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On the Weighted Logarithmic Mean of Accretive Matrices

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Abstract. The purpose of this paper is to introduce the weighted logarithmic mean of two accretive matrices. Among the obtained results, we present some inequalities about this weighted mean when the involved matrices are sectorial matrices. Our approach allowed us to derive a new matrix mean which is connected to the Heinz matrix mean.

1. Introduction

Means arise in various contexts and contribute as good tools for solving many scientific problems. It has been proved, throughout a lot of works, that the mean-theory is useful in theoretical point of view as well as in practical purposes. Recently, the mean-theory attracts many mathematicians by its nice properties and various applications.

• As usual, we understand by (binary) mean a map m between two positive numbers such that $\min(a, b) \le m(a, b) \le \max(a, b)$ for any $a, b > 0$. Among the standard means, we recall the following:

$$a\nabla b =: \frac{a+b}{2};\ \ a\sharp b =: \sqrt{ab};\ \ a!b =: \frac{2ab}{a+b},\ \ L(a,b) =: \frac{b-a}{\log b - \log a},\ L(a,a) = a,$$

which are known as the arithmetic mean, the geometric mean, the harmonic mean and the logarithmic mean, respectively. The following chain of inequalities is well-known in the literature

$$a!b \le a\sharp b \le L(a,b) \le a\nabla b. \tag{1}$$

• Let m_λ be a binary map indexed by $\lambda \in [0, 1]$. We say that m_λ is a weighted mean if the following assertions are satisfied:
(i) m_λ is a mean, for any $\lambda \in [0, 1]$,
(ii) $m_{1/2} =: m$ is a symmetric mean,
(iii) $m_\lambda(a, b) = m_{1-\lambda}(b, a)$ for any $a, b > 0$ and $\lambda \in [0, 1]$.

It is obvious that, (iii) implies (ii). The mean $m =: m_{1/2}$ is called the associated symmetric mean of m_λ. The standard weighted means are the following

$$a\nabla_\lambda b =: (1-\lambda)a + \lambda b;\ \ a\sharp_\lambda b =: a^{1-\lambda}b^\lambda;\ \ a!_\lambda b =: \Big((1-\lambda)a^{-1} + \lambda b^{-1}\Big)^{-1},$$

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which are called the λ-weighted arithmetic mean, the λ-weighted geometric mean and the λ-weighted harmonic mean, respectively. For $\lambda = 1/2$ they coincide with $a\nabla b$, $a\sharp b$ and $a!b$, respectively. These weighted means satisfy the following inequalities

$$a!_\lambda b \leq a\sharp_\lambda b \leq a\nabla_\lambda b. \tag{2}$$

• Another weighted mean, introduced in [9], is the weighted logarithmic mean defined, for $a, b > 0, a \neq b$, by

$$L_\lambda(a,b) =: \frac{1}{\log a - \log b}\left(\frac{1-\lambda}{\lambda}\Big(a - a^{1-\lambda}b^\lambda\Big) + \frac{\lambda}{1-\lambda}\Big(a^{1-\lambda}b^\lambda - b\Big)\right), \tag{3}$$

with $L_\lambda(a,a) = a$. It is not hard to check that $L_{\frac{1}{2}}(a,b) = L(a,b)$, $L_0(a,b) =: \lim_{\lambda\downarrow 0} L_\lambda(a,b) = a$ and $L_1(a,b) =: \lim_{\lambda\uparrow 1} L_\lambda(a,b) = b$. One can see that L_λ satisfies the conditions (i),(ii) and (iii). It has been shown in [9, Theorem 2.4, Theorem 3.1] that the following inequalities hold true

$$a\sharp_\lambda b \leq L_\lambda(a,b) \leq (a\sharp_\lambda b)\nabla(a\nabla_\lambda b) \leq a\nabla_\lambda b. \tag{4}$$

The present paper will be organized as follows: Section 2 is focused to recall some basic notions about accretive matrices. In Section 3, we collect some weighted means of accretive matrices that have been introduced in the literature. Section 4 deals with the definition and properties of the weighted logarithmic mean of two accretive matrices. This weighted mean is an extension of (3) when the positive scalar numbers a and b are, respectively, replaced by two accretive matrices A and B. At the end, we derive a new matrix mean and we show that it can be connected to the so-called Heinz matrix mean.

