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Variational Inequalities with the Logistic Type Nonlinearities and Dependence on the Gradient

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Abstract. In this paper, we study the following variational inequality

$$\begin{cases} u \in K, \\ \langle Au, v-u\rangle + \int\limits_\Omega g(x,u)(v-u) \geq \int\limits_\Omega f(x,u,\nabla u)(v-u), \forall v \in K, \end{cases}$$

where $K = \{u \in W_0^{1,p}(\Omega) : u(x) \geq 0\}$, A is the p- Laplacian and the function g is increasing in the second variable.
By constructing the solution operator for an associate variational inequality, we reduce the problem to a fixed point equation. Then, we apply the fixed point index to prove the existence of the nontrivial solution of the problem.

1. Introduction

We consider the following problem

$$\begin{cases} u \in K, \\ \langle Au, v-u\rangle + \int\limits_\Omega g(x,u)(v-u) \geq \int\limits_\Omega f(x,u,\nabla u)(v-u), \forall v \in K, \end{cases} \tag{1.1}$$

where Ω is a bounded domain with smooth boundary in $\mathbb{R}^N$, f, g are Caratheodory functions satisfying some suitable conditions which will be clarified later, $K = \{u \in W_0^{1,p}(\Omega) : u \geq 0\}$ and $< Au, \phi >= \int\limits_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\phi$ is the p-Laplacian.

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The problem (1.1) is a natural extension of the following generalized logistic equation

$$-\Delta_p u = f(x, u, \nabla u) - g(x, u) \text{ in } \Omega, u = 0 \text{ on } \partial\Omega,$$

which has received a great deal of attention from mathematicians due to many important applications in reaction - diffusion processes and biological models. See for example [6–9, 12, 13, 17] and references therein.

Elliptic variational inequalities of the form

$$\begin{cases} u \in K_1, \\ \langle A_1 u, v - u\rangle \geq \int_\Omega F_1(x, u, \nabla u)(v - u)dx, \forall v \in K_1, \end{cases} \tag{1.2}$$

in which $A_1 : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is an operator of Leray - Lions type, $F_1 : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Caratheodory function and $K_1 \subset W_0^{1,p}(\Omega)$ is a convex set have arisen in Physics, mechanics, engineering, control, optimization and other fields and have been widely studied. To investigate them, mathematicians have applied different methods such as approximations, variational, sub - supersolution, bifurcation, topological index methods. See [2, 3, 10, 11, 14, 15, 18].

In this paper, we will restrict our attention to the problem (1.1) (thus $F_1(x, u, v) = f(x, u, v) - g(x, u)$ in (1.2)) when the function $g(x, u)$ is increasing in the second variable. This condition includes, as a special case, the following condition which has been imposed in the literature [3].

- There is a positive number M such that the function $F_1(x, u, v) + Mu$ is increasing with respect to the variable u, for every $v \in \mathbb{R}^N$.

It is also closely related to the following condition of [10]

- $F_1(x, u) = F(x, u, u)$ with the function F be increasing in the second variable and decreasing in the third variable.

In those cases, the existence of a nontrivial solution can be proved by sub - supersolution method or by the fixed point theorems for increasing operators in ordered spaces. In studying problem (1.1) we do not impose any monotonicity condition on the function f, hence, we can not apply fixed point theorem of increasing operators. Also, unboundedness in x- variable of f, g makes it difficult to construct sub - supersolution of the problem. In present paper we shall use the solution operator of an associated variational inequality to reduce the problem (1.1) to a fixed point equation and then we apply the theory of fixed point index to prove the existence of nontrivial solutions of the problem. We consider the case of $(p-1)-$ sublinear growth of the function f in Theorem 3.1 and the case of asymptotically $(p-1)-$ linear growth in Theorem 3.2 and Theorem 3.3. In Example and Remark 4 we give some simple cases of the functions f and g for Theorems 3.1, 3.2, 3.3 to hold and for compairing of our results with the some previous results in the literature.

2. A reduction to fixed point problem

We shall use the following theorems due to L. Boccardo et al. [1]

Theorem 2.1. *Suppose that $g : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Caratheodory function satisfying the following conditions*

(i) *$g(x, 0) = 0, u \longmapsto g(x, u)$ is increasing function, for a.e $x \in \Omega$;*

(ii) *for all $t > 0$ there exists a function $\varphi_t \in L^1(\Omega)$ such that* $\sup_{|u|\leqslant t} |g(x, u)| \leqslant \varphi_t(x)$.

Then, for any $z \in W^{-1,p'}(\Omega)$, the problem

$$\begin{cases} u \in K, g(x,u) \in L^1(\Omega), ug(x,u) \in L^1(\Omega), \\ \langle Au, v-u\rangle + \int\limits_\Omega g(x,u)(v-u) \geq \langle z, v-u\rangle, \forall v \in K \cap L^\infty(\Omega), \end{cases} \tag{2.1}$$

has a unique solution u satisfying

$$\langle Au, u\rangle + \int\limits_\Omega g(x,u)u = \langle z, u\rangle. \tag{2.2}$$

In addition, if u_1, u_2 are two solutions with respect to z_1, z_2 in (2.1) *then $u_1 g(x,u_2)$ and $u_2 g(x,u_1)$ belong to $L^1(\Omega)$ and*

$$\langle Au_1 - Au_2, u_1 - u_2\rangle + \int\limits_\Omega \Big[g(x,u_1) - g(x,u_2)\Big](u_1 - u_2) \leqslant \langle z_1 - z_2, u_1 - u_2\rangle. \tag{2.3}$$

Theorem 2.2. *Let $u_0 \in W_0^{1,p}(\Omega)$ and μ be a positive Radon measure. Suppose that $h \in L^1(\Omega)$ satisfying*

$$\mu + h \in W_0^{-1,p'}(\Omega), u_0 \geq \theta, hu_0 \geq v \in L^1(\Omega).$$

Then, we have $hu_0 \in L^1(\Omega), u_0 \in L^1(\Omega, \mu)$, and

$$\langle \mu + h, u_0\rangle = \int\limits_\Omega u_0 d\mu \geq \int\limits_\Omega hu_0 dx.$$

We also recall the following properties of the operator $A = -\Delta_p$ (see [16]).

