Filomat 36:12 (2022), 4023–4040
https://doi.org/10.2298/FIL2212023K

Published by Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Renormalized Self-Intersection Local Time for Sub-Bifractional Brownian Motion

Nenghui Kuang[a], Bingquan Liu[b]

[a]*School of Mathematics and Computing Science, Hunan University of Science and Technology, Xiangtan, Hunan 411201, P.R. China*
[b]*School of Mathematics and Information Science, Weinan Normal University, Weinan, 714000, P.R. China*

Abstract. Let $S^{H,K} = \{S^{H,K}(t), t \geq 0\}$ be a $d-$dimensional sub-bifractional Brownian motion with indices $H \in (0,1)$ and $K \in (0,1]$. Assuming $d \geq 2$, as $HKd < 1$, we mainly prove that the renormalized self-intersection local time

$$\int_0^t \int_0^s \delta(S^{H,K}(s) - S^{H,K}(r))drds - \mathbf{E}\left[\int_0^t \int_0^s \delta(S^{H,K}(s) - S^{H,K}(r))drds\right]$$

exists in L^2, where $\delta(x)$ is the Dirac delta function for $x \in \mathbf{R}^d$.

1. Introduction and main results

Recently, El-Nouty and Journé (2013) introduced the process $S_0^{H,K} = \{S_0^{H,K}(t), t \geq 0\}$ with indices $H \in (0,1)$ and $K \in (0,1]$, named the sub-bifractional Brownian motion and defined as follows:

$$S_0^{H,K}(t) = \frac{1}{2^{(2-K)/2}}(B^{H,K}(t) + B^{H,K}(-t)),$$

where $\{B^{H,K}(t), t \in \mathbf{R}\}$ is a two-sided bifractional Brownian motion with indices $H \in (0,1)$ and $K \in (0,1]$, namely, $\{B^{H,K}(t), t \in \mathbf{R}\}$ is a centered Gaussian process, starting from zero, with covariance

$$\mathbf{E}[B_t^{H,K} B_s^{H,K}] = \frac{1}{2^K}\left[(|t|^{2H} + |s|^{2H})^K - |t-s|^{2HK}\right],$$

with $H \in (0,1)$ and $K \in (0,1]$.

Clearly, the sub-bifractional Brownian motion is a centered Gaussian process such that $S_0^{H,K}(0) = 0$, with probability 1, and $\mathrm{Var}[S_0^{H,K}(t)] = (2^K - 2^{2HK-1})t^{2HK}$. Note that since $(2H-1)K - 1 < K - 1 \leq 0$, it follows that $2HK - 1 < K$. We can easily verify that $S_0^{H,K}$ is self-similar with index HK. When $K = 1$, $S_0^{H,1}$ is the sub-fractional Brownian motion. For more on sub-fractional Brownian motion, we can see Kuang and Xie (2015,2017), Kuang and Liu (2015,2018) and so on. Straightforward computations show that for all $s, t \geq 0$,

$$\mathbf{E}[S_0^{H,K}(t)S_0^{H,K}(s)] = (t^{2H} + s^{2H})^K - \frac{1}{2}(t+s)^{2HK} - \frac{1}{2}|t-s|^{2HK}, \tag{1.1}$$

2020 *Mathematics Subject Classification*. 60G22; 60J55
Keywords. Sub-bifractional Brownian motion; self-intersection local time; renormalization
Received: 13 February 2021; Revised: 17 February 2022; Accepted: 11 March 2022
Communicated by Miljana Jovanović
Research supported by the Natural Science Foundation of Hunan Province under Grant 2021JJ30233 and by the Natural Science Foundation of Hunan University of Science and Technology under Grant E54018.
Email addresses: knh1552@163.com (Nenghui Kuang), 215733251@qq.com (Bingquan Liu)

and

$$C_1|t-s|^{2HK} \leq \mathbf{E}[(S_0^{H,K}(t) - S_0^{H,K}(s))^2] \leq C_2|t-s|^{2HK}, \tag{1.2}$$

where

$$C_1 = \min\{2^K - 1, 2^K - 2^{2HK-1}\}, \quad C_2 = \max\{1, 2 - 2^{2HK-1}\}. \tag{1.3}$$

(See El-Nouty and Journé (2013)). Kuang (2019) investigated the collision local time of two independent sub-bifractional Brownian motions. Kuang and Li (2022) obtained Berry-Esséen bounds and proved the almost sure central limit theorem for the quadratic variation of the sub-bifractional Brownian motion. For more on the sub-bifractional Brownian motion, we can see Kuang and Xie (2022) and Xie and Kuang (2022).

A $d-$dimensional sub-bifractional Brownian motion $S^{H,K} = \{S^{H,K}(t), t \geq 0\}$ is defined by

$$S^{H,K}(t) = \left(S_1^{H,K}(t), S_2^{H,K}(t), \cdots, S_d^{H,K}(t)\right),$$

where $S_1^{H,K}, S_2^{H,K}, \cdots, S_d^{H,K}$ are independent copies of $S_0^{H,K}$.

The self-intersection local time, as an important topic of probability theory, has been widely considered. For example, the self-intersection local time of the Brownian motion has been studied by many authors (see Albeverio et al. (1997), He et al. (1995) and Hu (1996)). In the case of fractional Brownian motion, the reader can refer to Hu (2001), Jung and Markowsky (2014, 2015). For the case of bifractional Brownian motion, Jiang and Wang (2009) considered self-intersection local time and collision local time of bifractional Brownian motion.

