition theorem of Besicovitch's type for the φ-mixed multifractal generalizations of Hausdorff and packing measures.

Definition 4.1. *Let $E \subset \mathrm{supp}(\mu)$ be a Borel set with $0 < \mathscr{H}^{\mathbf{q},t}_{\mu,\varphi}(E) < +\infty$. For backgrounds the readers may refer to [1, 31–33].*

1. *A point $x \in E$ is called $\mathscr{H}^{\mathbf{q},t}_{\mu,\varphi}$-regular point of E if $D^{\mathbf{q},t}_{\mu,\varphi}(x,E)$ exists. Otherwise x is a $\mathscr{H}^{\mathbf{q},t}_{\mu,\varphi}$-irregular point.*
2. *E is said to be $\mathscr{H}^{\mathbf{q},t}_{\mu,\varphi}$-regular if $\mathscr{H}^{\mathbf{q},t}_{\mu,\varphi}$-almost all its points are $\mathscr{H}^{\mathbf{q},t}_{\mu,\varphi}$-regular.*
3. *E is said to be $\mathscr{H}^{\mathbf{q},t}_{\mu,\varphi}$-irregular if $\mathscr{H}^{\mathbf{q},t}_{\mu,\varphi}$-almost all its points are $\mathscr{H}^{\mathbf{q},t}_{\mu,\varphi}$-irregular.*

Remark 4.2. *Similarly, we define the regularities for the relative φ-mixed multifractal packing measure $\mathscr{P}^{\mathbf{q},t}_{\mu,\varphi}$ by replacing $\mathscr{H}^{\mathbf{q},t}_{\mu,\varphi}$ in Definition 4.1 above by $\mathscr{P}^{\mathbf{q},t}_{\mu,\varphi}$.*

Lemma 4.3. *Let $E \subset \mathrm{supp}(\mu)$ be a Borel subset and $F \subseteq E$ be $\mathscr{H}^{\mathbf{q},t}_{\mu,\varphi}$-measurable.*

1. *Whenever $\mathscr{H}^{\mathbf{q},t}_{\mu,\varphi}(E) < \infty$ we have*

$$\overline{D}^{\mathbf{q},t}_{\mu,\varphi}(x,E) = \overline{D}^{\mathbf{q},t}_{\mu,\varphi}(x,F) \text{ and } \underline{D}^{\mathbf{q},t}_{\mu,\varphi}(x,E) = \underline{D}^{\mathbf{q},t}_{\mu,\varphi}(x,F), \text{ for } \mathscr{H}^{\mathbf{q},t}_{\mu,\varphi}\text{-a.e. } x \in F.$$

2. *Whenever $\mathscr{P}^{\mathbf{q},t}_{\mu,\varphi}(E) < \infty$ we have*

$$\overline{\Delta}^{\mathbf{q},t}_{\mu,\varphi}(x,E) = \overline{\Delta}^{\mathbf{q},t}_{\mu,\varphi}(x,F) \text{ and } \underline{\Delta}^{\mathbf{q},t}_{\mu,\varphi}(x,E) = \underline{\Delta}^{\mathbf{q},t}_{\mu,\varphi}(x,F), \text{ for } \mathscr{H}^{\mathbf{q},t}_{\mu,\varphi}\text{-a.e. } x \in F.$$

Proof. Let $\nu \in \mathscr{P}(\mathbb{R}^d)$ and define the measure ν_F by

$$\nu_F(A) = \nu(F \cap A),$$

for all Borel set A. Assume that $\mathscr{H}^{\mathbf{q},t}_{\mu,\varphi}(E) < \infty$. We will prove that for $\mathscr{H}^{\mathbf{q},t}_{\mu,\varphi}$-a.e. $x \in F$,

$$\overline{d}^{\mathbf{q},t}_{\mu,\varphi}(x,\nu) = \overline{d}^{\mathbf{q},t}_{\mu,\varphi}(x,\nu_F) \text{ and } \underline{d}^{\mathbf{q},t}_{\mu,\varphi}(x,\nu) = \underline{d}^{\mathbf{q},t}_{\mu,\varphi}(x,\nu_F). \tag{4.1}$$

