Filomat 36:12 (2022), 4123–4137
https://doi.org/10.2298/FIL2212123G

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Fixed Point Results via Modified ω-Distance and an Application to Networks Communication

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Abstract. In this paper, we establish some fixed point results in quasi-metric structures via modified ω-distance and using one control function, known as Jachymski function. Our results generalize the fixed point result of Alegre and Marín [Topology Appl. 203:32-41, 2016], and give an affirmative answer to the natural question of Alegre Gil et al. [Results Math. 74(4):1-9, 2019]. Apart from these, we utilize one of our results to study the solvability of a certain kind of fractal difference equation of networks communication.

1. Introduction and preliminaries

The concept of ω-distance in metric spaces was first introduced by Kada et al. [7] in 1996, and then using this concept, many mathematicians have obtained several fixed point results, where in the contraction condition, the metric is replaced by the ω-distance, (see [3, 8, 12] and the references therein). Besides these, in 2000, Park [10] proposed the idea of ω-distance in quasi-metric spaces and obtained some fixed point results. The study of fixed points via ω-distance on quasi-metric spaces was further continued by many researchers, (see [3, 9] and the references therein). It is notable that if d is a metric on a nonempty set X, then d itself becomes a ω-distance on (X, d) but this fact need not be true in case of quasi-metric, i.e., if d is a quasi-metric on X, then d is not necessarily a ω-distance on (X, d). Due to these facts, one can think, whether the idea ω-distance on quasi-metric spaces can be modified in such a way that the function d itself becomes a ω-distance on a quasi-metric space (X, d). This question was answered in 2016 by Alegre and Marín [2]. In [2], Alegre and Marín improved the definition of ω-distance on a quasi-metric space, and designate it by modified ω-distance ($m\omega$-distance).

Definition 1.1. *Suppose that (X, d) is a quasi-metric space and a function $q : X \times X \to [0, \infty)$ satisfies the succeeding properties:*

(a) $q(x, z) \le q(x, y) + q(y, z)$ *for all* $x, y, z \in X$;
(b) *for each* $x \in X$, *the function* $q(x, \cdot) : X \to [0, \infty)$ *is lower semicontinuous on* $(X, \tau_{d^{-1}})$;

2020 *Mathematics Subject Classification*. Primary 47H10, 54H25.
Keywords. Quasi-metric space; modified ω-distance; Jachymski function; fractal operator
Received: 25 March 2021; Accepted: 16 May 2022
Communicated by Vladimir Rakočević
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(c) for any $\varepsilon > 0$, there exists $\delta > 0$ such that $x, y, z \in X$ and $q(y, x) \leq \delta$, $q(x, z) \leq \delta$ imply $d(y, z) \leq \varepsilon$.

Then q is called a modified ω-distance ($m\omega$-distance) on (X, d).

If further, for each $x \in X$, the function $q(\cdot, x) : X \to [0, \infty)$ is lower semicontinuous on $(X, \tau_{d^{-1}})$, then q is called strong $m\omega$-distance on (X, d).

For examples of $m\omega$-distances and strong $m\omega$-distances, and some of its properties, one is referred to [2, 5].

After introducing mw-distance, Alegre and Marín [2] obtained a fixed point result using a control function, known as Jachymski function, in which the contraction condition is used via $m\omega$-distance.

Definition 1.2. *[3] A function $\varphi : [0, \infty) \to [0, \infty)$ satisfying the following, is called a Jachymski function:*

(a) $\varphi(0) = 0$;
(b) for any $\varepsilon > 0$, there exists $\delta > 0$ such that $t > 0$ and $\varepsilon < t < \varepsilon + \delta$ imply $\varphi(t) \leq \varepsilon$.

Theorem 1.3. *[2] Suppose that (X, d) is a complete quasi-metric space, $T : X \to X$ a map, there exist a strong-$m\omega$-distance q on (X, d) and a Jachymski function φ with $\varphi(t) < t$ for all $t > 0$ such that $q(Tx, Ty) \leq \varphi(q(x, y))$ for all $x, y \in X$. Then T has a unique fixed point z (say) with $q(z, z) = 0$.*

Following this initial paper, some mathematicians have obtained several results related to fixed point utilizing $m\omega$-distance, see [1, 5, 13]. An important one of such results is due to Alegre et al. [5], where they modified and extended Theorem 1.3 by replacing the control function (Jachymski function) by Bianchini-Grandolfi gauge function.

Definition 1.4. *[4, 11] A function $\varphi : [0, \infty) \to [0, \infty)$ satisfying the following, is called a Bianchini-Grandolfi gauge function:*

(a) φ is nondecreasing;
(b) $\sum_{n=0}^{\infty} \varphi^n(t) < \infty$ for all $t > 0$.

One may note that if φ is a Bianchini-Grandolfi gauge function, then $\varphi(t) < t$ for all $t > 0$.

Theorem 1.5. *[5] Suppose that (X, d) is a complete quasi-metric space, $T : X \to X$ a q-lower semicontinuous map, there exist a strong-$m\omega$-distance q on (X, d) and a Bianchini-Grandolfi gauge function φ such that*

$$q(Tx, Ty) \leq \varphi(\max\{q(x, y), q(x, Tx), q(y, Ty)\})$$

and

$$q(Tx, Ty) \leq \varphi(\max\{q(x, y), q(Tx, x), q(Ty, y)\})$$

for all $x, y \in X$. Then T has a unique fixed point.

Theorem 1.6. *[5] Suppose that (X, d) is a complete quasi-metric space, $T : X \to X$ a map, there exist a strong-$m\omega$-distance q on (X, d) and a Bianchini-Grandolfi gauge function φ such that*

$$q(Tx, Ty) \leq \varphi(\max\{q(x, y), q(x, Tx)\})$$

and

$$q(Tx, Ty) \leq \varphi(\max\{q(x, y), q(Tx, x)\})$$

for all $x, y \in X$. Then T has a unique fixed point.

From Theorem 1.6, it follows that Theorem 1.3 remains true if we replace the Jachymski function by Bianchini-Grandolfi gauge function. It is noteworthy to mention that φ is a Jachymski function if is Bianchini-Grandolfi gauge function but not conversely, see [5] for details. So from Theorems 1.5 and 1.6, it can't be concluded that these results hold true if the Bianchini-Grandolfi gauge function is replaced by Jachymski function. So a natural question appears in our mind whether Theorems 1.5 and 1.6 can be settled for Jachymski functions, and this is mentioned in [5]. Motivated by these facts, in this paper, we prove the analogous versions of Theorem 1.5 and 1.6 using Jachymski functions.

Apart from these, it is familiar to all that in contraction principle of fixed point theory, if some results are proved using the displacements $d(x, y)$, $d(x, Tx)$ and $d(y, Ty)$, then some of these can also be established using the displacements $d(x, y)$, $d(x, Ty)$ and $d(y, Tx)$. Due to these, we establish another fixed point result involving $m\omega$-distance, Jachymski function and the displacements $q(x, y)$, $q(x, Ty)$, $q(y, Tx)$, $q(Tx, y)$ and $q(Ty, x)$.