Varadhan (1969) studied the renormalized self-intersection local time of the planar Brownian motion. This result has been extended by Rosen (1987) to the (planar) fractional Brownian motion. Hu and Nualart (2005) extended the result to $d-$dimensional fractional Brownian motion. Chen et al. (2018) studied renormalized self-intersection local time of bifractional Brownian motion. But there exists the same mistake in Hu and Nualart (2005) and Chen et al. (2018), namely, (56) in Hu and Nualart (2005) and (3.18) in Chen et al. (2018) are wrong (see Remark 1 below). We will use the different method to study the renormalized self-intersection local time of a $d-$dimensional sub-bifractional Brownian motion.

In this paper, we investigate the local time and the renormalized self-intersection local time of a $d-$dimensional sub-bifractional Brownian motion $S^{H,K}$. They are defined respectively as follows: for $t > 0$, the local time

$$l_t^{H,K}(x) := \int_0^t \delta\left(S^{H,K}(s) - x\right) ds, \tag{1.4}$$

and the self-intersection local time

$$\alpha_t := \int_D \delta\left(S^{H,K}(s) - S^{H,K}(r)\right) drds, \tag{1.5}$$

where $D = \{(r,s) : 0 < r < s < t\}$ and $\delta(x)$ is the Dirac delta function for $x \in \mathbf{R}^d$, and

$$\delta(x) = \lim_{\epsilon \to 0} p_\epsilon(x) = (2\pi)^{-d} \int_{\mathbf{R}^d} \exp\{i\langle \xi, x\rangle\} d\xi,$$

and

$$p_\epsilon(x) := (2\pi\epsilon)^{-\frac{d}{2}} \exp\left\{-\frac{|x|^2}{2\epsilon}\right\} = (2\pi)^{-d} \int_{\mathbf{R}^d} \exp\left\{i\langle \xi, x\rangle - \frac{\epsilon|\xi|^2}{2}\right\} d\xi. \tag{1.6}$$

The approximated self-intersection local time of sub-bifractional Brownian motion is defined by

$$\alpha_{t,\epsilon} := \int_D p_\epsilon\left(S^{H,K}(s) - S^{H,K}(r)\right) drds. \tag{1.7}$$

Chen et al. (2015) obtained sufficient and necessary conditions for the existence of the local times, collision local times, and self-intersection local times for anisotropic Gaussian random fields. However a sharp upper bound of second moment of the local time for anisotropic Gaussian random fields is not obtained.

Now we state our main results as follows.

Theorem 1.1. Assuming $HKd < 1$, we have, for any $x \in \mathbf{R}^d$,

$$\mathbf{E}\left[\left|l_t^{H,K}(x)\right|^2\right] \le \frac{2\Gamma^2(1-HKd)}{k^d(2\pi)^d\Gamma(3-2HKd)}t^{2-2HKd}, \tag{1.8}$$

where $l_t^{H,K}(x)$ is given by (1.4), $\Gamma(\alpha)$ is a Gamma function defined by $\Gamma(\alpha) := \int_0^\infty t^{\alpha-1}e^{-t}dt$, and k is a constant depending on H and K.

Theorem 1.2. Let $S^{H,K} = \{S^{H,K}(t), t \ge 0\}$ be a d−dimensional sub-bifractional Brownian motion with indices $H \in (0,1)$ and $K \in (0,1]$. Assuming $d \ge 2$ and $HKd < 1$, we have, the renormalized self-intersection local time $\alpha_{t,\epsilon} - \mathbf{E}[\alpha_{t,\epsilon}]$ converges in L^2 as $\epsilon \to 0$, where $\alpha_{t,\epsilon}$ is given by (1.7).

In what follows, we will use k to denote unspecified positive and finite constants whose value may be different in each occurrence.

2. Some useful lemmas

In this section, we give some useful lemmas in order to prove the Theorems 1.1-1.2.

Lemma 2.1. For all constants $0 < a < b$, $S_0^{H,K}$ is strongly locally φ-nondeterministic on $I = [a,b]$ with $\varphi(r) = r^{2HK}$. That is, there exist positive constants c_1 and r_0 such that for all $t \in I$ and all $0 < r \le \min\{t, r_0\}$,

$$\mathrm{Var}\{S_0^{H,K}(t)|S_0^{H,K}(s) : s \in I, r \le |s-t| \le r_0\} \ge c_1\varphi(r). \tag{2.1}$$

Proof. See Kuang (2019).

From the local nondeterminism (see Berman(1973), Xiao(2007)), we have the following property: if $0 \le t_1 < t_2 < \cdots < t_n < t$, then there is a constant $k > 0$ such that

$$\mathrm{Var}\left[\sum_{i=2}^n u_i(S_0^{H,K}(t_i) - S_0^{H,K}(t_{i-1}))\right] \ge k\sum_{i=2}^n u_i^2|t_i - t_{i-1}|^{2HK}, \tag{2.2}$$

for any $u_i \in \mathbf{R}, i = 2, 3, \dots, n$.