Indeed, we already now that

$$\underline{d}^{\mathbf{q},t}_{\mu,\varphi}(x,\nu) \geq \underline{d}^{\mathbf{q},t}_{\mu,\varphi}(x,\nu_F) \qquad \text{and} \qquad \overline{d}^{\mathbf{q},t}_{\mu,\varphi}(x,\nu) \geq \overline{d}^{q,t}_{\mu,\varphi}(x,\nu_F).$$

Denote next $\lambda(A) = \nu(A \setminus F)$ for all $A \subseteq X$. Then,

$$\nu(A) = \nu\Big(A \cap (F^c \cup F)\Big) = \nu(A \setminus F) + \nu(A \cap F) = \lambda(A) + \nu_F(A).$$

It holds that

$$\underline{d}^{\mathbf{q},t}_{\mu,\varphi}(x,\nu) \leq \underline{d}^{\mathbf{q},t}_{\mu,\varphi}(x,\nu_F) + \overline{d}^{\mathbf{q},t}_{\mu,\varphi}(x,\lambda) \text{ and } \overline{d}^{\mathbf{q},t}_{\mu,\varphi}(x,\nu) \leq \overline{d}^{\mathbf{q},t}_{\mu,\varphi}(x,\nu_F) + \overline{d}^{\mathbf{q},t}_{\mu,\varphi}(x,\lambda).$$

Consequently, it suffices to show that $\overline{d}^{\mathbf{q},t}_{\mu,\varphi}(x,\lambda) = 0$. So, for $k \in \mathbb{N}$, let

$$F_k = \left\{x \in F;\ \overline{d}^{\mathbf{q},t}_{\mu,\varphi}(x,\lambda) \geq \frac{1}{k}\right\}.$$

By (3.1), we immediately conclude that

$$\frac{1}{k\xi}\mathscr{H}^{\mathbf{q},t}_{\mu,\varphi}(F_k) \leq \lambda(F_k) = \nu(F_k \setminus F) = \nu(\emptyset) = 0, \qquad \text{for all } k \geq 1,$$

which yields that $\overline{d}^{\mathbf{q},t}_{\mu,\varphi}(x,\lambda) = 0$, for $\mathscr{H}^{\mathbf{q},t}_{\mu,\varphi}$-a.e. $x \in F$ and thus leads to (4.1).
Now, in (4.1), taking $\nu = \mathscr{H}^{\mathbf{q},t}_{\mu,\varphi_{\llcorner E}}$ $\Big($resp. $\nu = \mathscr{P}^{\mathbf{q},t}_{\mu,\varphi_{\llcorner E}}\Big)$, we obtain Assertion 1 $\Big($resp. Assertion 2$\Big)$ of the Lemma. □

Lemma 4.4. *Let E be a Borel subset of* $\mathrm{supp}(\mu)$ *with $\mathscr{P}^{\mathbf{q},t}_{\mu,\varphi}(E) < \infty$ and $F = \Big\{x \in E;\ \overline{\Delta}^{\mathbf{q},t}_{\mu,\varphi}(x,E) < +\infty\Big\}$. If G is a Borel subset of F such that $\mathscr{H}^{\mathbf{q},t}_{\mu,\varphi}(G) = 0$, then $\mathscr{P}^{\mathbf{q},t}_{\mu,\varphi}(G) = 0$.*

Proof. It follows from (3.1) by taking $\nu = \mathscr{P}^{\mathbf{q},t}_{\mu,\varphi_{\llcorner E}}$. □

Theorem 4.5. *Let E be a Borel subset of* $\mathrm{supp}(\mu)$, *φ satisfying (2.1) and $\mu \in \mathscr{P}_D(\mathbb{R}^d)$.*

1. *If $\mathscr{H}^{\mathbf{q},t}_{\mu,\varphi}(E) < +\infty$, then the set of $\mathscr{H}^{\mathbf{q},t}_{\mu,\varphi}$-regular points of E is $\mathscr{H}^{\mathbf{q},t}_{\mu,\varphi}$-regular and the set of $\mathscr{H}^{\mathbf{q},t}_{\mu,\varphi}$-irregular points of E is $\mathscr{H}^{\mathbf{q},t}_{\mu,\varphi}$-irregular.*
2. *If $\mathscr{P}^{\mathbf{q},t}_{\mu,\varphi}(E) < +\infty$, then the set of $\mathscr{P}^{\mathbf{q},t}_{\mu,\varphi}$-regular points of E is $\mathscr{P}^{\mathbf{q},t}_{\mu,\varphi}$-regular and the set of $\mathscr{P}^{\mathbf{q},t}_{\mu,\varphi}$-irregular points of E is $\mathscr{P}^{\mathbf{q},t}_{\mu,\varphi}$-irregular.*

Proof.