On the other hand, we apply one of our results in the communication networks domain. It is creating the usage of the mathematical modelling method and through the operator based on some types of well-known techniques, where the basis point (node) is mapped to the target point, and the target point is conforming to the fixed point of the metric space, that is called the fixed point of the network space. Application instances of the network fixed-point theory will be presented to this work and examples have exposed that the fixed point theory is a promising theoretical backup related for the network association, dynamic normal and symmetry of the space-air-ground joined network.

Before going to our main results, we recollect some important definitions and notions in quasi-metric space theory, which will be useful in our main results. At first, we recall the definition quasi-metric spaces.

Definition 1.7. *A quasi-metric on a non-empty set X is a function $d : X \times X \to [0, \infty)$ which satisfies the following conditions:*

(i) $d(x, y) = d(y, x) = 0$ *if and only if* $x = y$;

(ii) $d(x, z) \leq d(x, y) + d(y, z)$ *for all* $x, y, z \in X$.

Remark 1.8. *Let (X, d) be a quasi-metric space. Then the function $d^{-1} : X \times X \to [0, \infty)$ defined by $d^{-1}(x, y) = d(y, x)$ for all $x, y \in X$ is again a quasi-metric on X and this quasi-metric is known as conjugate quasi-metric. Also, the function $d^s : X \times X \to [0, \infty)$ defined by $d^s(x, y) = \max\left\{d(x, y), d(y, x)\right\}$ for all $x, y \in X$, is a metric on X.*

We finish this section by recalling the completeness of a quasi-metric space.

Definition 1.9. *A quasi-metric space (X, d) is said to be complete if every Cauchy sequence $\{x_n\}$ in the metric space (X, d^s) converges with respect to the metric d^{-1}, i.e., if there exists $u \in X$ such that $d(x_n, u) \to 0$ as $n \to \infty$.*

2. Fixed point results

In the beginning of this section, we extend Theorem 1.5 using Jachymski function instead of Bianchini-Grandolfi gauge function. Before this, we need the following definition:

Definition 2.1. *[5] Let T be a self-map and q a $m\omega$-distance on a quasi-metric space (X, d). The T is said to be q-lower semicontinuous if the map $x \mapsto q(x, Tx)$ is lower semicontinuous on (X, d^s).*

Theorem 2.2. *Let T be a q-lower semicontinuous self-map on a complete quasi-metric space (X, d). Further, assume that there exists a strong mw-distance q on X and a Jachymski function φ such that*

$$q(Tx, Ty) \leq \varphi(\max\{q(x, y), q(x, Tx), q(y, Ty)\}) \tag{1}$$

and

$$q(Tx, Ty) \leq \varphi(\max\{q(x, y), q(Tx, x), q(Ty, y)\}) \tag{2}$$

for all $x, y \in X$, and $\varphi(t) < t$ for all $t > 0$. Then the following hold:

(i) T has a unique fixed point z (say) and $q(z,z) = 0$;

(ii) for any $x_0 \in X$, the Picard's iterative sequence $\{x_n\}$ converges to z in (X, d^s).

Proof. Choose $x_0 \in X$ arbitrarily and consider the sequences $\{x_n\}$ in X, $\{\alpha_n\}$, $\{\beta_n\}$ in $\mathbb{R}_0^+$, defined by $x_n = T^n x_0$, $\alpha_n = q(x_n, x_{n+1})$, $\beta_n = q(x_{n+1}, x_n)$ for $n \in \mathbb{N}$. We first show that $\lim_{n\to\infty} \alpha_n = 0$ and $\lim_{n\to\infty} \beta_n = 0$. We have

$$\begin{aligned}\alpha_{n+1} &= q(Tx_n, Tx_{n+1}) \\ &\leq \varphi(\max\{q(x_n, x_{n+1}), q(x_n, x_{n+1}), q(x_{n+1}, x_{n+2})\}) \quad (3)\\ &= \varphi(\max\{\alpha_n, \alpha_n, \alpha_{n+1}\}). \quad (4)\end{aligned}$$

If $\alpha_{n+1} = 0$, then clearly $\alpha_{n+1} \leq \alpha_n$. So we assume $\alpha_{n+1} \neq 0$. If $\max\{\alpha_n, \alpha_n, \alpha_{n+1}\} = \alpha_{n+1}$, then we have $\alpha_{n+1} < \alpha_{n+1}$, a contradiction. So $\max\{\alpha_n, \alpha_n, \alpha_{n+1}\} = \alpha_n$ and so

$$\alpha_{n+1} \leq \alpha_n. \quad (5)$$

Similarly, we have

$$\beta_{n+1} \leq \beta_n. \quad (6)$$

Hence there exist $\alpha, \beta \in \mathbb{R}_0^+$ such that

$$\lim_{n\to\infty} \alpha_n = \alpha \text{ and } \lim_{n\to\infty} \beta_n = \beta.$$

Next, we show that $\alpha = 0$. If $\alpha_{n_0} = 0$ for some $n_0 \in \mathbb{N}$, then from (5), it follows that $\alpha_n = 0$ for $n \geq n_0$ and so $\alpha = 0$. So we assume that $\alpha_n > 0$ for all n. If $\alpha_{n+1} = \alpha_n$ for some n, then from (4), we get

$$\alpha_n \leq \varphi(\max\{\alpha_n, \alpha_n, \alpha_n)\} = \varphi(\alpha_n) < \alpha_n,$$

a contraction. So $\alpha_{n+1} < \alpha_n$ for all n, which gives

$$\alpha < \alpha_n \text{ for all } n. \quad (7)$$

Now if $\alpha > 0$, then there exists $\delta > 0$ such that for $t > 0$, $\alpha < t < \alpha + \delta$ implies $\varphi(t) \leq \alpha$. Since $\lim_{n\to\infty} \alpha_n = \alpha$, there exists $N \in \mathbb{N}$ such that

$$\alpha < \alpha_n < \alpha + \delta \quad \text{for all } n \geq N,$$

which implies

$$\varphi(\alpha_n) \leq \alpha \quad \text{for all } n \geq N,$$

and from this we get

$$\alpha_{n+1} \leq \varphi(\max\{\alpha_n, \alpha_n, \alpha_{n+1}) = \varphi(\alpha_n\}) \leq \alpha,$$

a contradiction to (7). So we must have $\alpha = 0$. Similarly we have $\beta = 0$.