Lemma 2.2. Let

$$\lambda := \mathrm{Var}\left[S_0^{H,K}(s) - S_0^{H,K}(r)\right], \rho := \mathrm{Var}\left[S_0^{H,K}(s') - S_0^{H,K}(r')\right],$$

and

$$\mu := \mathrm{Cov}\left(S_0^{H,K}(s) - S_0^{H,K}(r), S_0^{H,K}(s') - S_0^{H,K}(r')\right).$$

Case 1: If $(r, s, r', s') \in D_1 := \{(r, s, r', s')|0 < r < r' < s < s' < t\}$, denoting $a = r' - r, b = s - r', c = s' - s$, then we have

$$\text{(1)}\quad C_1(a+b)^{2HK} \le \lambda = \lambda_1 \le C_2(a+b)^{2HK},\ \ C_1(b+c)^{2HK} \le \rho = \rho_1 \le C_2(b+c)^{2HK}, \tag{2.3}$$

where C_1 and C_2 are given by (1.3).

(2) There exists a positive constant k, such that

$$\lambda_1\rho_1 - \mu_1^2 \ge k\left[(a+b)^{2HK}c^{2HK} + (b+c)^{2HK}a^{2HK}\right], \tag{2.4}$$

where $\mu = \mu_1$.

(3) When $0 < 2HK < 1$, there exists a positive constant k, such that

$$\mu = \mu_1 \le k\left(a^{2HK} + b^{2HK} + c^{2HK}\right). \tag{2.5}$$

Case 2: If $(r,s,r',s') \in D_2 := \{(r,s,r',s')|0 < r < r' < s' < s < t\}$, denoting $a = r' - r, b = s' - r', c = s - s'$, then we have

$$(1) \quad C_1(a+b+c)^{2HK} \le \lambda = \lambda_2 \le C_2(a+b+c)^{2HK}, \;\; C_1 b^{2HK} \le \rho = \rho_2 \le C_2 b^{2HK}, \tag{2.6}$$

where C_1 and C_2 are given by (1.3).

(2) There exists a positive constant k, such that

$$\lambda_2\rho_2 - \mu_2^2 \ge kb^{2HK}\left(a^{2HK} + c^{2HK}\right), \tag{2.7}$$

where $\mu = \mu_2$.

(3) When $0 < 2HK < 1$, there exists a positive constant k, such that

$$\mu = \mu_2 \le kb^{2HK}. \tag{2.8}$$

Case 3: If $(r,s,r',s') \in D_3 := \{(r,s,r',s')|0 < r < s < r' < s' < t\}$, denoting $a = s - r, b = r' - s, c = s' - r'$, then we have

$$(1) \quad C_1 a^{2HK} \le \lambda = \lambda_3 \le C_2 a^{2HK}, \;\; C_1 c^{2HK} \le \rho = \rho_3 \le C_2 c^{2HK}, \tag{2.9}$$

where C_1 and C_2 are given by (1.3).

(2) There exists a positive constant k, such that

$$\lambda_3\rho_3 - \mu_3^2 \ge ka^{2HK}c^{2HK}, \tag{2.10}$$

where $\mu = \mu_3$.

(3) When $0 < 2HK < 1$, for $\alpha, \beta > 0$ with $\alpha + \beta = 1$, there exists a positive constant k, such that

$$\mu = \mu_3 \le kb^{2\alpha(HK-1)}(ac)^{\beta(HK-1)+1}. \tag{2.11}$$

Proof. The proof of this lemma is given in the Appendix since its proof is long.

Let $D^2 := \{(r,s,r',s')|0 < r < s < t, 0 < r' < s' < t\}$, then $D^2 \cap \{r < r'\} = D_1 \cup D_2 \cup D_3$, where D_1, D_2 and D_3 are given in Lemma 2.2. Let $\delta_i = \lambda_i\rho_i - \mu_i^2$, and $\Theta_i = \delta_i^{-\frac{d}{2}} - (\lambda_i\rho_i)^{-\frac{d}{2}}, i = 1,2,3$. Then we have the following three lemmas whose proofs are also given in the Appendix.

Lemma 2.3. If $(r,s,r',s') \in D_1$. Assume $d \ge 2$ and $HKd < \frac{3}{2}$, we have

$$\int_{D_1} \Theta_1 drdsdr'ds' < +\infty. \tag{2.12}$$

Lemma 2.4. If $(r,s,r',s') \in D_2$, we decompose the region $D_2 := I_1 \cup I_2 \cup I_3$, where $I_1 = \{b \ge \eta_1 a\}, I_2 = \{b \ge \eta_2 c\}, I_3 = \{b < \eta_1 a, b < \eta_2 c\}$ for some fixed but arbitrary $\eta_1 > 0$ and $\eta_2 > 0$. Assume $d \ge 2$, we have

(1) as $HKd < \frac{3}{2}$,

$$\int_{I_1 \cup I_2} \Theta_2 drdsdr'ds' < +\infty. \tag{2.13}$$

(2) as $HKd < 1$,

$$\int_{I_3} \Theta_2 drdsdr'ds' < +\infty. \tag{2.14}$$

Lemma 2.5. If $(r,s,r',s') \in D_3$, we decompose the region $D_3 := J_1 + J_2 + J_3 + J_4$, where $J_1 = \{a \ge \eta_1 b, c \ge \eta_2 b\}, J_2 = \{a < \eta_1 b, c < \eta_2 b\}, J_3 = \{a \ge \eta_1 b, c < \eta_2 b\}, J_4 = \{a < \eta_1 b, c \ge \eta_2 b\}$ for some fixed but arbitrary $\eta_1 > 0$ and $\eta_2 > 0$. Assume $d \ge 2$, we have