2. Background material and basic notions

Let $n \geq 2$ be an integer. Throughout this paper, the notation $\mathbb{M}_n$ refers to the space of $n \times n$ complex matrices.

• Every $A \in \mathbb{M}_n$ can be written in the following form

$$A = \Re A + i\Im A, \quad \text{with } \Re A = \frac{A + A^*}{2} \text{ and } \Im A = \frac{A - A^*}{2i}, \tag{5}$$

where the notation A^* refers to the adjoint of A. The decomposition (5) is known, in the literature, as the Cartesian decomposition of A and the matrices $\Re A$ and $\Im A$ are called the real part and the imaginary part of A, respectively. It is clear that $A^* = \Re A - i\Im A$ for any $A \in \mathbb{M}_n$.

• As usual, if $A \in \mathbb{M}_n$ is Hermitian, i.e. $A^* = A$, we say that A is positive (in short $A \geq 0$) if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathbb{C}^n$ and, A is strictly positive (in short $A > 0$) if A is positive and invertible. For $A, B \in \mathbb{M}_n$ Hermitian, we write $A \leq B$ or $B \geq A$ for meaning that $B - A$ is positive. We say that A, defined by (5), is accretive if its real part $\Re A$ is strictly positive. It is clear that if A and B are accretive then so is $A + B$ but, in general, AB may be not accretive. In particular, A accretive does not ensure that A^k is accretive, for $k \geq 2$ integer. However, it is well known that every accretive matrix $A \in \mathbb{M}_n$ is invertible and A^{-1} is also accretive.

• We also need to define the sector S_θ on the complex plane by the formulae

$$S_\theta = \Big\{z \in \mathbb{C} : \ \Re z > 0, \ |\Im z| \leq (\Re z)\tan\theta\Big\},$$

for some $\theta \in [0, \pi/2)$.

Otherwise, the numerical range of $A \in \mathbb{M}_n$ is defined as follows

$$W(A) = \Big\{x^*Ax : \ x \in \mathbb{C}^n, \ x^*x = 1\Big\}.$$

We say that A is a sector matrix whenever $W(A) \subset S_\theta$, for some $\theta \in [0, \pi/2)$. It is obvious that if A is a sector matrix then A is accretive. For further details about the properties of accretive matrices and sector matrices, we refer the reader to [3, 5, 6, 8, 10, 11] and the related references cited therein.

3. Means of accretive matrices

As already pointed before, every accretive matrix $A \in \mathbb{M}_n$ is invertible and A^{-1} is also accretive. Further, it is easy to check that the set of all accretive matrices is a convex cone of $\mathbb{M}_n$. Throughout this paper, except contrary mention, $A, B \in \mathbb{M}_n$ are accretive matrices and $\lambda \in [0,1]$.

• The following expressions

$$A\nabla_\lambda B =: (1-\lambda)A + \lambda B, \quad A!_\lambda B =: \Big((1-\lambda)A^{-1} + \lambda B^{-1}\Big)^{-1} = \Big(A^{-1}\nabla_\lambda B^{-1}\Big)^{-1}, \tag{6}$$

are known, in the literature, as the λ-weighted arithmetic mean and the λ-weighted harmonic mean of A and B, respectively. The λ-weighted geometric mean of A and B is defined by, see [10]

$$A\sharp_\lambda B =: \int_0^1 A!_t B\, d\nu(t), \tag{7}$$

where, for fixed $\lambda \in [0,1]$, $\nu_\lambda(t)$ is the probability measure on $(0,1)$ defined by

$$d\nu_\lambda(t) = \frac{\sin(\lambda\pi)}{\pi}\frac{t^{\lambda-1}}{(1-t)^\lambda}dt. \tag{8}$$