Next, we show that $\{x_n\}$ is Cauchy in the metric space (X, d^s). Let $\varepsilon > 0$ be arbitrary. Then there exists $\delta > 0$ such that $t > 0$ and $\varepsilon < t < \varepsilon + \delta$ imply $\varphi(t) \leq \varepsilon$, and $q(y, x) \leq \delta$ and $q(x, z) \leq \delta$ imply $q(y, z) \leq \frac{\varepsilon}{2}$. Again there exists $\delta_1 > 0$ such that $t > 0$ and $\frac{\delta}{2} < t < \frac{\delta}{2} + \delta_1$ imply $\varphi(t) \leq \frac{\delta}{2}$. Without loss of generality, we assume that $\delta_1 < \delta < \varepsilon$. Since $\lim_{n\to\infty} \alpha_n = 0 = \lim_{n\to\infty} \beta_n$, there exists $N \in \mathbb{N}$ such that

$$q(x_n, x_{n+1}) < \frac{\delta_1}{16} \text{ and } q(x_{n+1}, x_n) < \frac{\delta_1}{16} \text{ for all } n \geq N. \quad (8)$$

Let $k \geq N$ be arbitrary. Then by induction on n, we show that $q(x_k, x_{k+n}) < \frac{\delta_1}{4} + \frac{\delta}{2}$ for all $n \in \mathbb{N}$. If $n = 1$, the result follows from (8). Let $q(x_k, x_{k+n}) < \frac{\delta_1}{4} + \frac{\delta}{2}$ for some n. Then we have two cases.

Case I: Let $q(x_k, x_{k+n}) > \frac{\delta}{2}$. Then

$$\begin{aligned}\frac{\delta}{2} &< \max\{q(x_k, x_{k+n}), q(x_k, x_{k+1}), q(x_{k+n}, x_{k+n+1})\}\\ &< \max\left\{\frac{\delta_1}{4} + \frac{\delta}{2}, \frac{\delta_1}{16}, \frac{\delta_1}{16}\right\}\\ &< \frac{\delta}{2} + \delta_1.\end{aligned}$$

So

$$\begin{aligned}q(x_{k+n}, x_{k+n+1}) &\leq \varphi(\max\{q(x_k, x_{k+n}), q(x_k, x_{k+1}), q(x_{k+n}, x_{k+n+1})\})\\ &\leq \frac{\delta}{2}.\end{aligned}$$

Therefore,

$$\begin{aligned}q(x_k, x_{k+n+1}) &\leq q(x_k, x_{k+1}) + q(x_{k+1}, x_{k+n+1})\\ &< \frac{\delta_1}{16} + \frac{\delta}{2} < \frac{\delta_1}{4} + \frac{\delta}{2}.\end{aligned}$$

Case II: Let $q(x_k, x_{k+n}) \leq \frac{\delta}{2}$. Therefore,

$$\begin{aligned}q(x_k, x_{k+n+1}) &\leq q(x_k, x_{k+1}) + q(x_{k+1}, x_{k+n+1})\\ &\leq q(x_k, x_{k+1}) + \varphi(\max\{q(x_k, x_{k+n}), q(x_k, x_{k+1}), q(x_{k+n}, x_{k+n+1}\})\\ &\leq q(x_k, x_{k+1}) + \max\{q(x_k, x_{k+n}), q(x_k, x_{k+1}), q(x_{k+n}, x_{k+n+1})\}\\ &< \frac{\delta_1}{16} + \max\left\{\frac{\delta}{2}, \frac{\delta_1}{16}, \frac{\delta_1}{16}\right\}\\ &< \frac{\delta_1}{4} + \frac{\delta}{2}.\end{aligned}$$

Thus $q(x_k, x_{k+n}) < \frac{\delta_1}{4} + \frac{\delta}{2}$ for all $n \in \mathbb{N}$ and $k \geq N$. In a similar way, we can show that $q(x_{k+n}, x_n) < \frac{\delta_1}{4} + \frac{\delta}{2}$ for all $n \in \mathbb{N}$ and $k \geq N$.

Let $i, j \in \mathbb{N}$ be such that $j \geq i \geq N$. Then $i = N + n,\ j = N + m$ for some $m \geq n$. Then

$$q(x_i, x_N) = q(x_{N+n}, x_N) < \frac{\delta_1}{4} + \frac{\delta}{2} < \delta$$

and

$$q(x_N, x_j) = q(x_N, x_{N+m}) < \frac{\delta_1}{4} + \frac{\delta}{2} < \delta,$$

which imply that $d(x_i, x_j) \leq \frac{\varepsilon}{2} < \varepsilon$. Similarly we have $d(x_j, x_i) < \varepsilon$. Thus $\{x_n\}$ is a Cauchy sequence in (X, d^s). So there exists $z \in X$ such that $d(x_n, z) \to 0$ as $n \to \infty$.

Let $\varepsilon > 0$ be arbitrary. Then there exists $\delta > 0$ with $\delta < \frac{\varepsilon}{2}$ and $N \in \mathbb{N}$ such that

$$q(x_n, x_m) < \delta + \frac{\varepsilon}{2} \text{ for all } m \geq n \geq N.$$

Since for $n \geq N$, the mapping $q(x_n, \cdot)$ is lower semi-continuous on $(X, \tau_{d^{-1}})$ and $d(x_m, z) \to 0$ as $m \to \infty$, there exists $N_1 \in \mathbb{N}$ such that

$$q(x_n, z) - q(x_n, x_m) < \varepsilon \text{ for all } m \geq N_1.$$

Therefore, for $m \geq n \geq \max\{N, N_1\}$, we have

$$\begin{aligned} q(x_n, z) &< q(x_n, x_m) + \varepsilon \\ &< \delta + \frac{\varepsilon}{2} + \varepsilon < 2\varepsilon. \end{aligned}$$

So $q(x_n, z) \to 0$ as $n \to \infty$. Again since $q(\cdot, x_n)$ is lower semi-continuous, we can similarly show that $q(z, x_n) \to 0$ as $n \to \infty$. Thus

$$\begin{aligned} & q(z,z) \leq q(z, x_n) + q(x_n, z) \to 0 \text{ as } n \to \infty \\ \implies & q(z,z) = 0. \end{aligned}$$

Next, our goal is to prove $d(z, x_n) \to 0$ as $n \to \infty$. For any $\varepsilon > 0$, we get $\delta > 0$ with $\delta < \varepsilon$ such that $q(y, x) \leq \delta$ and $q(x, z) \leq \delta$ imply $d(y, z) \leq \frac{\varepsilon}{2}$. For sufficiently large m and n, we have $q(z, x_m) < \delta$ and $q(x_m, x_n) < \delta$, and so $d(z, x_n) < \varepsilon$.. Thus we reach to our goal. Since T is q-lower semicontinuous, the map $g(x) = q(x, Tx)$ is lower semicontinuous on (X, d^s). Since $d^s(x_n, z) \to 0$ as $n \to \infty$, we have

$$\begin{aligned} & g(z) \leq \liminf g(x_n) \\ \implies & q(z, Tz) \leq \liminf q(x_n, Tx_n) = 0 \\ \implies & q(z, Tz) = 0. \end{aligned}$$

Therefore,

$$\begin{aligned} q(x_{n+1}, Tz) &\leq \varphi(\max\left\{q(x_n, z), q(x_n, x_{n+1}), q(z, Tz)\right\}) \\ &\leq \max\left\{q(x_n, z), q(x_n, x_{n+1}), q(z, Tz)\right\} \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

Since $q(\cdot, z)$ is lower semi-continuous on $(X, \tau_{d^{-1}})$, for any $\varepsilon > 0$, we get $N_2 \in \mathbb{N}$ such that

$$\begin{aligned} & q(Tz, z) - q(x_n, z) < \varepsilon \\ & \implies q(Tz, z) < q(x_n, z) + \varepsilon \text{ for all } n \geq N_2. \end{aligned}$$

So $q(Tz, z) = 0$ and hence $q(Tz, Tz) = 0$. This gives $d(z, Tz) = 0 = d(Tz, z)$. Thus $Tz = z$, i.e., z is a fixed point of T.