Proposition 2.3. 1. *The mapping A is continuous, strictly monotone, bounded and of type S^+, that is, if $(u_n) \subset W_0^{1,p}(\Omega)$ such that*

$$u_n \xrightarrow{w} u \text{ and } \limsup\langle Au_n, u_n - u\rangle \leq 0,$$

then $u_n \to u$ strongly in $W_0^{1,p}(\Omega)$.

2. *If $u, v \in W_0^{1,p}(\Omega)$ and satisfying $\langle Au - Av, (u-v)^+\rangle \leq 0$ then $u \leq v$ a.e. in Ω. Here, $u^+ = max\{u, \theta\}$.*

Lemma 2.4. *Suppose that u is a solution of the problem* (2.1) *and $v \in K$. We have*

(i) $\Big\langle Au - z, (tu - v)^+\Big\rangle + \int\limits_\Omega g(x,u)(tu-v)^+ \leqslant 0, \ \forall t \geq 0;$

(ii) $\Big\langle Au - z, (tv - u)^+\Big\rangle + \int\limits_\Omega g(x,u)(tv-u)^+ \geq 0 \quad \text{if} \quad vg(x,u) \in L^1(\Omega);$

(iii) $\langle Au - z, v - u\rangle + \int\limits_\Omega g(x,u)(v-u) \geq 0 \quad \text{if} \quad vg(x,u) \in L^1(\Omega).$

Proof. We will prove that all assumptions in Theorem 2.2 hold true with $\mu = Au - z + g(x,u), h = -g(x,u)$ and a suitable u_0.

(i) Choosing $u_0 = tu - (tu - v)^+ = \min\{tu, v\}$, we then have

$$hu_0 = -g(x,u)\min\{tu, v\} \geq -g(x,u)tu \in L^1(\Omega).$$

It follows from Theorem 2.2 that

$$\langle Au - z, tu - (tu - v)^+\rangle + \int_\Omega g(x,u)[tu - (tu - v)^+] \geq 0. \tag{2.4}$$

Multiplying t with (2.2) and then inserting this into (2.4) we will get (i).

(ii) Choosing $u_0 = u + (tv - u)^+ = \max\{u, tv\}$ and setting $\Omega_1 = \{u \geq tv\}$ and $\Omega_2 = \{u < tv\}$, we have $hu_0 = -g(x,u)u$ in Ω_1 and $hu_0 = -g(x,u)tv$ in Ω_2. Therefore, $hu_0 = -g(x,u)\max\{u, tv\} \in L^1(\Omega)$. By Theorem 2.2 we have

$$\langle Au - z, u + (tv - u)^+\rangle + \int_\Omega g(x,u)[u + (tv - u)^+] \geq 0.$$

This along with (2.2) yields (ii).

(iii) The proof of (iii) can be done in the same manner by choosing $u_0 = v$ and we omit details. □

Lemma 2.5. *Suppose that $g : \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$ is a Caratheodory function and satisfies the following conditions*

(g1) $g(x,0) = 0$, and $g(x,u)$ is an increasing function with respect to u for a.e $x \in \Omega$;

(g2) there exist $a > 0, \beta < p^* - 1$ and $b \in L^{(p^*)'}(\Omega)$ so that $g(x,u) \leqslant au^\beta + b(x)$.

Then, for all $z \in K_1 = \{z \in W^{-1,p'}(\Omega) : \langle z, u\rangle \geq 0, \forall u \in K\}$, the problem

$$\begin{cases} u \in K, \\ \langle Au, v - u\rangle + \int_\Omega g(x,u)(v - u) \geq \langle z, v - u\rangle, \forall v \in K, \end{cases} \tag{2.5}$$

has a unique solution u, satisfying (2.2) *and*

$$\langle Au, v\rangle + \int_\Omega g(x,u)v \geq \langle z, v\rangle,\ \forall v \in K. \tag{2.6}$$

Moreover, the solutions u_1, u_2 of (2.5)*, corresponding to $z = z_1, z_2$, satisfy* (2.3).

Proof. It is easy to see that the function g extended for $u \in (-\infty, 0]$ by putting $g(x,u) = -g(x,-u)$ satisfies the conditions of Theorem 2.1. Therefore, the problem (2.1) has a unique solution. Since $|u|^\beta \in L^{\frac{p^*}{\beta}}(\Omega) \subset L^{(p^*)'}(\Omega)$ and condition (g2), it follows that $g(x,u) \in L^{(p^*)'}(\Omega)$. For $v \in K$ we have $vg(x,u) \in L^1(\Omega)$, hence by (iii) of Lemma 2.4, (2.5) holds true. Finally, from (2.2) and (2.5) we obtain (2.6) □

Lemma 2.6. *Let $P : K_1 \to K$ be a mapping which maps each $z \in K_1$ into $P(z) = u$, a unique solution of problem* (2.5)*. Then the following statements are true:*

(i) *P is increasing, that is $z_1 \leq z_2$ implies $P(z_1) \leq P(z_2)$. Here, $z_1 \leq z_2$ means that $\langle z_2 - z_1, u\rangle \geq 0, \forall u \in K$.*

(ii) *P is continuous and bounded that is if M is bounded then so is $P(M)$.*

(iii) *If $\delta > (p^*)'$ then $P : L^\delta(\Omega) \to W_0^{1,p}(\Omega)$ is compact.*

(iv) *If $z_n \to z \neq \theta$ and $t_n \to \infty$ then $\|P(t_n z_n)\| \to \infty$.*

Proof. (i) Let $z_1 \leqslant z_2$ and $u_i = P(z_i), i = 1, 2$. By Lemma 2.4, we have

$$\left\langle Au_1 - z_1, (u_1 - u_2)^+\right\rangle + \int_\Omega g(x, u_1)(u_1 - u_2)^+ dx \leqslant 0,$$

and

$$\left\langle Au_2 - z_2, (u_1 - u_2)^+\right\rangle + \int_\Omega g(x, u_2)(u_1 - u_2)^+ dx \geq 0.$$

We then have

$$\left\langle Au_1 - Au_2, (u_1 - u_2)^+\right\rangle + \int_{\Omega_1} \Big[g(x, u_1) - g(x, u_2)\Big](u_1 - u_2) + \left\langle z_2 - z_1, (u_1 - u_2)^+\right\rangle \leqslant 0, \quad (2.7)$$

where $\Omega_1 = \{x : u_1(x) \geq u_2(x)\}$. Since the second and the third terms in (2.7) are nonnegative, we have $\left\langle Au_1 - Au_2, (u_1 - u_2)^+\right\rangle \leqslant 0$. Therefore, $(u_1 - u_2)^+ = 0$, or equivalently, $u_1 \leqslant u_2$ a.e in Ω.