For $\lambda = 1/2$, the previous matrix means are simply denoted by $A\nabla B$, $A!B$ and $A\sharp B$, respectively. An explicit form of $A\sharp_\lambda B$ is given by, [1] (for $\lambda = 1/2$, see also [3])

$$A\sharp_\lambda B = A^{1/2}\Big(A^{-1/2}BA^{-1/2}\Big)^\lambda A^{1/2}, \tag{9}$$

where the power X^λ of a matrix X is defined by the Dunford integral representation in Functional Calculus, as usual. The following inequalities

$$A!_\lambda B \le A\sharp_\lambda B \le A\nabla_\lambda B \tag{10}$$

hold true whenever A and B are strictly positive. We have the following relationships

$$A\nabla_\lambda B = B\nabla_{1-\lambda}A, \quad A!_\lambda B = B!_{1-\lambda}A, \quad A\sharp_\lambda B = B\sharp_{1-\lambda}A.$$

It is clear that $A\nabla_\lambda B, A!_\lambda B$ and $A\sharp_\lambda B$ are accretive whenever A and B are. Further, it is obvious that $\Re\Big(A\nabla_\lambda B\Big) = \Big(\Re A\Big)\nabla_\lambda\Big(\Re B\Big)$. Otherwise, we have [10]

$$\Re\Big(A!_\lambda B\Big) \ge \Big(\Re A\Big)!_\lambda\Big(\Re B\Big), \quad \Re\Big(A\sharp_\lambda B\Big) \ge \Big(\Re A\Big)\sharp_\lambda\Big(\Re B\Big). \tag{11}$$

• The logarithmic mean of two accretive matrices A and B is defined by, [11]

$$L(A,B) =: \int_0^1 A\sharp_t B\, dt. \tag{12}$$

Clearly, $L(A,B)$ is also accretive and we have $L(A,B) = L(B,A)$. Further, the following inequality holds [11]

$$\Re L(A,B) \ge L\Big(\Re A, \Re B\Big). \tag{13}$$

We also have the following result, see [11] for instance,

$$A!B \le A\sharp B \le L(A,B) \le A\nabla B, \tag{14}$$

whenever $A, B \in \mathbb{M}_n$ are strictly positive. An analog of (14) for sector matrices was proved in [11, Theorem 3.5] and reads as follows:

$$(\cos\theta)^2\Re(A!B) \le (\cos\theta)^2\Re(A\sharp B) \le \Re L(A,B) \le (\sec\theta)^2\Re(A\nabla B), \tag{15}$$

provided that $A, B \in \mathbb{M}_n$ are accretive with $W(A), W(B) \subset S_\theta$ for some $\theta \in [0, \pi/2)$.

Also, reverses of (11) and (13) can be found in [11, Lemma 3.3, Proposition 3.4] and are given in what follows:

$$\Re(AmB) \leq (\sec\theta)^2 (\Re A) m (\Re B), \tag{16}$$

whenever $A, B \in \mathbb{M}_n$ are accretive with $W(A), W(B) \subset S_\theta$ for some $\theta \in [0, \pi/2)$. Here, m denotes one the three matrix means $!_\lambda$, $\sharp_\lambda$ and L.

• Let m be one of the previous matrix means i.e. $m \in \{\nabla_\lambda, !_\lambda, \sharp_\lambda, L\}$. It is not hard to check that the relationship

$$m(T^*AT, T^*BT) = T^* m(A, B) T \tag{17}$$

holds whenever $T \in \mathbb{M}_n$ is invertible. In particular, we have

$$m(A, B) = A^{1/2} m\Big(I, A^{-1/2} B A^{-1/2}\Big) A^{1/2}, \tag{18}$$

where I refers the identity matrix of $\mathbb{M}_n$. In the case where $A, B \in \mathbb{M}_n$ are strictly positive, we summarize what previous by saying that the above means are monotone matrix means in the sense of Kubo-Ando [4].

• Finally, for $A, B \in \mathbb{M}_n$ accretive and $\lambda \in [0, 1]$, the Heinz matrix mean of A and B is defined by

$$HZ_\lambda(A, B) =: \Big(A \sharp_\lambda B\Big) \nabla \Big(B \sharp_\lambda A\Big) =: \frac{A \sharp_\lambda B + B \sharp_\lambda A}{2}. \tag{19}$$

Some basic properties of $HZ_\lambda(A, B)$ could be immediately deduced from those of $A \sharp_\lambda B$. In particular, $HZ_\lambda(A, B)$ is symmetric in A and B and $HZ_\lambda(A, B) = HZ_{1-\lambda}(A, B)$. For further properties of $HZ_\lambda(A, B)$ we can consult [2, 7] and the related references cited therein.