(1) as $HKd < \frac{3}{2}$,

$$\int_{J_1} \Theta_3 drdsdr'ds' < +\infty. \tag{2.15}$$

(2) as $HKd < 1$,

$$\int_{J_2+J_3+J_4} \Theta_3 dr ds dr^{'} ds^{'} < +\infty. \tag{2.16}$$

Remark 1. We first point out the mistake of (56) in Hu and Nualart (2005). They claimed that: as $d \geq 2$, there exists a constant $0 < k < 1$ such that

$$\Theta_i \leq k\mu_i^2(\lambda_i\rho_i)^{-\frac{d}{2}-1}, \tag{2.17}$$

for $i = 2, 3$. In fact, (2.17) does not hold. Because we can prove that, for any $0 \leq x \leq 1$ and $\alpha < 0$,

$$(1-x)^{\alpha} \geq 1 - \alpha x. \tag{2.18}$$

Let $f(x) = (1-x)^{\alpha} - 1 + \alpha x, x \in [0, 1]$. Then

$$f^{'}(x) = \alpha[1 - (1-x)^{\alpha-1}].$$

By $0 \leq x \leq 1$ and $\alpha < 0$, we have $(1-x)^{1-\alpha} < 1$. Thus

$$1 - (1-x)^{\alpha-1} = 1 - \frac{1}{(1-x)^{1-\alpha}} = \frac{(1-x)^{1-\alpha} - 1}{(1-x)^{1-\alpha}} < 0,$$

which implies $f(x)$ is increasing on $[0, 1]$. Hence we obtain $f(x) \geq f(0) = 0$, (2.18) holds.

We take $x = \frac{\mu_i^2}{\lambda_i\rho_i}$ and $\alpha = -\frac{d}{2} < 0$. By Cauchy-Schwartz' inequality, $\frac{\mu_i^2}{\lambda_i\rho_i} \leq 1$, which shows $x = \frac{\mu_i^2}{\lambda_i\rho_i} \in [0, 1]$. Therefore, by (2.18),

$$\left(1 - \frac{\mu_i^2}{\lambda_i\rho_i}\right)^{-\frac{d}{2}} \geq 1 + \frac{d}{2}\frac{\mu_i^2}{\lambda_i\rho_i}.$$

We obtain

$$\Theta_i = \left[\left(1 - \frac{\mu_i^2}{\lambda_i\rho_i}\right)^{-\frac{d}{2}} - 1\right](\lambda_i\rho_i)^{-\frac{d}{2}} \geq \frac{d}{2}\frac{\mu_i^2}{\lambda_i\rho_i}(\lambda_i\rho_i)^{-\frac{d}{2}} = \frac{d}{2}\mu_i^2(\lambda_i\rho_i)^{-\frac{d}{2}-1} \geq \mu_i^2(\lambda_i\rho_i)^{-\frac{d}{2}-1},$$

since $d \geq 2$. This implies (2.17) does not hold.

Secondly, Hu and Nualart (2005) used (56) to prove the results similar to (2.14) in Lemma 2.4 and (2.16) in Lemma 2.5. Here we use the different method to prove (2.14) and (2.16). Hence the condition for (2.14) or (2.16) is $HKd < 1$ but not $HKd < \frac{3}{2}$.

Thirdly, (3.18) in Chen et al. (2018) cited from (56) in Hu and Nualart (2005). Hence the obtained results in Hu and Nualart (2005) and in Chen et al. (2018) may be not correct.

3. Proofs of Theorems

In this section, we will prove Theorems 1.1-1.2.

Proof of Theorem 1.1. By (1.4) and (1.6), we get

$$\mathbf{E}\left[\left|l_t^{H,K}(x)\right|^2\right] = \mathbf{E}\left[\int_{[0,t]^2} \delta\left(S^{H,K}(s) - x\right)\delta\left(S^{H,K}(r) - x\right) dr ds\right]$$

$$= \frac{1}{(2\pi)^{2d}} \int_{[0,t]^2} \int_{\mathbf{R}^{2d}} \mathbf{E}\left\{\exp\left(i\left(\left\langle S^{H,K}(s) - x, \xi\right\rangle + \left\langle S^{H,K}(r) - x, \eta\right\rangle\right)\right)\right\} d\xi d\eta dr ds$$

$$= \frac{1}{(2\pi)^{2d}} \int_{[0,t]^2} \int_{\mathbf{R}^{2d}} \mathbf{E}\left\{\exp\left(i\sum_{m=1}^{d}\left(\left(S_m^{H,K}(s) - x_m\right)\xi_m + \left(S_m^{H,K}(r) - x_m\right)\eta_m\right)\right)\right\} d\xi d\eta dr ds$$

$$= \frac{1}{(2\pi)^{2d}} \int_{[0,t]^2} \int_{\mathbf{R}^{2d}} \exp\left\{-\frac{1}{2}\mathrm{Var}\left[\sum_{m=1}^{d}\left(\left(S_m^{H,K}(s) - x_m\right)\xi_m + \left(S_m^{H,K}(r) - x_m\right)\eta_m\right)\right]\right\} d\xi d\eta dr ds$$