(ii) Let M be a bounded set in $W^{-1,p'}(\Omega)$ and $z \in M, u = P(z)$. By (2.2) we have $\|u\|^p \leqslant \langle z, u\rangle \leqslant \|z\|.\|u\|$, which implies that $\|u\| \leqslant \|z\|^{\frac{1}{p-1}}$. Therefore, $P(M)$ is bounded.

We will show that P is continuous by proving that if $\lim_{n\to\infty} z_n = z$ then sequence $u_n = P(z_n)$ has a subsequence, which converges to $P(z)$. Indeed, the sequence $u_n = P(z_n)$ is bounded, $W_0^{1,p}(\Omega)$ is reflexive and the embedding $W_0^{1,p}(\Omega) \to L^\gamma(\Omega)$ is compact as $\gamma < p^*$, we may assume, without loss of generality, that

$$u_n \to u \in K \quad \text{weakly in} \quad W_0^{1,p}(\Omega), \quad (2.8)$$

$$u_n \to u \quad \text{in} \quad L^\gamma(\Omega) \quad \text{with} \quad \gamma < p^*. \quad (2.9)$$

By assumption (g2), the Nemytskii operator $N_g : u \longrightarrow g(x, u(x))$ is continuous from $L^{\beta(p^*)'}(\Omega)$ to $L^{(p^*)'}(\Omega)$ and $\beta(p^*)' < p^*$. Therefore, by (2.9), we have

$$\lim_{n\to\infty} g(\cdot, u_n) = g(\cdot, u) \text{ in } L^{(p^*)'}(\Omega) \text{ and in } W^{-1,p'}(\Omega). \quad (2.10)$$

It follows from

$$\langle Au_n, u_n - u\rangle \leqslant -\int_\Omega g(\cdot, u_n)(u_n - u) + \langle z_n, u_n - u\rangle,$$

and (2.8), (2.10) and the fact that $\{z_n\}$ is convergent that

$$\limsup\langle Au_n, u_n - u\rangle = 0.$$

Since A is a mapping of S^+ class , we obtain $u_n \to u$ in $W_0^{1,p}(\Omega)$. Letting $n \to \infty$ in

$$\langle Au_n, v - u_n\rangle + \int_\Omega g(\cdot, u_n)(v - u_n) \geqslant \langle z_n, v - u_n\rangle, \forall v \in K$$

we get (2.5). Therefore $u = P(z)$.

(iii) Since the embedding $W_0^{1,p}(\Omega) \to L^\gamma(\Omega)$ is compact as $\gamma < p^*$, the embedding $L^\delta(\Omega) \to W^{-1,p'}(\Omega)$ is compact as $\delta > (p^*)'$. Since $P : W^{-1,p'}(\Omega) \to W_0^{1,p}(\Omega)$ is continuous, we get that $P : L^\delta(\Omega) \to W_0^{1,p}(\Omega)$ is compact as $\delta > (p^*)'$.

(iv) If the sequence $u_n = P(z_n)$ is bounded, then we may assume that it satisfies (2.8), (2.10). It follows from (2.6) that

$$\frac{1}{t_n}\left[\langle Au_n, v\rangle + \int_\Omega g(x,u_n)\right] \geq \langle z_n, v\rangle,\ \forall v \in K,$$

which by letting $n \to \infty$ gives $0 \geq \langle z_0, v\rangle\ \forall v \in K$, a contradiction to $z_0 \in K_1 \setminus \{\theta\}$. $\square$

Lemma 2.7. *[9] Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}$ be a Caratheodory function satisfying*

$$|f(x,u,v)| \leqslant g(x,|u|,|v|),$$

where $g : \Omega \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is a Caratheodory function which is increasing with respect to the second and the third variables, and satisfies the following condition

$$u \in L^{p^*}(\Omega), v \in L^p(\Omega) \Longrightarrow g(x,u,v) \in L^\delta(\Omega).$$

Then, the Nemytskii operator $N_f : u \longmapsto f(x,u(x),\nabla u(x))$ from $W_0^{1,p}(\Omega)$ to $L^\delta(\Omega)$ is continuous.

Corollary 2.8. *The Nemytskii operator N_f from $W_0^{1,p}(\Omega)$ to $L^\delta(\Omega)$ is continuous and bounded in the following cases*

(i) $\delta = \min\left\{\frac{p}{\gamma}; \frac{qp^*}{q\alpha+p^*}\right\}$, $|f(x,u,v)| \leqslant m(x)|u|^\alpha + c|v|^\gamma$, *where* $\alpha < p^*-1, \gamma < \frac{p}{(p^*)'}$ *and* $m \in L^q(\Omega)$ *with* $q > (\frac{p^*}{1+\alpha})'$,

(ii) $\delta = \frac{pp^*q}{pp^*+\alpha pq+\gamma p^* q}$, $|f(x,u,v)| \leqslant m(x)|u|^\alpha\left(1+c|v|^\gamma\right)$, *where* $\alpha < p^*-1, \gamma < p\left(1-\frac{1+\alpha}{p^*}\right)$ *and* $m \in L^q(\Omega)$ *with* $q > \frac{p}{p\left(1-\frac{1+\alpha}{p^*}\right)-\gamma}$.

Proof. (i) If $u \in L^{p^*}(\Omega)$ and $v \in L^p(\Omega)$ then $m(x)|u|^\alpha \in L^{\frac{qp^*}{q\alpha+p^*}}(\Omega)$ and $|v|^\gamma \in L^{\frac{p}{\gamma}}(\Omega)$. Therefore, $m(x)|u|^\alpha + |v|^\gamma \in L^\delta(\Omega)$, where $\delta = \min\left\{\frac{p}{\gamma}; \frac{qp^*}{q\alpha+p^*}\right\}$.