4. The weighted logarithmic mean of accretive matrices

As already pointed before, the aim of this section is to answer the following question:

Question: What should be the analog of (3) *when the positive real numbers a and b are replaced by two accretive matrices A and B, respectively?*

By virtue of its complicated form, (3) does not allow us to deduce its matrix version. For this, we try to find another equivalent form of (3) which is appropriate to be extended for matrix arguments. We have the following result.

Proposition 4.1. *For any $a, b > 0$ and $\lambda \in [0, 1]$,* (3) *is equivalent to*

$$L_\lambda(a, b) = (1-\lambda) L(a, a \sharp_\lambda b) + \lambda L(a \sharp_\lambda b, b). \tag{20}$$

Proof. Using the explicit form of $L(a, b) = \frac{a-b}{\log a - \log b}$, $a \neq b$, and starting from the right expression in (20) we get (3) after simple algebraic operations. The details are straightforward and therefore omitted here. □

Using (20), we are in the position to state the following central definition which introduces our first weighted matrix mean.

Definition 4.2. *Let $A, B \in \mathbb{M}_n$ be accretive and $\lambda \in [0, 1]$. The λ-weighted logarithmic mean of A and B is defined by*

$$L_\lambda(A, B) =: (1-\lambda) L(A, A \sharp_\lambda B) + \lambda L(A \sharp_\lambda B, B), \tag{21}$$

where $A \sharp_\lambda B$ is defined by (7) *and $L(A, B)$ is given by* (12).

Some basic properties of $L_\lambda(A, B)$ are embodied in what follows.

Proposition 4.3. *Let $A, B \in \mathbb{M}_n$ be accretive and $\lambda \in [0,1]$. Then the following assertions are met:*
(i) $L_\lambda(A,B)$ is accretive.
(ii) $L_\lambda(A,B) = L_{1-\lambda}(B,A)$.
(iii) For any $T \in \mathbb{M}_n$ invertible, we have

$$L_\lambda(T^*AT, T^*BT) = T^*L_\lambda(A,B)T. \tag{22}$$

(iv) If A and B are strictly positive then so is $L_\lambda(A,B)$.

Proof. (i) It follows from (21), with the fact that $L(A,B)$ and $A\sharp_\lambda B$ are accretive whenever A and B are.
(ii) The relationship $A\sharp_\lambda B = B\sharp_{1-\lambda}A$ when substituted in (21) immediately yields the desired result.
(iii) By (21), with (17) for $m = \sharp_\lambda$, we get (22).
(iv) If A and B are strictly positive then so are $L(A,B)$ and $A\sharp_\lambda B$, and thus (21) implies that $L_\lambda(A,B)$ is also strictly positive. □

In order to give more results, we need to state the following lemma which will be of great interest throughout this section.

Lemma 4.4. *For any $A, B \in \mathbb{M}_n$ accretive and $\lambda \in [0,1]$, one has*

$$L(A, A\sharp_\lambda B) = \int_0^1 A\sharp_{\lambda t}B\, dt, \quad L(A\sharp_\lambda B, B) = \int_0^1 B\sharp_{(1-\lambda)t}A\, dt, \tag{23}$$

and hence,

$$L_\lambda(A,B) = \int_0^1 \Big(A\sharp_{\lambda t}B\Big)\nabla_\lambda\Big(B\sharp_{(1-\lambda)t}A\Big)\, dt. \tag{24}$$

Proof. By (12) we have

$$L(A, A\sharp_\lambda B) = \int_0^1 A\sharp_t\Big(A\sharp_\lambda B\Big)dt.$$

According to (9) we get

$$A\sharp_t\Big(A\sharp_\lambda B\Big) = A^{1/2}\Big(I\sharp_t A^{-1/2}\Big(A\sharp_\lambda B\Big)A^{-1/2}\Big)A^{1/2} = A^{1/2}\Big(I\sharp_t\Big(A^{-1/2}BA^{-1/2}\Big)^\lambda\Big)A^{1/2}, \tag{25}$$

and by (9) again we obtain

$$A\sharp_t\Big(A\sharp_\lambda B\Big) = A^{1/2}\Big(A^{-1/2}BA^{-1/2}\Big)^{\lambda t}A^{1/2} = A\sharp_{\lambda t}B.$$

The second relation about $L(A\sharp_\lambda B, B)$ can be obtained by the same way, or simply we use that of $L(A, A\sharp_\lambda B)$ with the relationship $A\sharp_\lambda B = B\sharp_{1-\lambda}A$.