$$= \frac{1}{(2\pi)^{2d}} \int_{[0,t]^2} \int_{\mathbf{R}^{2d}} \exp\left\{-\frac{1}{2}\mathrm{Var}\left[\sum_{m=1}^{d}\left(S_m^{H,K}(s)\xi_m + S_m^{H,K}(r)\eta_m\right)\right]\right\} d\xi d\eta dr ds$$

$$= \frac{2}{(2\pi)^{2d}} \int_{0\le r<s\le t} \int_{\mathbf{R}^{2d}} \prod_{m=1}^{d} \exp\left\{-\frac{1}{2}\mathrm{Var}\left[S_m^{H,K}(s)\xi_m + S_m^{H,K}(r)\eta_m\right]\right\} d\xi d\eta dr ds.$$

By (2.2), we have

$$\mathrm{Var}\left[S_m^{H,K}(s)\xi_m + S_m^{H,K}(r)\eta_m\right] = \mathrm{Var}\left[(S_m^{H,K}(s) - S_m^{H,K}(r))\xi_m + S_m^{H,K}(r)(\xi_m + \eta_m)\right]$$
$$\ge k\left(\xi_m^2|s-r|^{2HK} + (\xi_m+\eta_m)^2 r^{2HK}\right).$$

Hence,

$$\mathbf{E}\left[\left|l_t^{H,K}(x)\right|^2\right] \le \frac{2}{(2\pi)^{2d}} \int_{0\le r<s\le t} \int_{\mathbf{R}^{2d}} \prod_{m=1}^{d} \exp\left\{-\frac{k}{2}\left(\xi_m^2|s-r|^{2HK} + (\xi_m+\eta_m)^2 r^{2HK}\right)\right\} d\xi d\eta dr ds$$

$$= \frac{2}{(2\pi)^d k^d} \int_{0\le r<s\le t} \frac{1}{r^{HKd}(s-r)^{HKd}} dr ds \quad (\text{let } r = su)$$

$$= \frac{2}{(2\pi)^d k^d} \int_0^t s^{1-2HKd}\left[\int_0^1 u^{-HKd}(1-u)^{-HKd} du\right] ds$$

$$= \frac{2}{(2\pi)^d k^d} B(1-HKd, 1-HKd) \int_0^t s^{1-2HKd} ds$$

$$= \frac{2\Gamma^2(1-HKd)}{(2\pi)^d k^d \Gamma(3-2HKd)} t^{2-2HKd},$$

since $HKd < 1$. Thus we finished the proof.

Proof of Theorem 1.2. By (1.7) and the independence of $S_1^{H,K}, S_2^{H,K}, \cdots, S_d^{H,K}$, we have

$$\mathbf{E}[\alpha_{t,\epsilon}] = \mathbf{E}\left[\int_D p_\epsilon\left(S^{H,K}(s) - S^{H,K}(r)\right) dr ds\right]$$

$$= \frac{1}{(2\pi)^d} \int_D \int_{\mathbf{R}^d} \mathbf{E}\left[\exp\left\{i\sum_{m=1}^{d}\left(S_m^{H,K}(s) - S_m^{H,K}(r)\right)\xi_m\right\}\right] \exp\left\{-\frac{\epsilon|\xi|^2}{2}\right\} d\xi dr ds$$

$$= \frac{1}{(2\pi)^d} \int_D \int_{\mathbf{R}^d} \prod_{m=1}^{d} \exp\left\{-\frac{1}{2}\mathrm{Var}\left[\left(S_m^{H,K}(s) - S_m^{H,K}(r)\right)\xi_m\right]\right\} \exp\left\{-\frac{\epsilon|\xi|^2}{2}\right\} d\xi dr ds$$

$$= \frac{1}{(2\pi)^{d/2}} \int_D \left\{\mathrm{Var}\left[S_0^{H,K}(s) - S_0^{H,K}(r) + \epsilon\right]\right\}^{-\frac{d}{2}} dr ds$$

$$= \frac{1}{(2\pi)^{d/2}} \int_D (\lambda+\epsilon)^{-\frac{d}{2}} dr ds, \tag{3.1}$$

where λ is given in Lemma 2.2.

Similarly, we also get

$$\mathbf{E}[|\alpha_{t,\epsilon}|^2] = \frac{1}{(2\pi)^d} \int_{D^2} \left[(\lambda+\epsilon)(\rho+\epsilon) - \mu^2\right]^{-\frac{d}{2}} dr ds dr' ds', \tag{3.2}$$

where λ, ρ, μ are given in Lemma 2.2, and $D^2 = \{(r, s, r', s')|0 < r < s < t, 0 < r' < s' < t\}$.