For uniqueness of fixed point of T, let $u_1,\ u_2$ be two fixed points of T. Then

$$\begin{aligned} q(u_1, u_1) &\leq \varphi(\max\{q(u_1, u_1), q(u_1, u_1), q(u_1, u_1))\} \\ &= \varphi(q(u_1, u_1)), \end{aligned}$$

which implies that $q(u_1, u_1) = 0$. Similarly $q(u_2, u_2) = 0$. From these, we have

$$\begin{aligned} q(u_1, u_2) &\leq \varphi \max\{q(u_1, u_2), q(u_1, u_1), q(u_2, u_2)\}) \\ &= \varphi(q(u_1, u_2)). \end{aligned}$$

So $q(u_1, u_2) = 0$. Similarly $q(u_2, u_1) = 0$. Then proceeding as the argument in which we show that $z = Tz$, we can show that $u_1 = u_2$. This completes the proof. □

Subsequently, we prove the following theorem by making some changes in the contraction conditions (1) and (2), and relaxing the q-lower semicontinuity of T:

Theorem 2.3. *Suppose that (X, d) is a complete quasi-metric space and T is a self-map on X. Further, assume that there exists a strong mw-distance q on X and a Jachymski function φ such that*

$$q(Tx, Ty) \leq \varphi(\max\{q(x, y), q(x, Tx)\}) \tag{9}$$

and

$$q(Tx, Ty) \leq \varphi(\max\{q(x, y), q(Tx, x)\}) \tag{10}$$

for all $x, y \in X$, and $\varphi(t) < t$ for all $t > 0$. Then the following hold:

(i) T has a unique fixed point z (say) and $q(z, z) = 0$;

(ii) for any $x_0 \in X$, the Picard's iterative sequence $\{x_n\}$ converges to z in (X, d^s).

Proof. The proof of this theorem follows from the proofs of Theorem 2.2 and [5, Theorem 6]. □

Next, we prove the following theorem by modifying the contraction condition of Theorem 2.2. To prove this theorem, we need some some extra assumptions on φ. To be more specific, we need the following extra assumptions:

($E1$) φ is nondecreasing;
($E2$) $\limsup \varphi(x_n) \leq \varphi(\limsup x_n)$ for every sequence $\{x_n\}$ in $\mathbb{R}_0^+$.

Theorem 2.4. *Let T be a q-lower semicontinuous self-map on a complete quasi-metric space (X, d). Assume that there exist a strong mw-distance q on X, a Jachymski function φ satisfying ($E1$), ($E2$), and three constants $a, b, c \in \mathbb{R}_0^+$ such that*

$$q(Tx, Ty) \leq \varphi(aq(x, y) + bq(x, Ty) + cq(y, Tx)) \tag{11}$$

and

$$q(Tx, Ty) \leq \varphi(aq(x, y) + bq(Tx, y) + cq(Ty, x)) \tag{12}$$

for all $x, y \in X$, $\varphi(t) < t$ for all $t > 0$ and $a + b + c = 1$. Along with these, suppose that there exists $x_0 \in X$ such that $q(T^n x_0, T^n x_0) \leq \varphi(q(T^{n-1}x_0, T^{n-1}x_0))$ for all $n \in \mathbb{N}$ and the set $\left\{q(T^n x_0, T^m x_0) : n, m \in \mathbb{N}\right\}$ is bounded. Then T has has a unique fixed point z (say) satisfying $q(z, z) = 0$.

Proof. If $a = 1$, then the result follows from [2, Theorem 2]. Therefore, we now assume that $a < 1$.

Let us consider the sequence $\{x_n\}$ defined by $x_n = T^n x_0$ for all n. Put $\alpha_n = q(x_n, x_{n+1})$, $\beta_n = q(x_{n+1}, x_n)$ and $\gamma_n = q(x_n, x_n)$ for all n. Since the set $\left\{q(T^n x_0, T^m x_0) : n, m \in \mathbb{N}\right\}$ is bounded, we have $D_1 = \limsup \alpha_n < \infty$, $D_2 = \limsup \beta_n < \infty$ and $D_3 = \limsup \gamma_n < \infty$. By given condition, we have

$$\begin{aligned} &q(x_{n+1}, x_{n+1}) \leq \varphi(q(x_n, x_n)) \\ &\quad \Longrightarrow \gamma_{n+1} \leq \varphi(\gamma_n). \end{aligned}$$

Taking lim sup in both sides of above inequality and using ($E1$), ($E2$), we get $D_3 \leq \varphi(D_3)$, from which we get $D_3 = 0$.

Now

$$\begin{aligned} q(Tx_n, Tx_{n+1}) &\leq \varphi(aq(x_n, x_{n+1}) + bq(x_n, Tx_{n+1}) + cq(x_{n+1}, Tx_n)) \\ &\leq \varphi(aq(x_n, x_{n+1}) + bq(x_n, x_{n+1}) + bq(x_{n+1}+, x_{n+1}) + cq(x_{n+1}, x_{n+1})) \\ \Longrightarrow \alpha_{n+1} &\leq \varphi(a\alpha_n + b\alpha_n + b\alpha_{n+1} + c\gamma_{n+1}). \end{aligned}$$

Taking lim sup in both sides of above inequality and using ($E1$), ($E2$), we get

$$\begin{aligned} &D_1 \leq \varphi(aD_1 + 2bD_1 + cD_3) \\ \Longrightarrow\ &D_1 \leq \varphi(aD_1 + 2bD_1). \end{aligned} \tag{13}$$

Again we have

$$\begin{aligned} q(Tx_n, Tx_{n+1}) &\le \varphi(aq(x_n, x_{n+1}) + bq(Tx_n, x_{n+1}) + cq(Tx_{n+1}, x_n)) \\ &\le \varphi(aq(x_n, x_{n+1}) + bq(x_{n+1}, x_{n+1}) + cq(x_{n+2}+, x_{n+1}) + cq(x_{n+1}, x_n)) \\ \Longrightarrow\ \alpha_{n+1} &\le \varphi(a\alpha_n + b\gamma_n + c\beta_{n+1} + c\beta_n). \end{aligned}$$

Taking lim sup in both sides of above inequality and using $(E1)$, $(E2)$, $D_3 = 0$, we get

$$D_1 \le \varphi(aD_1 + 2cD_2). \tag{14}$$

Similarly interchanging x_n and x_{n+1} and using (11) and (12), we can obtain

$$D_2 \le \varphi(aD_2 + 2cD_1), \tag{15}$$
$$D_2 \le \varphi(aD_2 + 2bD_2). \tag{16}$$