The two previous relationships when substituted in (21) immediately imply (24). □

We have the following result.

Proposition 4.5. *Let $A, B \in \mathbb{M}_n$ be strictly positive. Then we have*

$$A\sharp_\lambda B \le (A\sharp_{\lambda/2}B)\sharp_\lambda\Big(A\sharp_{\frac{1+\lambda}{2}}B\Big) \le L_\lambda(A,B) \le (A\sharp_\lambda B)\nabla(A\nabla_\lambda B) \le A\nabla_\lambda B. \tag{26}$$

Proof. We first prove the two right inequalities in (26). By (21), with the right inequality in (14), we get

$$L_\lambda(A,B) \le (1-\lambda)A\nabla(A\sharp_\lambda B) + \lambda(A\sharp_\lambda B)\nabla B.$$

This, with the definition of ∇, implies that

$$L_\lambda(A,B) \le (A\sharp_\lambda B)\nabla(A\nabla_\lambda B),$$

which, with the right inequality in (10), yields

$$L_\lambda(A,B) \le (A\sharp_\lambda B)\nabla(A\nabla_\lambda B) \le (A\nabla_\lambda B)\nabla(A\nabla_\lambda B) = A\nabla_\lambda B.$$

We now show the two left inequalities in (26). By (21) again, with the left inequality in (14), we obtain

$$L_\lambda(A,B) \ge (1-\lambda)A\sharp(A\sharp_\lambda B) + \lambda(A\sharp_\lambda B)\sharp B.$$

As for proving (25), one can check that $A\sharp(A\sharp_\lambda B) = A\sharp_{\lambda/2}B$ and $(A\sharp_\lambda B)\sharp B = A\sharp_{\frac{1+\lambda}{2}}B$ which, when substituted in the last inequality, yields

$$L_\lambda(A,B) \ge (1-\lambda)A\sharp_{\lambda/2}B + \lambda A\sharp_{\frac{1+\lambda}{2}}B = (A\sharp_{\lambda/2}B)\nabla_\lambda\Big(A\sharp_{\frac{1+\lambda}{2}}B\Big) \ge (A\sharp_{\lambda/2}B)\sharp_\lambda\Big(A\sharp_{\frac{1+\lambda}{2}}B\Big).$$

Again, as for (25), we show that the right expression of this latter equality is simply reduced to $A\sharp_\lambda B$, so completing the proof. □

Remark 4.6. *(i) We can prove the right inequality in* (26) *via* (24) *by using* (10)*, and we obtain*

$$L_\lambda(A,B) \le \int_0^1 \Big(A\nabla_{\lambda t}B\Big)\nabla_\lambda\Big(B\nabla_{(1-\lambda)t}A\Big)\,dt,$$

which, by the linearity of ∇_λ, gives after integration and then a simple algebraic manipulation

$$L_\lambda(A,B) \le \Big(A\nabla_{\lambda/2}B\Big)\nabla_\lambda\Big(B\nabla_{(1-\lambda)/2}A\Big) = A\nabla_\lambda B.$$

(ii) We can also show the inequalities (26) *by another different way. In fact, the matrix means $A\sharp_\lambda B, L_\lambda(A,B)$ and $A\nabla_\lambda B$ are monotone means in the sense of Kubo-Ando [4] i.e. satisfy a similar relationship as* (18)*. By the techniques of functional Calculus, the proof of* (26) *can be reduced to that of its scalar version which can be found in [9].*

The following result may be stated as well.