By (3.1) and (3.2), we obtain

$$\begin{aligned}
&\mathbf{E}\left[(\alpha_{t,\epsilon_1} - \mathbf{E}(\alpha_{t,\epsilon_1}))(\alpha_{t,\epsilon_2} - \mathbf{E}(\alpha_{t,\epsilon_2}))\right] \\
&= \mathbf{E}\left[\alpha_{t,\epsilon_1}\alpha_{t,\epsilon_2}\right] - \mathbf{E}[\alpha_{t,\epsilon_1}]\mathbf{E}[\alpha_{t,\epsilon_2}] \\
&= \frac{1}{(2\pi)^d}\int_{D^2}\left\{\left[(\lambda+\epsilon_1)(\rho+\epsilon_2)-\mu^2\right]^{-\frac{d}{2}} - \left[(\lambda+\epsilon_1)(\rho+\epsilon_2)\right]^{-\frac{d}{2}}\right\} drdsdr'ds'.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\lim_{\epsilon_1,\epsilon_2\to 0}\mathbf{E}\left[(\alpha_{t,\epsilon_1} - \mathbf{E}(\alpha_{t,\epsilon_1}))(\alpha_{t,\epsilon_2} - \mathbf{E}(\alpha_{t,\epsilon_2}))\right] \\
&= \frac{1}{(2\pi)^d}\int_{D^2}\left[\left(\lambda\rho-\mu^2\right)^{-\frac{d}{2}} - (\lambda\rho)^{-\frac{d}{2}}\right] drdsdr'ds'.
\end{aligned}$$

Consequently, by Loève's criterion of mean-square convergence (see Chen et al.(2018)), a necessary and sufficient condition for the convergence of $\alpha_{t,\epsilon} - \mathbf{E}[\alpha_{t,\epsilon}]$ in L^2 is that

$$\Xi_t := \int_{D^2}\left[\left(\lambda\rho-\mu^2\right)^{-\frac{d}{2}} - (\lambda\rho)^{-\frac{d}{2}}\right] drdsdr'ds' < +\infty. \tag{3.3}$$

Since $D^2 \cap \{r < r'\} = D_1 \cup D_2 \cup D_3$, it suffices to prove that

$$\int_{D_i}\Theta_i drdsdr'ds' < +\infty, \quad i = 1, 2, 3. \tag{3.4}$$

By Lemmas 2.3-2.5, we obtain that (3.4) holds as $HKd < 1$. Therefore the proof of Theorem 1.2 is complete.

4. Appendix

In this section we prove Lemmas 2.2-2.5.

Proof of Lemma 2.2 By (1.2), we obtain (2.3), (2.6) and (2.9) easily.

For Case 1, by (2.2), we have

$$\begin{aligned}
&\mathrm{Var}\left[u\left(S^{H,K}_s - S^{H,K}_r\right) + v\left(S^{H,K}_{s'} - S^{H,K}_{r'}\right)\right] \\
&= \mathrm{Var}\left[u\left(S^{H,K}_{r'} - S^{H,K}_r\right) + (u+v)\left(S^{H,K}_s - S^{H,K}_{r'}\right) + v\left(S^{H,K}_{s'} - S^{H,K}_s\right)\right] \\
&\geq k\left[u^2a^{2HK} + (u+v)^2b^{2HK} + v^2c^{2HK}\right].
\end{aligned}$$

We also have

$$u^2\lambda_1 + 2uv\mu_1 + v^2\rho_1 \geq k\left[u^2a^{2HK} + (u+v)^2b^{2HK} + v^2c^{2HK}\right].$$

This implies that

$$(\lambda_1 - ka^{2HK} - kb^{2HK})u^2 + 2v(\mu_1 - kb^{2HK})u + v^2(\rho_1 - kb^{2HK} - kc^{2HK}) \geq 0,$$

which gives

$$(\mu_1 - kb^{2HK})^2 - (\lambda_1 - ka^{2HK} - kb^{2HK})(\rho_1 - kb^{2HK} - kc^{2HK}) \leq 0,$$

namely,

$$\begin{aligned}
\lambda_1\rho_1 - \mu_1^2 \geq &-2\mu_1kb^{2HK} + k^2b^{4HK} + \lambda_1(kb^{2HK} + kc^{2HK}) + \rho_1(ka^{2HK} + kb^{2HK}) \\
&-k^2(a^{2HK} + b^{2HK})(b^{2HK} + c^{2HK}).
\end{aligned}$$

Since $\mu_1 \le \sqrt{\lambda_1\rho_1} \le \frac{\lambda_1+\rho_1}{2}$, and by (2.3), we get

$$\begin{aligned}\lambda_1\rho_1-\mu_1^2 &\ge \lambda_1 kc^{2HK}+\rho_1 ka^{2HK}-k^2(a^{2HK}b^{2HK}+a^{2HK}c^{2HK}+b^{2HK}c^{2HK})\\ &\ge C_1k(a+b)^{2HK}c^{2HK}+C_1k(b+c)^{2HK}a^{2HK}-k^2(a^{2HK}b^{2HK}+a^{2HK}c^{2HK}+b^{2HK}c^{2HK})\\ &\ge C_1k\left[(a+b)^{2HK}c^{2HK}+(b+c)^{2HK}a^{2HK}\right]-2k^2\left[(a+b)^{2HK}c^{2HK}+(b+c)^{2HK}a^{2HK}\right]\\ &= k(C_1-2k)\left[(a+b)^{2HK}c^{2HK}+(b+c)^{2HK}a^{2HK}\right], \end{aligned} \tag{A.1}$$

where we use the inequality

$$(a+b)^{2HK}c^{2HK}+(b+c)^{2HK}a^{2HK}\ge\frac{1}{2}\left(a^{2HK}b^{2HK}+a^{2HK}c^{2HK}+b^{2HK}c^{2HK}\right).$$

Thus (2.4) holds by replacing $k(C_1-2k)$ in $(A.1)$ with k.