1. Put $F = \Big\{x \in E;\ D^{\mathbf{q},t}_{\mu,\varphi}(x,E) = 1\Big\}$. Since $F \subset E$ and $\mathscr{H}^{\mathbf{q},t}_{\mu,\varphi}(E) < \infty$, from Theorem 3.2, we have $\overline{D}^{\mathbf{q},t}_{\mu,\varphi}(x,F) = 1$, for $\mathscr{H}^{\mathbf{q},t}_{\mu,\varphi}$-a.e. $x \in F$. So, we only have to prove that $\underline{D}^{\mathbf{q},t}_{\mu,\varphi}(x,F) = 1$, for $\mathscr{H}^{\mathbf{q},t}_{\mu,\varphi}$-a.e. $x \in F$. Lemma 4.3 implies that
$$\underline{D}^{\mathbf{q},t}_{\mu,\varphi}(x,F) = \underline{D}^{\mathbf{q},t}_{\mu,\varphi}(x,E), \quad \text{for } \mathscr{H}^{\mathbf{q},t}_{\mu,\varphi}\text{-a.e. } x \in F.$$
We therefore conclude that $\underline{D}^{\mathbf{q},t}_{\mu,\varphi}(x,F) = 1$, for $\mathscr{H}^{\mathbf{q},t}_{\mu,\varphi}$-a.e. $x \in F$. Again, by Lemma 4.3,
$$\overline{D}^{\mathbf{q},t}_{\mu,\varphi}(x,E \setminus F) = \overline{D}^{\mathbf{q},t}_{\mu,\varphi}(x,E), \quad \text{for } \mathscr{H}^{\mathbf{q},t}_{\mu,\varphi}\text{-a.e. } x \in E \setminus F$$
and
$$\underline{D}^{\mathbf{q},t}_{\mu,\varphi}(x,E \setminus F) = \underline{D}^{\mathbf{q},t}_{\mu,\varphi}(x,E), \quad \text{for } \mathscr{H}^{\mathbf{q},t}_{\mu,\varphi}\text{-a.e. } x \in E \setminus F.$$
Finally, it follows that
$$\mathscr{H}^{\mathbf{q},t}_{\mu,\varphi}\Big(\Big\{x \in E \setminus F,\ D^{\mathbf{q},t}_{\mu,\varphi}(x,E \setminus F) = 1\Big\}\Big) = 0.$$
2. The proof is similar to assertion 1.

□

5. Application

Our purpose in this section is to establish a necessary condition for a strong regularity with the φ-mixed multifractal generalizations of Hausdorff and packing measures.

Definition 5.1. *Let $(X, \mathscr{F}, \mu)$ be a measure space and E, F in $\mathscr{F}$. We say that E is a subset of F μ-almost everywhere and write $E \subseteq F$ μ-a.e., if $\mu(F \setminus E) = 0$.*

Consider the following sets

$$F = \Big\{x \in E;\ D^{\mathbf{q},t}_{\mu,\varphi}(x, E) = 1\Big\} \quad \text{and} \quad G = \Big\{x \in E;\ \Delta^{\mathbf{q},t}_{\mu,\varphi}(x, E) = 1\Big\}.$$

Theorem 5.2. *Let φ be as in (2.1) and $\mu \in \mathscr{P}_D(\mathbb{R}^d)$. Let E be a $\mathscr{P}^{\mathbf{q},t}_{\mu,\varphi}$-measurable set with $\mathscr{P}^{\mathbf{q},t}_{\mu,\varphi}(E) < \infty$ and $\overline{\Delta}^{\mathbf{q},t}_{\mu,\varphi}(x, E) < +\infty$ for all $x \in E$. The following assertions are equivalent for any measurable subset B of E.*

1. $\mathscr{H}^{\mathbf{q},t}_{\mu,\varphi}(B) = \mathscr{P}^{\mathbf{q},t}_{\mu,\varphi}(B)$.
2. $\mathscr{P}^{\mathbf{q},t}_{\mu,\varphi}\Big(F \setminus B\Big) = 0$.
3. $\mathscr{P}^{\mathbf{q},t}_{\mu,\varphi}\Big(G \setminus B\Big) = 0$.