Proposition 4.7. *Let $A,B \in \mathbb{M}_n$ be accretive and $\lambda \in [0,1]$. Then*

$$\Re L_\lambda(A,B) \ge L_\lambda\Big(\Re A, \Re B\Big). \tag{27}$$

Proof. By (24) we have

$$\Re L_\lambda(A,B) = \int_0^1 \Re\Big(A\sharp_{\lambda t}B\Big)\nabla_\lambda\Re\Big(B\sharp_{(1-\lambda)t}A\Big)\,dt, \tag{28}$$

which, with the help of the second inequality in (11), yields

$$\Re L_\lambda(A,B) \ge \int_0^1 \Big(\Re A\sharp_{\lambda t}\Re B\Big)\nabla_\lambda\Big(\Re B\sharp_{(1-\lambda)t}\Re A\Big)\,dt,$$

and whence (27) by (24) again. □

Using Lemma 4.4 again and assuming that the matrices A and B are sectorial, a reverse of (27) is recited in the following.

Proposition 4.8. *Let $A, B \in \mathbb{M}_n$ be accretive with $W(A), W(B) \subset S_\theta$ for some $\theta \in [0, \pi/2)$. For any $\lambda \in [0,1]$ there holds:*

$$\Re L_\lambda(A,B) \leq (\sec\theta)^2 L_\lambda\Big(\Re A, \Re B\Big). \tag{29}$$

Proof. According to (28), with the help of (16) for $m = \sharp_\lambda$, we get

$$\Re L_\lambda(A,B) \leq (\sec\theta)^2 \int_0^1 \Big(\Re A \sharp_{\lambda t} \Re B\Big) \nabla_\lambda \Big(\Re B \sharp_{(1-\lambda)t} \Re A\Big)\, dt$$

and by (24) again we obtain (29). □

The following result gives an analog of (26), and even more, when the matrices A and B are assumed to be sectorial.

Theorem 4.9. *Let $A, B \in \mathbb{M}_n$ be accretive with $W(A), W(B) \subset S_\theta$ for some $\theta \in [0, \pi/2)$. Then, for any $\lambda \in [0,1]$, we have*

$$(\cos\theta)^2 \Re(A\sharp_\lambda B) \leq \Re L_\lambda(A,B) \leq (\sec\theta)^2 L\Big(\Re A \nabla_\lambda \Re B, \Re A \sharp_\lambda \Re B\Big) \leq (\sec\theta)^2 \Re(A\nabla_\lambda B). \tag{30}$$

Proof. By (27) and the left inequality in (26) we get

$$\Re L_\lambda(A,B) \geq L_\lambda(\Re A, \Re B) \geq \Re A \sharp_\lambda \Re B,$$

which, with (16) for $m = \sharp_\lambda$, implies the first inequality of (30). We now prove the second and third inequalities of (30). By (21) with (23), and the relation $L(A,B) = L(B,A)$, one can easily check that

$$\begin{aligned} \Re L_\lambda(A,B) &= (1-\lambda)\Re L(A, A\sharp_\lambda B) + \lambda \Re L(B, A\sharp_\lambda B) \\ &\leq (\sec\theta)^2 \Big\{(1-\lambda) L(\Re A, \Re A \sharp_\lambda \Re B) + \lambda L(\Re B, \Re A \sharp_\lambda \Re B)\Big\}. \end{aligned} \tag{31}$$

It is well known that, for any $\lambda \in (0,1)$, the map $X \longmapsto X^\lambda$ is matrix concave for $X > 0$. By (9), also valid for $A, B > 0$, and the standard techniques in Functional Calculus, we then deduce that the map $X \longmapsto X\sharp_\lambda B$, for fixed $B > 0$, is also matrix concave. This, with (12), immediately implies that $X \longmapsto L(X, B)$, for fixed $B > 0$, is matrix concave. This latter information when applied to the last expression of (31) allows us to obtain

$$\Re L_\lambda(A,B) \leq (\sec\theta)^2 L\Big((1-\lambda)\Re A + \lambda \Re B, \Re A \sharp_\lambda \Re B\Big),$$

or, equivalently,

$$\Re L_\lambda(A,B) \leq (\sec\theta)^2 L\Big(\Re A \nabla_\lambda \Re B, \Re A \sharp_\lambda \Re B\Big).$$

This, with the right inequality in (14) and then (10), implies that

$$\Re L_\lambda(A,B) \leq (\sec\theta)^2 \Big(\Re A \nabla_\lambda \Re B\Big) \nabla \Big(\Re A \sharp_\lambda \Re B\Big) \leq (\sec\theta)^2 \Re\Big(A \nabla_\lambda B\Big).$$

The second and third inequalities in (30) are established and the proof of the theorem is finished. □

5. About a new symmetric matrix mean

As already pointed before, we will derive here a new matrix mean from the previous weighted logarithmic mean in a simple way. Precisely, we state the following definition.