In order to prove (2.5), let $e=r$, by (1.1), we have

$$\begin{aligned}\mu =& \left[(s^{2H}+(s')^{2H})^K-(s^{2H}+(r')^{2H})^K-(r^{2H}+(s')^{2H})^K+(r^{2H}+(r')^{2H})^K\right]\\ &+\frac{1}{2}\left[(s+r')^{2HK}-(s+s')^{2HK}+(r+s')^{2HK}-(r+r')^{2HK}\right]\\ &+\frac{1}{2}\left[|s-r'|^{2HK}-|s-s'|^{2HK}+|r-s'|^{2HK}-|r-r'|^{2HK}\right].\end{aligned} \tag{A.2}$$

Hence,

$$\begin{aligned}\mu_1 =& \Big\{\left[(e+a+b)^{2H}+(e+a+b+c)^{2H}\right]^K-\left[(e+a+b)^{2H}+(e+a)^{2H}\right]^K\\ &-\left[e^{2H}+(e+a+b+c)^{2H}\right]^K+\left[e^{2H}+(e+a)^{2H}\right]^K\Big\}\\ &+\frac{1}{2}\left[(2e+2a+b)^{2HK}-(2e+2a+2b+c)^{2HK}+(2e+a+b+c)^{2HK}-(2e+a)^{2HK}\right]\\ &+\frac{1}{2}\left[b^{2HK}-c^{2HK}+(a+b+c)^{2HK}-a^{2HK}\right]\\ :=& \Delta_{1,1}+\Delta_{1,2}+\Delta_{1,3}.\end{aligned}$$

For $\Delta_{1,1}$, we obtain

$$\begin{aligned}\Delta_{1,1} &= \int_e^{e+a+b} d\left\{\left[x^{2H}+(e+a+b+c)^{2H}\right]^K-\left[x^{2H}+(e+a)^{2H}\right]^K\right\}\\ &= 2HK\int_e^{e+a+b} x^{2H-1}\left\{\left[x^{2H}+(e+a+b+c)^{2H}\right]^{K-1}-\left[x^{2H}+(e+a)^{2H}\right]^{K-1}\right\}dx\\ &\le 0,\end{aligned} \tag{A.3}$$

since $0<K\le 1$.

For $\Delta_{1,2}$, we get

$$\begin{aligned}\Delta_{1,2} &= \frac{1}{2}\left[(2e+2a+b)^{2HK}-(2e+2a+2b+c)^{2HK}+(2e+a+b+c)^{2HK}-(2e+a)^{2HK}\right]\\ &\le \frac{1}{2}\left[(2e+a+b+c)^{2HK}-(2e+a)^{2HK}\right]\\ &\le \frac{1}{2}(b+c)^{2HK}\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}(b^2+2bc+c^2)^{HK} \\
&\leq \frac{1}{2}[2(b^2+c^2)]^{HK} \\
&\leq 2^{HK-1}(b^{2HK}+c^{2HK}),
\end{aligned}
\tag{A.4}
$$

since $0<2HK<1$, where we use the inequality $x^\alpha - y^\alpha \leq |x-y|^\alpha$ for any $x>0, y>0, 0<\alpha<1$.

For $\Delta_{1,3}$, we deduce,

$$
\begin{aligned}
\Delta_{1,3} &= \frac{1}{2}\left[(a+b+c)^{2HK}+b^{2HK}-a^{2HK}-c^{2HK}\right] \\
&= \frac{1}{2}\left[(a^2+b^2+c^2+2ab+2ac+2bc)^{HK}+b^{2HK}-a^{2HK}-c^{2HK}\right] \\
&\leq \frac{1}{2}\left[(3a^2+3b^2+3c^2)^{HK}+b^{2HK}-a^{2HK}-c^{2HK}\right] \\
&\leq \frac{3^{HK}-1}{2}\left(a^{2HK}+c^{2HK}\right)+\frac{3^{HK}+1}{2}b^{2HK} \\
&\leq \frac{3^{HK}+1}{2}\left(a^{2HK}+b^{2HK}+c^{2HK}\right).
\end{aligned}
\tag{A.5}
$$

Therefore (2.5) holds from $(A.3)$, $(A.4)$ and $(A.5)$.

For Case 2, by (2.2), we have

$$
\begin{aligned}
&\mathrm{Var}\left[u\left(S_s^{H,K}-S_r^{H,K}\right)+v\left(S_{s'}^{H,K}-S_{r'}^{H,K}\right)\right] \\
&= \mathrm{Var}\left[u\left(S_{r'}^{H,K}-S_r^{H,K}\right)+(u+v)\left(S_{s'}^{H,K}-S_{r'}^{H,K}\right)+u\left(S_s^{H,K}-S_{s'}^{H,K}\right)\right] \\
&\geq k\left[u^2a^{2HK}+(u+v)^2b^{2HK}+u^2c^{2HK}\right].
\end{aligned}
$$

We also have

$$
u^2\lambda_2+2uv\mu_2+v^2\rho_2 \geq k\left[u^2a^{2HK}+(u+v)^2b^{2HK}+u^2c^{2HK}\right].
$$