(ii) If $u \in L^{p^*}(\Omega), v \in L^p(\Omega)$ then we get $|u|^\alpha \in L^{\frac{p^*}{\alpha}}(\Omega), 1+c|v|^\gamma \in L^{\frac{p}{\gamma}}(\Omega)$. Hence, $m(x)|u|^\alpha\left(1+c|v|^\gamma\right) \in L^\delta(\Omega)$, where $\frac{1}{\delta} = \frac{1}{q} + \frac{\alpha}{p^*} + \frac{\gamma}{p}$.
Therefore, by the Lemma 2.7, the operator N_f is continuous from $W_0^{1,p}(\Omega)$ into $L^\delta(\Omega)$.
To see the boundedness of N_f, we apply the Holder inequality and we get

$$||N_f(u)||_\delta \leq ||m||_q||u^\alpha||_{\frac{p^*}{\alpha}} + C|||\nabla u|^\gamma||_{\frac{p}{\gamma}} \leq ||m||_q||u||_{p^*}^\alpha + C||u||^\gamma$$

for the case (i) and

$$||N_f(u)||_\delta \leq ||m||_q|||u|^\alpha||_{\frac{p^*}{\alpha}}||1+C|\nabla u|^\gamma||_{\frac{p}{\gamma}} \leq C\,(1+||u||^{\alpha+\gamma})$$

for the case (ii). $\square$

Remark 2.9. If the condition (g1)-(g2) and one of the conditions (i) and (ii) of Lemma 2.8 are satisfied then the operator $P \circ N_f$ is completely continuous from $W_0^{1,p}(\Omega)$ to $W_0^{1,p}(\Omega)$. By definition of the operator P, the problem (1.1) is reduced to the fixed point problem $u = P \circ N_f(u)$.

We now recall some preliminaries on ordered spaces and the fixed point index.

Let E be a Banach space ordered by the cone $K \subset E$, that is, K is a closed convex subset such that $\lambda K \subset K$ for all $\lambda \geq 0$, $K \cap (-K) = \{\theta\}$ and ordering in E is defined by $x \leq y$ *iff* $y - x \in K$.

If D is a bounded relatively open subset of K and $F : \overline{D} \to K$ is a compact operator such that $F(u) \neq u, \forall u \in \partial D$, then the fixed point index $i(F,D,K)$ of F on D with respect to K is well-defined. This fixed point index admits all usual properties of the Leray - Schauder degree. In the sequence, we will use the following important results on computation of the index. See [5, 9].

Proposition 2.10. *Assume that D is a bounded relatively open subset of K and $F : \overline{D} \to K$ is a compact operator satisfying $F(u) \neq u, \forall u \in \partial D$. If there exits $u_0 \in K \setminus \{\theta\}$ such that*

$$u \neq F(u) + tu_0, \forall t > 0, \forall u \in \partial D,$$

then $i(F, D, K) = 0$.

Proposition 2.11. *Let (E, K) and (E_1, K_1) be the ordered Banach spaces and $N : K \to K_1$ be a continuous, bounded operator, $P : K_1 \to K$ be a compact operator, $P(\theta) = \theta$. Let $D \subset E$ be a bounded open subset containing θ.*

1. *If $u \neq P[tN(u)], \forall t \in [0, 1], \forall u \in \partial D \cap K$, then $i(P \circ N, D, K) = 1$.*
2. *Suppose that there exists $u_0 \in K_1 \setminus \{\theta\}$ such that*

$$u \neq P[N(u) + \lambda u_0],\ \forall \lambda \geq 0,\ \forall u \in K \cap \partial\Omega$$

and if $\{t_n\} \subset (0, \infty), \{z_n\} \subset K_1, t_n \to \infty, z_n \to u_0$ then $\lim_{n\to\infty} \|P(t_n z_n)\| = \infty$.
Then $i(P \circ N, D, K) = 0$.

Here, we use the notation $i(P \circ N, D, K)$ instead of $i(P \circ N, D \cap K, K)$.

3. Main results

In this section we shall order space $W_0^{1,p}(\Omega)$ by the cone K defined in (1.1).

Theorem 3.1. *Suppose that the Caratheodory function $g : \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$ satisfies (g1)-(g2). Also assume that the Caratheodory function $f : \Omega \times \mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{R}^+$ satisfies the following conditions*

(H0) There exist $\alpha < p - 1, m(x) \in L^q(\Omega), q > \left(\frac{p^*}{1+\alpha}\right)'$ such that $f(x, u, v) \leqslant m(x)u^\alpha + c|v|^\gamma$.

(H1) Either of two following conditions holds

a. $\gamma < p - 1$.

b. There exists a number $\beta_1 > 1$ such that $\gamma < \frac{\beta_1 - 1}{\beta_1} p, ug(x, u) \geqslant a|u|^{\beta_1}$.

(H2) There exist an open set $\Omega_0 \Subset \Omega$ and numbers $m_0 > 0, m_1 > 0, \varepsilon > 0, \alpha_1 < p - 1, \beta_2 > \alpha_1$ such that
$f(x, u, v) \geq m_0 u^{\alpha_1}$ and $g(x, u) \leqslant m_1 u^{\beta_2},\ \forall x \in \Omega_0, u \in [0; \varepsilon], v \in \mathbb{R}^N$.

Then, problem (1.1) has a positive solution.

Proof. We will proceed the proof of Theorem 3.1 in three steps.
Step 1. We show that there exists a number R large enough such that

$$u \neq P[tN_f u],\ \forall t \in [0; 1],\ \forall u \geqslant \theta,\ \|u\| = R.$$

Assume by contradiction that we can find $\{t_n\} \subset [0; 1]$, and $u_n \geqslant \theta, \|u_n\| \to \infty$ such that $u_n = P[tN_f u_n]$. By (2.2), we get

$$\langle Au_n, u_n\rangle + \int_\Omega g(x, u_n)u_n \leqslant \int_\Omega f(x, u_n, \nabla u_n)u_n,$$

which implies that

$$\|u_n\|^p + \int_\Omega g(x, u_n)u_n \leqslant \int_\Omega m(x)u_n^{1+\alpha} + c\int_\Omega |\nabla u_n|^\gamma u_n. \tag{3.1}$$