If $D_1 \ne 0, D_2 \ne 0$, then from (13)-(16), we get

$$D_1 < aD_1 + 2bD_1,\ D_1 < aD_1 + 2cD_2,\ D_2 < aD_2 + 2cD_1,\ D_2 < aD_2 + 2bD_2.$$

Adding the above four inequalities, we obtain

$$2D_1 + 2D_2 < 2D_1 + 2D_2,$$

which is a contradiction, and this contradiction ensures that at least one of D_1 and D_2 must be 0. Without loss of generality, we assume that $D_1 = 0$. Then from (15), we get $D_2 \le \varphi(aD_2)$. If $D_2 \ne 0$, then we get

$$D_2 \le \varphi(aD_2) \le aD_2 < D_2,$$

which leads to a contradiction. Therefore, $D_1 = 0 = D_2$, that is,

$$\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = \lim_{n\to\infty} \gamma_n = 0. \tag{17}$$

Next, we show that $\{x_n\}$ is a Cauchy sequence in the metric space (X, d^s). Let $\varepsilon > 0$ be arbitrary. Then we get $\delta > 0$ with $\delta < \varepsilon$ such that

$$t > 0 \text{ and } \varepsilon < t < \varepsilon + \delta \implies \varphi(t) \le \varepsilon$$

and

$$q(y, x) \le \delta,\ q(x, z) \le \delta \implies d(y, z) \le \frac{\varepsilon}{2}.$$

Again, we get $\delta_1 > 0$ with $\delta_1 < \delta$ such that

$$t > 0 \text{ and } \frac{\delta}{2} < t < \frac{\delta}{2} + \delta_1 \implies \varphi(t) \le \frac{\delta}{2}.$$

By (17), we get $N \in \mathbb{N}$ such that

$$q(x_n, x_{n+1}) < \frac{\delta_1}{16},\ q(x_{n+1}, x_n) < \frac{\delta_1}{16},\ q(x_n, x_n) < \frac{\delta_1}{16}, \text{ for all } n \ge N. \tag{18}$$

Let $k \ge n$ be arbitrary. Then we show by induction on n that

$$q(x_k, x_{k+n}) < \frac{\delta_1}{4} + \frac{\delta}{2},\ q(x_{k+n}, x_k) < \frac{\delta_1}{4} + \frac{\delta}{2}.$$

For $n = 1$, it follows from (18). We now assume that

$$q(x_k, x_{k+n}) < \frac{\delta_1}{4} + \frac{\delta}{2},\ q(x_{k+n}, x_k) < \frac{\delta_1}{4} + \frac{\delta}{2}$$

for some n. Then we have the following cases:

Case I: Let $q(x_k, x_{k+n}) > \frac{\delta}{2}$, $q(x_{k+n}, x_k) > \frac{\delta}{2}$. Therefore,

$$\begin{aligned} q(x_{k+1}, x_{k+n+1}) &\leq \varphi(aq(x_k, x_{k+n}) + bq(x_k, x_{k+n+1}) + cq(k_{k+n}, x_{k+1})) \\ &\leq \varphi(aq(x_k, x_{k+n}) + bq(x_k, x_{k+n}) + bq(x_{k+n}, x_{k+n+1}) \\ &+ cq(k_{k+n}, x_k) + cq(k_k, x_{k+1})). \end{aligned} \tag{19}$$

Also since $\frac{\delta}{2} < q(x_k, x_{k+n})$, $q(x_{k+n}, x_k)$, we have

$$\begin{aligned} &(a + b + c)\frac{\delta}{2} < aq(x_k, x_{k+n}) + bq(x_k, x_{k+n}) + cq(x_{k+n}, x_k) \\ &\implies \frac{\delta}{2} < aq(x_k, x_{k+n}) + bq(x_k, x_{k+n}) + bq(x_{k+n}, x_{k+n+1}) \\ &\quad + cq(k_{k+n}, x_k) + cq(k_k, x_{k+1}) \\ &\quad < a\left(\frac{\delta_1}{4} + \frac{\delta}{2}\right) + b\left(\frac{\delta_1}{4} + \frac{\delta}{2}\right) + b\frac{\delta_1}{16} + a\left(\frac{\delta_1}{4} + \frac{\delta}{2}\right) + c\frac{\delta_1}{16} \\ &\quad < \frac{\delta}{2} + \delta_1. \end{aligned}$$

Therefore,

$$\varphi(aq(x_k, x_{k+n}) + bq(x_k, x_{k+n}) + bq(x_{k+n}, x_{k+n+1}) + cq(k_{k+n}, x_k) + cq(k_k, x_{k+1})) \leq \frac{\delta}{2},$$

and hence from (19), we get

$$q(x_{k+1}, x_{k+n+1}) \leq \frac{\delta}{2}.$$

Therefore,

$$\begin{aligned} q(x_k, x_{k+n+1}) &\leq q(x_k, x_{k+1}) + q(x_{k+1}, x_{k+n+1}) \\ &< \frac{\delta}{2} + \frac{\delta_1}{16} < \frac{\delta}{2} + \frac{\delta_1}{4}. \end{aligned}$$

Similarly, we can show that

$$q(x_{k+n+1}, x_k) < \frac{\delta}{2} + \frac{\delta_1}{4}.$$

Case II: Let $q(x_k, x_{k+n}) > \frac{\delta}{2}$ and $q(x_{k+n}, x_k) \leq \frac{\delta}{2}$. Therefore,

$$\begin{aligned} (a + b)\frac{\delta}{2} &< aq(x_k, x_{k+n}) + bq(x_k, x_{k+n}) \\ &< aq(x_k, x_{k+n}) + bq(x_k, x_{k+n}) + bq(x_{k+n}, x_{k+n+1}) + cq(x_{k+n}, x_k) + cq(x_k, x_{k+1}) \\ &< a\left(\frac{\delta_1}{4} + \frac{\delta}{2}\right) + b\left(\frac{\delta_1}{4} + \frac{\delta}{2}\right) + b\frac{\delta_1}{16} + c\frac{\delta}{2} + c\frac{\delta_1}{16} \\ &< \frac{\delta}{2} + \delta_1. \end{aligned}$$

Then we have two subcases:

Subcase I: Let

$$\frac{\delta}{2} < aq(x_k, x_{k+n}) + bq(x_k, x_{k+n}) + bq(x_{k+n}, x_{k+n+1}) + cq(x_{k+n}, x_k) + cq(x_k, x_{k+1}) < \frac{\delta}{2} + \delta_1.$$

Therefore, from (19), we get

$$q(x_{k+1}, x_{k+n+1}) \leq \frac{\delta}{2}.$$

Continuing similar to Case I, we have

$$q(x_k, x_{k+n+1}) < \frac{\delta}{2} + \frac{\delta_1}{4}.$$

Subcase II: Let

$$aq(x_k, x_{k+n}) + bq(x_k, x_{k+n}) + bq(x_{k+n}, x_{k+n+1}) + cq(x_{k+n}, x_k) + cq(x_k, x_{k+1}) \leq \frac{\delta}{2}.$$