Definition 5.1. *Let $A, B \in \mathbb{M}_n$ be accretive. We set*

$$\mathcal{T}(A,B) =: \int_0^1 L_t(A,B)\, dt. \tag{32}$$

Some elementary properties of $\mathcal{T}(A,B)$ can be immediately deduce from those of the weighted logarithmic mean. First, we left to the reader the routine task for formulating an analog of Proposition 4.3 for $\mathcal{T}(A,B)$. Further basic properties of $\mathcal{T}(A,B)$ may be summarized in the following result.

Proposition 5.2. *Let $A,B\in\mathbb{M}_n$ be accretive. Then the following assertions hold true:*
(i) If A and B are strictly positive then $\mathcal{T}(A,B)$ interpolates $L(A,B)$ and $A\nabla B$ in the sense that

$$L(A,B)\le\mathcal{T}(A,B)\le A\nabla B. \tag{33}$$

(ii) $\Re\mathcal{T}(A,B)\ge\mathcal{T}\Big(\Re A,\Re B\Big)$.
(iii) If $A,B\in\mathbb{M}_n$ are sectorial with $W(A),W(B)\subset S_\theta$ then

$$\Re\mathcal{T}(A,B)\le(\sec\theta)^2\mathcal{T}\Big(\Re A,\Re B\Big).$$

(iv) With the same hypothesis as in (iii), we have

$$(\cos\theta)^2\Re L(A,B)\le\Re\mathcal{T}(A,B)\le(\sec\theta)^2\Re(A\nabla B).$$

Proof. They follow, by simple integrations, from (26), (27), (29) and (30), respectively. The details are straightforward and therefore omitted here. □

Now, we will see that Lemma 4.4 could be a good tool again for proving a more interesting result which explores a relationship between $\mathcal{T}(A,B)$ and the Heinz matrix mean $HZ_\lambda(A,B)$. It reads as follows.

Theorem 5.3. *Let $A,B\in\mathbb{M}_n$ be accretive. Then we have*

$$\mathcal{T}(A,B)=2\int_0^1\Big(t-1-\log t\Big)HZ_t(A,B)\,dt, \tag{34}$$

or, equivalently,

$$\mathcal{T}(A,B)=\int_0^1 HZ_t(A,B)\,d\nu(t), \tag{35}$$

where ν is the probability measure on $(0,1)$ defined by

$$d\nu(t)=:2\Big(t-1-\log t\Big)\,dt.$$

Proof. By (32), with (20) and then (23), we get

$$\mathcal{T}(A,B)=\int_0^1\int_0^1(1-t)A\sharp_{ts}B\,ds\,dt+\int_0^1\int_0^1 t\,B\sharp_{(1-t)s}A\,ds\,dt=:J_1+J_2.$$

Computing the first integral, by setting $ts=u$, we obtain

$$J_1=:\int_0^1\int_0^1(1-t)A\sharp_{ts}B\,ds\,dt=\int_0^1\int_0^t\frac{1-t}{t}A\sharp_uB\,du\,dt,$$

which, with the standard Fubini formula for double integral, implies that

$$\begin{aligned}J_1&=\int_0^1\int_0^t\frac{1-t}{t}A\sharp_uB\,du\,dt=\int_0^1\int_u^1\frac{1-t}{t}A\sharp_uB\,dt\,du\\&=\int_0^1\Big[\log t-t\Big]_{t=u}^1A\sharp_uB\,du=\int_0^1\Big(u-1-\log u\Big)A\sharp_uB\,du.\end{aligned}$$

By similar way (or shortly use a simple change of variables) one can check that

$$J_2 =: \int_0^1 \int_0^1 t\ B\sharp_{(1-t)s}A\ ds = \int_0^1 \Big(u - 1 - \log u\Big) B\sharp_u A\ du.$$

These, with the fact that $A\sharp_u B + B\sharp_u A = 2\ HZ_u(A,B)$ and $\mathcal{T}(A,B) = J_1 + J_2$, yields (34) and so (35). The proof is finished. □

From the previous theorem we can deduce another expression of $\mathcal{T}(A,B)$ which seems to be more symmetric in t and $1-t$. This symmetry generates an expression of $\mathcal{T}(A,B)$ in terms of $A\sharp_\lambda B$. It reads as follows.