If condition a. in (H1) holds, by applying (3.1), Holder's and Young's inequalities and the fact that second term in the left hand is nonnegative, we obtain

$$\|u_n\|^p \leqslant C_1\|u_n\|^{1+\alpha}_{(1+\alpha)q'} + \varepsilon\|u_n\|^p + C(\varepsilon)\|u_n\|^t_t, \text{ where } t = (\frac{p}{\gamma})'. \tag{3.2}$$

It follows from $(1+\alpha)q' < p^*, t < p$ and (3.2) that

$$\|u_n\|^p \leqslant C\Big(\|u_n\|^{1+\alpha} + \|u_n\|^t\Big),$$

which contradicts to $\|u_n\| \to \infty, 1+\alpha < p, t < p$.
If condition b. in (H1) holds, by (3.1) we get

$$\|u_n\|^p + a\|u_n\|^{\beta_1}_{\beta_1} \leqslant \int_\Omega m(x)u_n^{1+\alpha} + C\int_\Omega |\nabla u_n|^\gamma u_n. \tag{3.3}$$

By Young's inequality, we have

$$|\nabla u_n|^\gamma u_n \leqslant C(\varepsilon)|\nabla u_n|^{\gamma(\beta_1)'} + \varepsilon u_n^{\beta_1},$$

which along with (3.3) and $\gamma(\beta_1)' < p$ implies that

$$\|u_n\|^p + a\|u_n\|^{\beta_1}_{\beta_1} \leqslant C\|u_n\|^{(1+\alpha)}_{(1+\alpha)q'} + \varepsilon\|u_n\|^{\beta_1}_{\beta_1} + C(\varepsilon)\|u_n\|^{\gamma(\beta_1)'}. \tag{3.4}$$

Then, we obtain

$$\|u_n\|^p + a\|u_n\|^{\beta_1}_{\beta_1} \leqslant C\Big(\|u_n\|^{1+\alpha} + \|u_n\|^{\gamma(\beta_1)'}\Big)$$

which is a contradiction to $(1+\alpha) < p, \gamma(\beta_1)' < p$ and $\|u_n\| \to \infty$.

Step 2. We show that there exists $r > 0$ small enough such that

$$u \neq P \circ N_f(u) + tu_0, \forall t > 0, \forall u \geq \theta, \|u\| = r, \tag{3.5}$$

where u_0 is given as follows. Let $\overline{u}$ be a positive eigenfunction corresponding to the eigenvalue of the problem

$$-\Delta_p u(x) = \lambda|u|^{p-2}.u \text{ in } \Omega_0,\ u(x) = 0 \text{ on } \partial\Omega_0.$$

We define u_0 as $u_0 = c\overline{u}$ in Ω_0 where $c > 0$ small enough and $u_0 = 0$ in $\Omega \setminus \Omega_0$.
We now prove that (3.5) holds. Assume by contradiction that we can find $t_n > 0, u_n \geq \theta, \|u_n\| \to 0$ such that

$$u_n = P_0 \circ N_f(u_n) + t_n u_0.$$

Let λ_n be a maximal number such that $u_n \geq \lambda_n u_0$. Since $u_n \geq t_n u_0$, we get $\lambda_n \geq t_n > 0$. Due to the fact that $u_n \to \theta$, we have $\lambda_n \to 0$.
We choose a number σ such that $1 > \sigma > \max\{\frac{\alpha}{p-1}; \frac{\alpha_1}{\beta_2}\}$ and we will show that for n large enough

$$P \circ N_f(u_n) \geq \lambda_n^\sigma u_0. \tag{3.6}$$

It was shown in [2] that

$$\langle Au_0, \varphi\rangle \leq \int_\Omega u_0^{\alpha_1}\varphi, \forall\varphi \in K. \tag{3.7}$$

Setting $v = P \circ N_f(u_n)$, it follows from (ii) of Lemma 2.4 that

$$\langle Av, (\lambda_n^\sigma u_0 - v)^+\rangle \geq \int_\Omega \Big[- g(x,v) + f(x,u_n,\nabla u_n)\Big](\lambda_n^\sigma u_0 - v)^+. \tag{3.8}$$

Taking $\varphi = (\lambda_n^\sigma u_0 - v)^+$ in (3.7) and then multiplying (3.7) with $\lambda_n^{\sigma(p-1)}$ and finally taking the difference of (3.7) and (3.8), we get

$$\langle A(\lambda_n^\sigma u_0) - Av, (\lambda_n^\sigma u_0 - v)^+\rangle \leq \int_{\Omega_1} \left[\lambda_n^{\sigma(p-1)} u_0^{\alpha_1} + g(x,v) - f(x,u_n,\nabla u_n)\right](\lambda_n^\sigma u_0 - v), \tag{3.9}$$

where $\Omega_1 = \{t^\sigma u_0 \geq v\}$. Setting $h = \left[\lambda_n^{\sigma(p-1)} u_0^{\alpha_1} + g(x,v) - f(x,u_n,\nabla u_n)\right](\lambda_n^\sigma u_0 - v)$, we have $h = 0$ in $\Omega_1 \setminus \Omega_0$. In $\Omega_1 \cap \Omega_0$, we have

$$\begin{aligned} h &\leq [\lambda_n^{\sigma(p-1)} u_0 - m_0(\lambda_n u_0)^{\alpha_1} + m_1(\lambda_n^\sigma u_0)^{\beta_2}](\lambda_n^\sigma u_0 - v) \\ &= (\lambda_n u_0)^{\alpha_1}[\lambda_n^{\sigma(p-1)-\alpha_1} - m_0 + m_1\lambda_n^{\sigma\beta_2-\alpha_1} u_0^{\beta_2-\alpha_1}](\lambda_n^\sigma u_0 - v). \end{aligned}$$

Since u_0 is bounded and $\lambda_n \to 0$, $h \leq 0$ as n is sufficiently large. Therefore, from (3.9) we obtain

$$\langle A(\lambda_n^\sigma u_0) - Av, (\lambda_n^\sigma u_0 - v)^+\rangle \leq 0,$$

which proves $(\lambda_n^\sigma u_0 - v)^+ \leq 0$ or $\lambda_n^\sigma u_0 \leq v$ provided that n is sufficiently large.