Therefore,

$$\varphi(aq(x_k, x_{k+n}) + bq(x_k, x_{k+n}) + bq(x_{k+n}, x_{k+n+1}) + cq(x_{k+n}, x_k) + cq(x_k, x_{k+1})) \leq \varphi\left(\frac{\delta}{2}\right) < \frac{\delta}{2}.$$

Then by following Subcase I and case I, we have

$$q(x_k, x_{k+n+1}) < \frac{\delta}{2} + \frac{\delta_1}{4}.$$

Similarly we can show that

$$q(x_{k+n+1}, x_k) < \frac{\delta}{2} + \frac{\delta_1}{4}.$$

Case III: Let $q(x_k, x_{k+n}) \leq \frac{\delta}{2}$ and $q(x_{k+n}, x_k) > \frac{\delta}{2}$. This case is similar to Case II.
Case IV: Let $q(x_k, x_{k+n}) \geq \frac{\delta}{2}$ and $q(x_{k+n}, x_k) \geq \frac{\delta}{2}$. Therefore,

$$\begin{aligned} q(x_{k+1}, x_{k+n+1}) &\leq \varphi(aq(x_{k+1}, x_{k+n}) + bq(x_k, x_{k+n}) + cq(x_{k+n}, x_k) + bq(x_{k+n+}, x_{k+n+1}) \\ &\quad + cq(x_k, x_{k+1})) \\ &\leq \varphi\left(a\frac{\delta}{2} + b\frac{\delta}{2} + c\frac{\delta}{2} + (b+c)\frac{\delta_1}{16}\right) < \frac{\delta}{2} + \frac{\delta_1}{16}. \end{aligned}$$

So

$$q(x_k, x_{k+n+1}) < \frac{\delta_1}{16} + \frac{\delta}{2} + \frac{\delta_1}{16} < \frac{\delta}{2} + \frac{\delta_1}{4}.$$

Similarly we have

$$q(x_{k+n+1}, x_k) < \frac{\delta}{2} + \frac{\delta_1}{4}.$$

Thus

$$q(x_k, x_{k+n}),\ q(x_{k+n}, x_k) < \frac{\delta}{2} + \frac{\delta_1}{4} \text{ for all } n \text{ and for all } k \geq N. \tag{20}$$

Let $i, j \in \mathbb{N}$ be arbitrary with $j \geq i > N$. Then $i = N + n$, $j = N + m$ for some $n,\ m$ with $m \geq n$. Then

$$\begin{aligned} q(x_N, x_j) &= q(x_N, x_{N+n}) < \frac{\delta}{2} + \frac{\delta_1}{4} < \delta, \\ q(x_i, x_N) &= q(x_{N+m}, x_N) < \frac{\delta}{2} + \frac{\delta_1}{4} < \delta. \end{aligned}$$

Therefore,

$$d(x_i, x_j) \leq \frac{\varepsilon}{2} < \varepsilon.$$

In a similar way, we have

$$d(x_j, x_i) < \varepsilon.$$

Hence $\{x_n\}$ is Cauchy in (X, d^s) and so we get $z \in X$ such that $d(x_n, z) \to 0$ as $n \to \infty$. Let $\varepsilon >$ be arbitrary. Then by (20), we get $\delta > 0$ with $\delta < \frac{\varepsilon}{2}$ and $N \in \mathbb{N}$ such that

$$q(x_n, x_m) < \delta + \frac{\varepsilon}{2} \text{ for all } m \geq n \geq N.$$

Now for $n \geq N$, the mapping $q(x_n, \cdot)$ is lower semicontinuous on $(X, \tau_{d^{-1}})$ and $d(x_n, z) \to 0$ as $n \to \infty$, so we get $N_1 \in \mathbb{N}$ such that

$$q(x_n, z) - q(x_n, x_m) < \varepsilon \text{ for all } m \geq N_1.$$

Then for $m \geq n \geq \max\{N, N_1\}$, we have

$$q(x_n, z) < q(x_n, x_m) + \varepsilon < \delta + \frac{\varepsilon}{2} + \varepsilon < 2\varepsilon,$$

which shows that $q(x_n, z) \to 0$ as $n \to \infty$. Again $q(\cdot, x_n)$ is lower semicontinuous, so proceeding as above, we have $q(z, x_n) \to 0$ as $n \to \infty$. Therefore,

$$\begin{aligned} & q(z,z) \leq q(z, x_n) + q(x_n, z) \ \to \ 0 \text{ as } n \to \infty \\ \implies & q(z,z) = 0. \end{aligned}$$

Next, we show that $d(z, x_n) \to 0$ and $d(x_n, Tz) \to 0$ as $n \to \infty$. For this choose $\varepsilon > 0$ arbitrarily. Then we get $\delta > 0$ with $\delta < \varepsilon$ such that $q(y, x) \leq \delta$ and $q(x, z) \leq \delta$ implies $d(y, z) \leq \frac{\varepsilon}{2}$. For sufficiently large m, n, since $q(z, x_m) < \delta$ and $q(x_m, x_n) < \delta$ hold, we have $d(z, x_n) < \varepsilon$. This shows that $d(z, x_n) \to 0$ as $n \to \infty$. Hence $d^s(x_n, z) \to 0$ as $n \to \infty$.

Since T is q-lower semicontinuous, we have

$$\begin{aligned} & q(z, Tz) \leq \liminf q(x_n, Tx_n) \\ & \qquad = \liminf q(x_n, x_{n+1}) = 0 \\ \implies & q(z, Tz) = 0. \end{aligned}$$

Since $q(x_n, z) \to 0$ as $n \to \infty$, we get $N_3 \in \mathbb{N}$ such that $q(x_n, z) < \delta$ for all $n \geq N_3$ Also, we have $q(z, Tz) = 0 < \delta$. So

$$d(x_n, Tz) \leq \frac{\varepsilon}{2} \text{ for all } n \geq N_3.$$

Therefore, $d(x_n, Tz) \to 0$ as $n \to \infty$. Since $q(\cdot, z)$ is lower semi-continuous on $(X, \tau_{d^{-1}})$ and $d(x_n, Tz) \to 0$, we get $N_4 \in \mathbb{N}$ such that

$$\begin{aligned} & q(Tz, z) - q(x_n, z) < \varepsilon \text{ for all } n \geq N_4 \\ & \quad \implies q(Tz, z) < q(x_n, z) + \varepsilon. \end{aligned}$$

Then using the fact $q(x_n, z) \to 0$, we have $q(Tz, z) = 0$. Using the facts $q(z, Tz) = 0$ and $q(Tz, z) = 0$, we have $q(z, Tz) = 0$. Since $q(Tz, z) = 0 < \delta$ and $q(z, z) = 0 < \delta$, we have $d(Tz, z) \leq \frac{\varepsilon}{2}$, that is, $d(Tz, z) = 0$. Similarly we have $d(z, Tz) = 0$ Therefore, $Tz = z$, that is, z is a fixed point of T.