It is an easy consequence of the following lemmas.

Lemma 5.3. *Let φ be as in (2.1), $\mu \in \mathscr{P}_D(\mathbb{R}^d)$ and E be a Borel subset of* $\operatorname{supp}(\mu)$ *with $\mathscr{P}^{\mathbf{q},t}_{\mu,\varphi}(E) < \infty$. For all measurable subset B of E we have*

$$\mathscr{H}^{\mathbf{q},t}_{\mu,\varphi}(B) = \mathscr{P}^{\mathbf{q},t}_{\mu,\varphi}(B) \text{ if and only if } \mathscr{P}^{\mathbf{q},t}_{\mu,\varphi}\Big(G \setminus B\Big) = 0.$$

Proof. Without loss of the generality, we may assume that $\mathscr{P}^{\mathbf{q},t}_{\mu,\varphi}(B) > 0$. First, we suppose that $\mathscr{H}^{\mathbf{q},t}_{\mu,\varphi}(B) = \mathscr{P}^{\mathbf{q},t}_{\mu,\varphi}(B)$. By Theorem 3.2, we obtain $\Delta^{\mathbf{q},t}_{\mu,\varphi}(x, B) = 1$, for $\mathscr{P}^{\mathbf{q},t}_{\mu,\varphi}$-a.e. $x \in B$. By Lemmas 4.3 and 4.4, we obtain

$$\Delta^{\mathbf{q},t}_{\mu,\varphi}(x, B) = \Delta^{\mathbf{q},t}_{\mu,\varphi}(x, E), \text{ for } \mathscr{P}^{\mathbf{q},t}_{\mu,\varphi}\text{-a.e. } x \in B.$$

Now, assume that $\Delta^{\mathbf{q},t}_{\mu,\varphi}(x, E) = 1$, for $\mathscr{P}^{\mathbf{q},t}_{\mu,\varphi}$-a.e. $x \in B$. Then, we easily see that $\Delta^{\mathbf{q},t}_{\mu,\varphi}(x, E) = 1$, for $\mathscr{H}^{\mathbf{q},t}_{\mu,\varphi}$-a.e. $x \in B$. Using Lemma 4.3, we get

$$\Delta^{\mathbf{q},t}_{\mu,\varphi}(x, B) = \Delta^{\mathbf{q},t}_{\mu,\varphi}(x, E), \text{ for } \mathscr{H}^{\mathbf{q},t}_{\mu,\varphi}\text{-a.e. } x \in B.$$

We have $\Delta^{\mathbf{q},t}_{\mu,\varphi}(x, B) = 1$, for $\mathscr{H}^{\mathbf{q},t}_{\mu,\varphi}$-a.e. $x \in B$. From Lemma 4.4, we get $\Delta^{\mathbf{q},t}_{\mu,\varphi}(x, B) = 1$, for $\mathscr{P}^{\mathbf{q},t}_{\mu,\varphi}$-a.e. $x \in B$. Finally, Theorem 3.2 permits to get $\mathscr{H}^{\mathbf{q},t}_{\mu,\varphi}(B) = \mathscr{P}^{\mathbf{q},t}_{\mu,\varphi}(B)$. □

Lemma 5.4. *Let φ be as in (2.1), $\mu \in \mathscr{P}_D(\mathbb{R}^d)$ and E be a $\mathscr{P}^{\mathbf{q},t}_{\mu,\varphi}$-measurable set with $\mathscr{P}^{\mathbf{q},t}_{\mu,\varphi}(E) < \infty$. Then $\mathscr{P}^{\mathbf{q},t}_{\mu,\varphi}\Big(F \setminus G\Big) = 0$. Moreover, if $\overline{\Delta}^{\mathbf{q},t}_{\mu,\varphi}(x, E) < +\infty$ on E, we get $\mathscr{P}^{\mathbf{q},t}_{\mu,\varphi}\Big(G \setminus F\Big) = 0$.*