From (3.6), we obtain $u_n \geq P \circ N_f(u_n) \geq \lambda_n^\sigma u_0$. Therefore, by the choice of λ_n we get $\lambda_n^\sigma \leq \lambda_n$, which contradicts to $\sigma < 1, \lambda_n \to 0$.

Step 3. It follows from Step 1 and Step 2 and Propositions 2.10, 2.11 that

$$i(P \circ N_f, B(\theta, R), K) = 1, \text{ as } R \text{ is large enough},$$

and

$$i(P \circ N_f, B(\theta, r), K) = 0, \text{ as } r \text{ is small enough}.$$

Therefore, there exists $u \geq \theta$ such that $r \leq \|u\| \leq R$ and $u = P \circ N_f(u)$. This means that the problem (1.1) has a positive solution. □

Theorem 3.2. *Suppose that $N-1 < p < N$, $f : \Omega \times \mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{R}^+$, $g : \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$ are Caratheodory functions, satisfying condition (g1) and the following conditions*

(H3) a. $ug(x,u) \geq au^{\beta_2} - b(x)$ *with* $p < \beta_2 < p^*$, $b \in L^1_+(\Omega)$,
b. $g(x,u) \leq au^{\beta_2 - 1}$.

(H4) $f(x,u,v) \leq m(x)u^{p-1} + C|v|^\gamma, \forall u \geq 0, \forall v \in \mathbb{R}^N$ *with* $p - 1 \leq \gamma \leq \frac{\beta_2 - 1}{\beta_2} p$ *and* $m \in L^q_+(\Omega), q > \left(\frac{p^*}{p}\right)'$.

(H5) There exists a function $m_1 \in L^r_+(\Omega), m_1(x) \not\equiv 0, r > \frac{Np}{(p-1)(p+1-N)}$ *such that*

a. For all positive sequences $t_n \to 0, u_n \to u$, *and any bounded sequence* $\{v_n\} \subset \mathbb{R}^N$, *we have*

$$\lim_{n\to\infty} \frac{f(x, t_n u_n, t_n v_n)}{t_n^{p-1}} = m_1(x).u^{p-1}.$$

b. The principal eigenvalue λ_0 *of the problem*

$$\begin{cases} -\Delta_p u = \lambda m_1(x)|u|^{p-2}u \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

satisfies $\lambda_0 < 1$.

Then, problem (1.1) *has a positive solution.*

Proof. We split the proof of Theorem 3.2 into three steps.
Step 1. We show that there exists a sufficiently large number R such that

$$u \neq P[tN_f u], \ \forall t \in [0,1], \ \forall u \geqslant \theta, \ \|u\| = R.$$

Assume by contradiction that we can find $\{t_n\} \subset [0,1]$, and $u_n \geqslant \theta, \|u_n\| \to \infty$ such that $u_n = P[t_n N_f u_n]$. By (2.2), we get

$$\langle Au_n, u_n\rangle + \int_\Omega g(x,u_n)u_n \leqslant \int_\Omega f(x,u_n,\nabla u_n)u_n.$$

It follows from (H3)-(H4) that

$$\|u_n\|^p + a\|u_n\|_{\beta_2}^{\beta_2} \leq C_1 + \int_\Omega m u_n^p + C|\nabla u_n|^\gamma u_n. \tag{3.10}$$

By Holder's inequality, Young's inequality and $\gamma\beta_2' < p$ we get

$$\int_\Omega m.u_n^p \leq \|m\|_q \|u_n^p\|_{q'} = C_2\|u_n\|_{pq'}^p, \tag{3.11}$$

and

$$\int_\Omega |\nabla u_n|^\gamma u_n \leq \varepsilon\|u_n\|_{\beta_2}^{\beta_2} + C(\varepsilon)\||\nabla u_n|^\gamma\|_{\beta_2'}^{\beta_2'} \leq \varepsilon\|u_n\|_{\beta_2}^{\beta_2} + C(\varepsilon)\|u_n\|^{\gamma\beta_2'}. \tag{3.12}$$

From (3.10), (3.12) and $\gamma\beta_2' < p$ we have

$$\|u_n\|^p + \|u_n\|_{\beta_2}^{\beta_2} \leq C\Big(1 + \|u_n\|_{pq'}^p\Big). \tag{3.13}$$

We consider two cases.
Case 1. If $pq' \leq \beta_2$ then it follows from (3.13) that

$$\|u_n\|^p + \|u_n\|_{\beta_2}^{\beta_2} \leq C\Big(1 + \|u_n\|_{\beta_2}^p\Big).$$

which implies $\|u_n\|_{\beta_2} \to \infty$, a contradiction to $p < \beta_2$.
Case 2. Suppose that $pq' > \beta_2$. It follows from the assumption (H4) that $pq' < p^*$. By the interpolation inequality, we obtain

$$\|u_n\|_{pq'} \leq C\|u_n\|_{p^*}^\sigma.\|u_n\|_{\beta_2}^{1-\sigma} \leq C\|u_n\|^\sigma\|u_n\|_{\beta_2}^{1-\sigma}, \tag{3.14}$$

where $\sigma \in (0;1)$ satisfying

$$\frac{1}{\beta_2} - \frac{1}{pq'} = \sigma\Big(\frac{1}{\beta_2} - \frac{1}{p^*}\Big).$$

From (3.13), we obtain, for n be large enough,

$$\|u_n\| \leq C\|u_n\|_{pq'} \text{ and } \|u_n\|_{\beta_2} \leq C\|u_n\|_{pq'}^{\frac{p}{\beta_2}}. \tag{3.15}$$

From (3.14) and (3.15) we have

$$\|u_n\|_{pq'} \leq C\|u_n\|_{pq'}^{\sigma + (1-\sigma)\frac{p}{\beta_2}},$$

which is a contradiction to that $\|u_n\|_{pq'} \to \infty$ and $\sigma + (1-\sigma)\frac{p}{\beta_2} < 1$.
Step 2. We show that there exists a sufficiently small number r such that

$$u \neq P\Big[N_f(u) + tu_0\Big], \ \forall t \geq 0, \ \forall u \geq \theta, \ \|u\| = r,$$

where $u_0 \geq \theta, u_0 \neq \theta$. Assume by contradiction that we can find $t_n \geq 0, u_n \geq \theta, \|u_n\| \to 0$ such that

$$u_n = P[N_f(u_n) + t_n u_0].$$

We have

$$\langle Au_n, \varphi - u_n\rangle \geq \int_\Omega [f(x, u_n, \nabla u_n) - g(x, u_n) + t_n u_0](\varphi - u_n), \forall\varphi \in K, \tag{3.16}$$

and

$$\langle Au_n, u_n\rangle = \int_\Omega [f(x, u_n, \nabla u_n) - g(x, u_n) + t_n u_0]u_n. \tag{3.17}$$