For uniqueness, let u_1, u_2 be two fixed points of T. Then

$$\begin{aligned} & q(u_1, u_1) = q(Tu_1, Tu_1) \\ & \qquad \leq \varphi(aq(u_1, u_1) + bq(u_1, Tu_1) + cq(u_1, Tu_1)) \\ & \qquad = \varphi(q(u_1, u_1)) \\ \implies & q(u_1, u_1) = 0. \end{aligned}$$

Similarly we have $q(u_2, u_2) = 0$. Thus

$$\begin{aligned} q(u_1, u_2) &= q(Tu_1, Tu_2) \\ &\leq \varphi(aq(u_1, u_2) + bq(u_1, u_2) + cq(u_2, u_1)). \end{aligned}$$

If $q(u_1, u_2) + q(u_2, u_1) \neq 0$, then we have

$$q(u_1, u_2) < aq(u_1, u_2) + bq(u_1, u_2) + cq(u_2, u_1). \tag{21}$$

Similarly, we have

$$q(u_2, u_1) < aq(u_2, u_1) + bq(u_2, u_1) + cq(u_1, u_2). \tag{22}$$

Adding (21) and (22), we get

$$q(u_1, u_2) + q(u_2, u_1) < q(u_1, u_2) + q(u_2, u_1),$$

which leads to a contradiction. So we have $q(u_1, u_2) + q(u_2, u_1) = 0$, that is, $q(u_1, u_2) = 0 = q(u_2, u_1)$. Therefore, $d(u_1, u_2) = 0 = d(u_2, u_1)$. This completes the proof. □

Now we present some examples to validate the above two results.

Example 2.5. *Let us take $X = [0, 3]$ and define $d : X \times X \to \mathbb{R}$ by $d(x, y) = y - x$ if $y \geq x$ and $d(x, y) = 1$ if $y < x$; $q(x, y) = x$ for all $x, y \in X$. Then (X, d) is a quasi-metric space and q is a strong $m\omega$-distance on X.*

Next, we define $T : X \to X$ by $Tx = 0$ if $x \in [0, 1)$ and $Tx = \frac{3}{5}x$ if $x \in [1, 3]$. Also, we take a Jachymski function φ defined by $\varphi(t) = \frac{9}{10}t$. Then clearly $\varphi(t) < t$ for all $t > 0$. It can be easily verified that (1) and (2) hold for all $x, y \in X$ and T is q-lower semicontinuous. Therefore, by the conclusions of Theorem 2.2, we get a unique fixed point z of T satisfying $q(z, z) = 0$. Indeed, here $z = 0$.

Example 2.6. *Let us consider the normed lattice $(X, \preccurlyeq, \|\cdot\|)$, where $X = C[0, 1]$, $x \preccurlyeq y$ means $x(t) \leq y(t)$ for all $t \in [0, 1]$ and $\|\cdot\|$ is the sup norm. Let $X^+ = \{x \in X : x(t) \geq 0$ for all $t \in [0, 1]\}$, and define $\|x\|^+ = \|x \vee \mathbf{0}\|$, where $\mathbf{0}$ is the zero function and $d^+(x, y) = \|y - x\|^+$ for all $x, y \in X^+$. Then from [2, Example 5], it is known that (X^+, d^+) is a quasi-metric space, and the function $q : X^+ \times X^+ \to [0, \infty)$ defined by $q(x, y) = \|y\|$ is a strong mw-distance on X^+.*

Now we define a map $T : X^+ \to X^+$ by $(Tx)(t) = 0$ if $\|x\| \leq 6$ and $(Tx)(t) = 2t$ elsewhere. We choose a Jachymski function φ defined by $\varphi(t) = \frac{3}{4}t$. So $\varphi(t) < t$ for all $t > 0$. Then one can verify by simple calculations that T is q-lower semicontinuous and (1) and (2) hold for all $x, y \in X$. So by the conclusions of Theorem 2.2, T possesses a unique fixed point z satisfying $q(z, z) = 0$, and here z is the zero function.

Example 2.7. *Let $X = \mathbb{R}^2$ and define $p : X \to \mathbb{R}$ by*

$$p(x_1, x_2) = \begin{cases} x_1 + x_2, & \text{if } x_1, x_2 \geq 0 \\ 0, & \text{elsewhere.} \end{cases}$$

Then p is an asymmetric norm on X. Therefore, $d, q : X \times X \to \mathbb{R}$ defined by $d(x, y) = p(y - x)$ and $q(x, y) = p(x) + p(-y)$, are respectively a complete quasi-metric and a strong $m\omega$-distance on (X, d) respectively.

Next, we define a mapping $T : X \to X$ by

$$T(x_1, x_2) = \begin{cases} (x_1 - 1, 0), & \text{if } (x_1, x_2) \in C \\ (-1, 2), & \text{if } (x_1, x_2) \notin C, \end{cases}$$

where $C = \{(x_1, x_2) \in \mathbb{R}^2 : x_1, x_2 \geq 1$ and $2x_1 \leq x_2 + 3\}$. We choose a Jachymski function φ defined by $\varphi(t) = \frac{3}{4}t$. Then $\varphi(t) < t$ for all $t > 0$ and φ satisfies (E1), (E2). One can easily verify that T is q-lower semicontinuous and (11) and (12) hold for $a = b = c = \frac{1}{3}$ for all $x, y \in X$. Also, if we take $x_0 \in C$ then $q(T^n x_0, T^n x_0) \leq \varphi(q(T^{n-1}x_0, T^{n-1}x_0))$ for all $n \in \mathbb{N}$ and the set $\{q(T^n x_0, T^m x_0) : n, m \in \mathbb{N}\}$ is bounded. So by consequences of Theorem 2.4, it follows that T has a unique fixed point z of T with $q(z, z) = 0$. Note that here $z = (-1, 2)$.

3. Application

Measurement theory (MT) is mainly concerned about the investigation of how numbers are allocated to things and occurrences, and its distresses contain the classes of possessions that can be dignified, how diverse measures communicate to each other, and the issue of error in the measurement procedure. In this theory, it requests to optimize the minimum location to transform the information in the networks [6]. It has been recognized as one of the best processes in MT is by using the unique fixed point of a suitably defined operator. In this application, we aim to use Theorem 2.2. For this purpose, we shall use the modified fractional local calculus (fractal) [14].