Proof. Without loss of generality, we may assume that $\mathscr{P}^{\mathbf{q},t}_{\mu,\varphi}(G) > 0$. By using Theorem 4.5, we have $\Delta^{\mathbf{q},t}_{\mu,\varphi}(x, G) = 1$, for $\mathscr{P}^{\mathbf{q},t}_{\mu,\varphi}$-a.e. $x \in G$. From Theorem 3.2, we obtain $D^{\mathbf{q},t}_{\mu,\varphi}(x, G) = 1$, for $\mathscr{P}^{\mathbf{q},t}_{\mu,\varphi}$-a.e. $x \in G$. Hence, $D^{\mathbf{q},t}_{\mu,\varphi}(x, G) = 1$, for $\mathscr{H}^{\mathbf{q},t}_{\mu,\varphi}$-a.e. $x \in G$. Using Lemma 4.3, we get

$$\overline{D}^{\mathbf{q},t}_{\mu,\varphi}(x, G) = \overline{D}^{\mathbf{q},t}_{\mu,\varphi}(x, E) \text{ and } \underline{D}^{\mathbf{q},t}_{\mu,\varphi}(x, G) = \underline{D}^{\mathbf{q},t}_{\mu,\varphi}(x, E), \text{ for } \mathscr{H}^{\mathbf{q},t}_{\mu,\varphi}\text{-a.e. } x \in G.$$

Then $D^{\mathbf{q},t}_{\mu,\varphi}(x, E) = 1$, for $\mathscr{H}^{\mathbf{q},t}_{\mu,\varphi}$-a.e. $x \in G$. By Lemma 4.4, $D^{\mathbf{q},t}_{\mu,\varphi}(x, E) = 1$, for $\mathscr{P}^{\mathbf{q},t}_{\mu,\varphi}$-a.e. $x \in G$.
The remaining part may be proved by similar techniques. □

6. Appendix

Theorem 6.1 (Besicovitch covering theorem). *[8, 19]. There exists $\xi \in \mathbb{N}$ such that, for any subset A of $\mathbb{R}^d$ and any set of real numbers $(r_x)_{x\in A}$ satisfying*

1. $r_x > 0, \forall\, x \in A,$
2. $\sup\limits_{x\in A} r_x < \infty,$

there exists ξ countable or finite subfamilies $B_1, \ldots, B_\xi$ of $\left\{B(x, r_x),\ x \in A\right\}$, such that

1. $A \subset \bigcup_i \bigcup_{B\in B_i} B.$
2. *B_i is composed of disjoint sets.*

Lemma 6.2 (Vitali's lemma). *[19]. Let X be a bounded compact metric space and $\mathscr{B}$ a set of closed balls in X, such that*

$$\sup\left\{\dim(B),\ B \in \mathscr{B}\right\} < \infty.$$

Then, there exists a finite or countable sequence of disjoint balls $(B_i)_i \subset \mathscr{B}$, such that

$$\bigcup_{B\in\mathscr{B}} B \subset \bigcup_i \left(5B_i\right).$$

Theorem 6.3 (Vitali 1). *[19]. Let μ be a Radon measure on $\mathbb{R}^d$, $A \subset \mathbb{R}^d$ and $\mathscr{B}$ a family of closed balls, such that each point of A is the center of an arbitrarily small ball of $\mathscr{B}$, i.e.,*

$$\inf\left\{r,\ B(x, r) \in \mathscr{B}\right\} = 0, \text{ for } x \in A.$$

Then, there exists a family of disjoint balls $(B_i)_i \subset \mathscr{B}$, such that

$$\mu\Big(A \setminus \bigcup_i B_i\Big) = 0.$$

Theorem 6.4 (Vitali 2). *[17] Let X be a metric space, E a subset of X and $\mathscr{B}$ a family of fine cover of E. Then, there exists either*

1. *an infinite (centered closed ball) packing $\left\{B(x_i, r_i)\right\}_i \subset \mathscr{B}$, with $\inf\{r_i\} > 0$, or*
2. *a countable (possibly finite) centered closed ball packing $\left\{B(x_i, r_i)\right\}_i \subset \mathscr{B}$, such that for all $k \in \mathbb{N}$,*

$$E \setminus \bigcup_{i=1}^{k} B(x_i, r_i) \subset \bigcup_{i\geq k} B(x_i, 5r_i).$$

Acknowledgments

We would like to thank the anonymous referees for their valuable comments and suggestions that led to the improvement of the manuscript. This work was supported by Analysis, Probability & Fractals Laboratory LR18ES17, and the Laboratory of Algebra, Number Theory and Nonlinear Analysis LR18ES15.

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