From (3.16) and (3.17) we obtain

$$\langle Au_n, \varphi\rangle \geq \int_\Omega [f(x, u_n, \nabla u_n) - g(x, u_n) + t_n u_0]\varphi, \forall\varphi \in K. \tag{3.18}$$

Taking $z_n = \frac{u_n}{\|u_n\|}$, then $\{z_n\}$ is a bounded consequence. Since $W_0^{1,p}(\Omega)$ is a reflexive space, we may assume, without loss of generality, that $z_n \to z$ weakly in $W_0^{1,p}(\Omega)$ and $z_n \to z$ in $L^{\delta'}(\Omega)$, where δ is defined in Corollary 2.8 (i) and with $\alpha = p - 1$. By (H3)-(H4), we get

$$0 \leq \frac{f(x, u_n, \nabla u_n)}{\|u_n\|^{p-1}} \leq m(x)z_n^{p-1} + c\|u_n\|^{\gamma-p+1}|\nabla z_n|^\gamma,$$

and

$$0 \leq \frac{g(x, u_n)}{\|u_n\|^{p-1}} \leq az_n^{\beta_2-1}\|u_n\|^{\beta_2-p}.$$

Since the mapping $z \mapsto m(x)z^{p-1} + c|\nabla z|^\gamma, z \mapsto az^{\beta_2-1}$ maps bounded sets in $W_0^{1,p}(\Omega)$ into bounded sets in $L^\delta(\Omega)$, the sequence $\left\{\int_\Omega [f(x, u_n, \nabla u_n) - g(x, u_n)]\frac{\varphi}{\|u_n\|^{p-1}}\right\}$ is bounded. Moreover, from the fact that the mapping A is bounded and (3.17), the sequence $\left\{\frac{t_n}{\|u_n\|^{p-1}}\right\}$ is bounded; and hence we may assume $\frac{t_n}{\|u_n\|^{p-1}} \to t_0 \geq 0$.
It follows from (3.16) that

$$\langle Az_n, z_n - z\rangle \leq \int_\Omega [f(x, u_n, \nabla u_n) - g(x, u_n) + t_n u_0]\frac{(z_n - z)}{\|u_n\|^{p-1}}$$

Since the sequence $\left\{[f(x, u_n, \nabla u_n) - g(x, u_n) + t_n u_1]\frac{1}{\|u_n\|^{p-1}}\right\}$ is bounded in $L^\delta(\Omega)$ and by $\lim_{n\to\infty}(z_n - z) = 0$ in $L^{\delta'}(\Omega)$, we get $\limsup\langle Az_n, z_n - z\rangle \leq 0$.
Since A is a mapping of the type S^+, we have $z_n \to z \neq \theta$. Since $z_n \to z$ in $L^{p^*}(\Omega)$, $\nabla z_n \to \nabla z$ in $L^p(\Omega)$, we may assume $z_n \to z, \nabla z_n \to \nabla z$ and $|z_n| \leq z_0 \in L^{p^*}(\Omega), |\nabla z_n| \leq v_0 \in L^p(\Omega)$ a.e in Ω. This in combination with (H4)-H(5) implies that

$$\begin{aligned}\lim_{n\to\infty}\frac{f(x, u_n, \nabla u_n)\varphi}{\|u_n\|^{p-1}} &= \lim_{n\to\infty}\frac{f(x, \|u_n\|z_n, \|u_n\|\nabla z_n)\varphi}{\|u_n\|^{p-1}}\\ &= m_1(x)z^{p-1}\varphi \text{ a.e in } \Omega,\end{aligned}$$

and

$$\begin{aligned}\frac{|f(x, u_n, \nabla u_n)|}{\|u_n\|^{p-1}} &\leq m(x)z_n^{p-1} + c\|u_n\|^{\gamma-p+1}|\nabla z_n|^\gamma\\ &\leq m(x)z_0^{p-1} + cv_0^{p-1} \in L^\delta(\Omega) \subset L^{(p^*)'}(\Omega).\end{aligned}$$

By the Lebesgue's Dominated Convergence Theorem, we obtain

$$\lim_{n\to\infty}\int_\Omega \frac{f(x,u_n,\nabla u_n)\varphi}{\|u_n\|^{p-1}} = \int_\Omega m_1(x)z^{p-1}\varphi.$$

Similarly

$$\lim_{n\to\infty}\int_\Omega \frac{g(x,u_n)\varphi}{\|u_n\|^{p-1}} = 0.$$

Therefore, passing to the limit in (3.18), we get

$$\langle Az,\varphi\rangle \geq \int_\Omega (m_1(x)z^{p-1}+t_0u_0)\varphi, \forall\varphi\in K, \tag{3.19}$$

which proves

$$z \geq (-\Delta_p)^{-1}(m_1(x)z^{p-1}) := w. \tag{3.20}$$

Since $m(x)z^{p-1}\in L^s(\Omega)$ with $s=\frac{rp^*}{r(p-1)+p^*} > \frac{Np}{p-1}$, and the result of [4] we have $w\in intC_+$, where $C_+=\{u\in C_0^1(\overline{\Omega}), u(x)\geq 0\}$. Let φ_0 be a positive eigenfunction corresponding to the principal eigenvalue λ_0 and $s>0$ be a maximal number satisfying $z\geq s\varphi_0$. Then we get from (3.20) that

$$z \geq (\Delta_p)^{-1}(m_1(x)s^{p-1}\varphi_0^{p-1}) = \frac{s}{\lambda_0^{\frac{1}{p-1}}}\varphi_0.$$

Since $\lambda_0<1$, $\frac{s}{\lambda_0^{\frac{1}{p-1}}} > s$, which contradicts to the choice of s.