3.1. Fractal operator construction

Let $X = C_\alpha(0,1)$ be the space of local fractional continuous functions in the domain $I = (0,1)$ such that for all $\gamma > 0$, there is $\kappa > 0$ satisfying $|x - x_0| < \gamma$ whenever, $|t - t_0| < \kappa$. In communication, x is the distributor of information while x_0 is the target. The distance between the distributor and the target can be measured by the fractal $D^\alpha : X \times X \to [0,\infty)$ as follows:

$$D^\alpha(x, x_0) = \Gamma(\alpha+1)|x - x_0|, \quad \alpha \in (0,1]. \tag{23}$$

Note that the $|\cdot|$ indicates the minimum distance between x and x_0. Moreover, $D^\alpha(x,\cdot)$ achieves

$$D^\alpha(x,\cdot) = D^\alpha(x) = \Gamma(\alpha+1)x, \quad \alpha \in (0,1].$$

Define an operator $T_\alpha : X \to X$ by

$$T_\alpha x = \alpha\left(\frac{D^\alpha(x)}{\Gamma(\alpha+1)}\right), \quad \alpha \in (0,1].$$

We have the following result:

Theorem 3.1. *Consider the fractal space $X = C_\alpha(0,1)$ for $\alpha \in (0,1]$. Then (X, D^α) is a quasi-metric space and T_α has a unique fixed point.*

Proof. Consider the fractal space $X = C_\alpha(0,1)$ for $\alpha \in (0,1]$. Then for $x, x_0 \in X$ we have

(a) $D^\alpha(x,x_0) = \Gamma(\alpha+1)|x - x_0| \geq 0$ and $D^\alpha(x_0,x_0) = \Gamma(\alpha+1)|x_0 - x_0| = 0$;

(b) $D^\alpha(x,x_0) = \Gamma(\alpha+1)|x - x_0| = \Gamma(\alpha+1)|x_0 - x| = D^\alpha(x_0,x)$;

(c) $D^\alpha(x,x_{00}) = \Gamma(\alpha+1)|x - x_{00}| \leq \Gamma(\alpha+1)|x - x_0| + \Gamma(\alpha+1)|x_0 - x_{00}| = D^\alpha(x,x_0) + D^\alpha(x_0,x_{00})$.

Thus (X, D^α) is a quasi-metric space.

Define a target function (objective function) $Q : \mathbb{R}^2 \to [0,\infty)$ by $Q(x,x_0) = x$. Then, we obtain

(a) $Q(x,x_{00}) = x \leq x + x_0 = Q(x,x_0) + Q(x_0,x_{00})$ for all $x, x_0, x_{00} \in X$;

(b) define $Q(x,\cdot) : \mathbb{R} \to [0,\infty)$ by $Q(x,\cdot) = x$; thus, Q is a lower semicontinuous function on X.

(c) Let $x, x_0, x_{00} \in X$ such that $Q \leq \delta$ for all $x \in X$. Suppose that $\delta = \dfrac{\epsilon}{2\Gamma(\alpha+1)}$. Then we have

$$\begin{aligned} D^\alpha(x_0,x_{00}) &= \Gamma(\alpha+1)|x_0 - x_{00}| \\ &\leq \Gamma(\alpha+1)\left(Q(x_0,.) + Q(x_{00},.)\right) \\ &\leq \Gamma(\alpha+1)(2\delta) \\ &= \epsilon. \end{aligned}$$

Thus, we have $D^\alpha(x_0,x_{00}) \leq \epsilon$; consequently, Q is a strong $m\omega-$distance on (X, D^α).

Define a function $\Theta_\alpha : [0,\infty) \to [0,\infty)$ by

$$\Theta_\alpha(\chi) = \Gamma(\alpha+1)\chi, \quad \alpha \in (0,1).$$

Then, we get

(a) $\Theta_\alpha(0) = 0$;

(b) for any $\epsilon > 0$, if we choose $\delta = \left(\frac{1}{\Gamma(\alpha+1)} - 1\right)\epsilon$, then $\chi > 0$ and $\epsilon < \chi < \epsilon + \delta$ imply $\Theta_\alpha(\chi) \leq \epsilon$.

Thus Θ_α is a Jachymski function. Also for $\chi > 0$, we have $\Theta_\alpha(\chi) = \Gamma(\alpha+1)\chi < \chi$, $\alpha \in (0,1)$. Finally, we shall prove that the conditions (1) and (2) are satisfied by using the operator T_α, where

$$T_\alpha x = \alpha\left(\frac{D^\alpha(x)}{\Gamma(\alpha+1)}\right), \quad \alpha \in (0,1].$$

(a) It is clear that $Q(T_\alpha x, T_\alpha x_0) = T_\alpha x = \alpha x$. Moreover, we have $Q(x, x_0) = x$, $Q(x, T_\alpha x) = x$, $Q(x_0, T_\alpha x_0) = x_0$; then, we obtain

$$\begin{aligned}\Theta_\alpha\left(\max\{Q(x,x_0), Q(x,T_\alpha x), Q(x_0,T_\alpha x_0)\}\right) &= \Theta_\alpha\left(\max\{x, x_0\}\right)\\ &= \Gamma(\alpha+1)\max\{x, x_0\}.\end{aligned}$$

But

$$\begin{aligned}Q(T_\alpha x, T_\alpha x_0) &= \alpha x\\ &\leq \Gamma(\alpha+1)\max\{x, x_0\}\\ &= \Theta_\alpha\left(\max\{Q(x,x_0), Q(x,T_\alpha x), Q(x_0,T_\alpha x_0)\}\right), \quad \alpha \in (0,1];\end{aligned}$$

thus, condition (1) is achieved.

(b) Similarly, we have $Q(T_\alpha x, T_\alpha x_0) = T_\alpha x = \alpha x$, $Q(x, x_0) = x$, $Q(T_\alpha x, x) = \alpha x$, $Q(T_\alpha x_0, x_0) = \alpha x_0$ then,

$$\begin{aligned}\Theta_\alpha\left(\max\{Q(x,x_0), Q(T_\alpha x,x), Q(T_\alpha x_0,x_0)\}\right) &= \Theta_\alpha\left(\max\{x, \alpha x_0, \alpha x\}\right)\\ &= \Gamma(\alpha+1)\max\{x, \alpha x_0, \alpha x\}.\end{aligned}$$

Consequently, we obtain

$$Q(T_\alpha x, T_\alpha x_0) \leq \Theta_\alpha\left(\max\{Q(x,x_0), Q(T_\alpha x,x), Q(T_\alpha x_0,x_0)\}\right),$$

which indicates condition (2). As a conclusion, this shows that the operator T_α has a unique fixed point in X, where $Q(T_\alpha(0), T_\alpha(0)) = 0$ (see Theorem 2.2).

□

Example 3.2. *Let $\alpha = 0.5$ then we have $D^{0.5}x = 0.886x$ and $T_{0.5}x = 0.5x$, $Q(x,x_0) = x$ and $\Theta_{0.5}\chi = 0.886\chi < \chi$ such that*

$$Q(T_\alpha x, T_\alpha x_0) = 0.5x \leq 0.886x = \Theta_\alpha\left(\max\{Q(x,x_0), Q(x,T_\alpha x), Q(x_0,T_\alpha x_0)\}\right).$$

Similarly, we have

$$Q(T_\alpha x, T_\alpha x_0) = 0.5x \leq 0.886x = \Theta_\alpha\left(\max\{Q(x,x_0), Q(T_\alpha x,x), Q(T_\alpha x_0,x_0)\}\right).$$

Hence, in view of Theorem 3.1, T_α has a fixed point in $X = C_\alpha(0,1)$ where $\alpha = 0.5$.

Acknowledgement: The authors are thankful to the section editor for his valuable comments towards the improvement of the article. The second named author is funded by the Foundation for Science and Technology Development of Ton Duc Thang University (FOSTECT), website: http:// fostect.tdtu.edu.vn, under Grant FOSTECT.2019.14.

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