Corollary 5.4. *Let $A, B \in \mathbb{M}_n$ be accretive. Then we have*

$$\mathcal{T}(A,B) = -\int_0^1 \Big(1 + \log t(1-t)\Big) HZ_t(A,B)\ dt = -\int_0^1 \Big(1 + \log t(1-t)\Big) A\sharp_t B\ dt, \tag{36}$$

or, equivalently,

$$\mathcal{T}(A,B) = \int_0^1 HZ_t(A,B)\ d\mu(t) = \int_0^1 A\sharp_t B\ d\mu(t), \tag{37}$$

where μ is the probability measure on $(0,1)$ defined by

$$d\mu(t) =: -\Big(1 + \log t(1-t)\Big)\ dt.$$

Proof. If in (34) we make the change of variables $t = 1-s,\ s \in [0,1]$ and we use $HZ_{1-s}(A,B) = HZ_s(A,B)$ we obtain

$$\mathcal{T}(A,B) = 2\int_0^1 \Big(-t - \log(1-t)\Big) HZ_t(A,B)\ dt.$$

Adding side to side this latter equality and (34) we get the first relation of (36), and thus that of (37), after simple manipulations. The second relation of (36) and that of (37) follow from simple operations with a change of variables. The details are straightforward and therefore omitted here. □

Corollary 5.4 has, in its turn, an interesting consequence which gives us a relationship between the three matrix means $\mathcal{T}(A,B)$, $L(A,B)$ and $HZ_\lambda(A,B)$ (resp. $A\sharp_\lambda B$). It reads as follows.

Corollary 5.5. *For any $A, B \in \mathbb{M}_n$ accretive, we have:*

$$\mathcal{T}(A,B) = -L(A,B) - \int_0^1 \Big(\log t(1-t)\Big)\ HZ_t(A,B)\ dt = -L(A,B) - \int_0^1 \Big(\log t(1-t)\Big)\ A\sharp_t B\ dt. \tag{38}$$

Proof. By (12) and (19), with a simple change of variables, one can easily check that $\int_0^1 HZ_t(A,B)\ dt = L(A,B)$. Whence the desired result by (36). □

We have seen in Proposition 5.2 that some properties of $\mathcal{T}(A,B)$ can be simply deduced from those of $L_\lambda(A,B)$. In addition, Theorem 5.3 (resp. Corollary 5.4) tells us that we can also deduce some properties of $\mathcal{T}(A,B)$ from those of $HZ_\lambda(A,B)$ (resp. $A\sharp_\lambda B$). As an example, we present the following result.

Proposition 5.6. *Let $A, B \in \mathbb{M}_n$ be sectorial with $W(A), W(B) \subset S_\theta$. Then we have*

$$\omega\Big(\mathcal{T}(A,B)\Big) \le (\sec\theta)^3 \mathcal{T}\Big(\omega(A), \omega(B)\Big),$$

where $\omega(A)$ refers to the numerical radius of A defined by:

$$\omega(A) =: \max\Big\{|x^*Ax| :\ x \in \mathbb{C}^n,\ x^*x = 1\Big\}.$$

Proof. It is shown in [2] that, for all $t \in [0,1]$, we have

$$\omega\Big(HZ_t(A,B)\Big) \leq (\sec\theta)^3 HZ_t\Big(\omega(A),\omega(B)\Big).$$

Multiplying this latter inequality by $p(t) =: -\Big(1+\log t(1-t)\Big) \geq 0$ and integrating with respect to $t \in [0,1]$, with the subadditivity and the positive homogeneity of $A \longmapsto \omega(A)$, we get

$$\omega\Big(\int_0^1 p(t)HZ_t(A,B)\,dt\Big) \leq \int_0^1 p(t)\omega\Big(HZ_t(A,B)\Big)\,dt \leq (\sec\theta)^3 \int_0^1 p(t)HZ_t\Big(\omega(A),\omega(B)\Big)\,dt. \tag{39}$$

This, with Corollary 5.4, yields the desired result. □

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