Step 3. Using the argument that used in Step 3 in proof of Theorem 3.1, we conclude that the problem (1.1) has a positive solution. $\square$

Theorem 3.3. *Assume that $N-1<p<N$ and the conditions (g1), and (H3), (H5) in Theorem* 3.2 *hold true, and the condition (H4) is replaced by the following condition*

(H4') $f(x,u,v)\leq m(x)u^{p-1}(1+c|v|^\gamma)$ with $\gamma<p(1-\frac{p}{\beta_2}), m(x)\in L^q(\Omega), q>\frac{p^*p}{p^*(p-\gamma)-p^2}$.

Then, problem (1.1) *has a positive solution.*

Proof. The proof is similar to that of Theorem 3.2 with the minor difference in Step 1. We now sketch it here.

We will prove that there exists a sufficiently large number R so that

$$u\neq P[tN_f(u)],\ \forall t\in[0;1],\ \forall u\geq\theta,\ \|u\|=R.$$

Assume by contradiction that we can find sequences $\{t_n\},\{u_n\}$ such that $t_n\in[0,1], u_n\geq\theta, \|u_n\|\to\infty$ satisfying $u_n=P[t_nN_f(u_n)]$. By (2.2), we get

$$\langle Au_n,u_n\rangle + \int_\Omega g(x,u_n)u_n \leqslant \int_\Omega f(x,u_n,\nabla u_n)u_n.$$

By (H3) and (H4′), we obtain

$$
\begin{aligned}
\|u_n\|^p + a\|u_n\|_{\beta_2}^{\beta_2} &\le \int_\Omega b(x) + \int_\Omega m(x)u_n^p(1 + c|\nabla u_n|^\gamma) \\
&\le \|m\| \left\{ \int_\Omega u_n^{pq'}(1 + |\nabla u_n|^\gamma)^{q'} \right\}^{\frac{1}{q'}} + C \\
&\le C\|u_n^{pq'}\|_t^{\frac{1}{q'}} \|(1 + |\nabla u_n|^\gamma)^{q'}\|_s^{\frac{1}{q'}} \\
&\le C(1 + \|u_n\|^\gamma)\|u_n\|_{pq't}^p \\
&\le C\|u_n\|^\gamma \|u_n\|_{pq't}^p,
\end{aligned} \tag{3.21}
$$

where $s = \frac{p}{\gamma q'}, t = s' = (\frac{p}{\gamma q'})'$. We consider two cases.
Case 1. If $pq't \le \beta_2$ then by (3.21) we get

$$
\|u_n\|^p + \|u_n\|_{\beta_2}^{\beta_2} \le C\|u_n\|^\gamma \|u_n\|_{\beta_2}^p, \tag{3.22}
$$

which implies that

$$
\|u_n\|_{\beta_2} \le C\|u_n\|^{\frac{\gamma}{\beta_2 - p}},
$$

and that

$$
\|u_n\|^p \le C\|u_n\|^{\gamma + \frac{\gamma p}{\beta_2 - p}}.
$$

This contradicts to that $p > \gamma + \frac{\gamma p}{\beta_2 - p}$ and $\|u_n\| \to \infty$.

Case 2. If $\beta_2 < pq't$ then we have $\beta_2 < pq't < p^*$. By interpolation inequality, we get

$$
\|u_n\|_{pq't} \le \|u_n\|_{p^*}^\sigma \|u_n\|_{\beta_2}^{1-\sigma} \le \|u_n\|^\sigma \|u_n\|_{\beta_2}^{1-\sigma}, \tag{3.23}
$$

where σ is defined by $\frac{1}{\beta_2} - \frac{1}{pq't} = \sigma(\frac{1}{\beta_2} - \frac{1}{p^*})$. It follows from (3.21) that

$$
\|u_n\| \le C\|u_n\|_{pq't}^{\frac{p}{p-\gamma}} \text{ and } \|u_n\|_{\beta_2} \le \|u_n\|_{pq't}^{\frac{p^2}{\beta_2(p-\gamma)}}.
$$

This in combination with (3.23) gives

$$
\|u_n\|_{pq't} \le \|u_n\|_{pq't}^\lambda,
$$

where $\lambda = \sigma\frac{p}{p-\gamma} + (1 - \sigma)\frac{p^2}{\beta_2(p-\gamma)} < 1$, a contradiction to that $\|u_n\|_{pq't} \to \infty$.

At this stage, by the argument used in Step 2 and Step 3 in the proof of Theorem 3.2, we complete the proof of the theorem. □

4. Example and Remark

1. Let $f(x, u, v) = mu^\alpha + c|v|^\gamma, (x, u, v) \in \Omega \times [0, \infty) \times \mathbb{R}^N$, where $0 < \alpha < p - 1$ and m, c, γ are positive numbers. Conditions of Theorem 3.1 hold in the following cases
   a. $g(x, u) = \ln(1 + u^\beta)$ or $g(x, u) = u^\beta$ and $\gamma < p - 1, \alpha < \beta < p^* - 1$.
   b. $g(x, u) = u^\beta$ and $\gamma < \frac{\beta}{\beta+1}$, $\alpha < \beta < p^* - 1$.

2. Let $N-1 < p < N$, $g(x,u) = u^{\beta}$ with $p-1 < \beta < p^* - 1$ and λ_* be the principal eigenvalue of the problem

$$-\Delta_p u = \lambda |u|^{p-2} u \text{ in } \Omega, u = 0 \text{ on } \partial\Omega.$$

   a. Conditions of Theorem 3.2 are satisfied for the function $f(x,u,v) = mu^{p-1} + c|v|^{\gamma}$ with $p-1 < \gamma < \frac{\beta}{\beta+1}$ and $m > \lambda_*$.
   b. Conditions of Theorem 3.3 hold for the function $f(x,u,v) = mu^{p-1}(1+c|v|^{\gamma})$ with $0 < \gamma < p\left(1 - \frac{p}{\beta+1}\right)$ and $m > \lambda_*$.

Note that, these functions do not satisfy the following condition which has been proposed in the literature [14, 15]

$$\lim_{u\to 0} \frac{f(x,u,v)}{u^{p-1}} = m(x)$$

uniformly with respect to $x \in \Omega$ and to v in each bounded subset of $\mathbb{R}^N$.

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