This implies that

$$
(\lambda_2-ka^{2HK}-kb^{2HK}-kc^{2HK})u^2+2v(\mu_2-kb^{2HK})u+v^2(\rho_2-kb^{2HK})\geq 0,
$$

which gives

$$
(\mu_2-kb^{2HK})^2-(\lambda_2-ka^{2HK}-kb^{2HK}-kc^{2HK})(\rho_2-kb^{2HK})\leq 0,
$$

namely,

$$
\begin{aligned}
\lambda_2\rho_2-\mu_2^2 \geq& -2\mu_2kb^{2HK}+\lambda_2kb^{2HK}+\rho_2(ka^{2HK}+kb^{2HK}+kc^{2HK}) \\
&-k^2(a^{2HK}+c^{2HK})b^{2HK}.
\end{aligned}
$$

Since $\mu_2 \leq \sqrt{\lambda_2\rho_2} \leq \frac{\lambda_2+\rho_2}{2}$, and by (2.6), we get

$$
\begin{aligned}
\lambda_2\rho_2-\mu_2^2 &\geq \rho_2k(a^{2HK}+c^{2HK})-k^2(a^{2HK}+c^{2HK})b^{2HK} \\
&\geq C_1k(a^{2HK}+c^{2HK})b^{2HK}-k^2(a^{2HK}+c^{2HK})b^{2HK} \\
&= k(C_1-k)(a^{2HK}+c^{2HK})b^{2HK},
\end{aligned}
\tag{A.6}
$$

Thus (2.7) holds by replacing $k(C_1-k)$ in $(A.6)$ with k.

In order to prove (2.8), let $e=r$, by (1.1) and $(A.2)$, we have

$$
\mu_2 = \left\{\left[(e+a+b+c)^{2H}+(e+a+b)^{2H}\right]^K-\left[(e+a+b+c)^{2H}+(e+a)^{2H}\right]^K\right.
$$

$$
\begin{aligned}
&\quad -\left[e^{2H}+(e+a+b)^{2H}\right]^{K}+\left[e^{2H}+(e+a)^{2H}\right]^{K}\Big\}\\
&+\frac{1}{2}\left[(2e+2a+b+c)^{2HK}-(2e+2a+2b+c)^{2HK}+(2e+a+b)^{2HK}-(2e+a)^{2HK}\right]\\
&+\frac{1}{2}\left[(b+c)^{2HK}-c^{2HK}+(a+b)^{2HK}-a^{2HK}\right]\\
:=&\ \Delta_{2,1}+\Delta_{2,2}+\Delta_{2,3}.
\end{aligned}
$$

For $\Delta_{2,1}$, we obtain

$$
\begin{aligned}
\Delta_{2,1} &= \int_{e}^{e+a+b+c} d\left\{\left[x^{2H}+(e+a+b)^{2H}\right]^{K}-\left[x^{2H}+(e+a)^{2H}\right]^{K}\right\}\\
&= 2HK\int_{e}^{e+a+b+c} x^{2H-1}\left\{\left[x^{2H}+(e+a+b)^{2H}\right]^{K-1}-\left[x^{2H}+(e+a)^{2H}\right]^{K-1}\right\}dx\\
&\leq 0, \qquad (A.7)
\end{aligned}
$$

since $0<K\leq 1$.

For $\Delta_{2,2}$, we get

$$
\begin{aligned}
\Delta_{2,2} &= \frac{1}{2}\left[(2e+2a+b+c)^{2HK}-(2e+2a+2b+c)^{2HK}+(2e+a+b)^{2HK}-(2e+a)^{2HK}\right]\\
&\leq \frac{1}{2}\left[(2e+a+b)^{2HK}-(2e+a)^{2HK}\right]\\
&\leq \frac{1}{2}b^{2HK}, \qquad (A.8)
\end{aligned}
$$

since $0<2HK<1$.

For $\Delta_{2,3}$, we deduce,

$$
\begin{aligned}
\Delta_{2,3} &= \frac{1}{2}\left[(a+b)^{2HK}-a^{2HK}+(b+c)^{2HK}-c^{2HK}\right]\\
&\leq \frac{1}{2}\left(b^{2HK}+b^{2HK}\right)\\
&= b^{2HK}, \qquad (A.9)
\end{aligned}
$$

since $0<2HK<1$. Therefore (2.8) holds from $(A.7)$, $(A.8)$ and $(A.9)$.

For Case 3, by (2.2), we have

$$
\mathrm{Var}\left[u\left(S_{s}^{H,K}-S_{r}^{H,K}\right)+v\left(S_{s'}^{H,K}-S_{r'}^{H,K}\right)\right]\geq k\left(u^{2}a^{2HK}+v^{2}c^{2HK}\right).
$$

We also have

$$
u^{2}\lambda_{3}+2uv\mu_{3}+v^{2}\rho_{3}\geq k\left(u^{2}a^{2HK}+v^{2}c^{2HK}\right).
$$

This implies that

$$
(\lambda_{3}-ka^{2HK})u^{2}+2v\mu_{3}u+v^{2}(\rho_{3}-kc^{2HK})\geq 0,
$$

which gives

$$
\mu_{3}^{2}-(\lambda_{3}-ka^{2HK})(\rho_{3}-kc^{2HK})\leq 0,
$$

namely,

$$
\lambda_{3}\rho_{3}-\mu_{3}^{2}\geq \lambda_{3}kc^{2HK}+\rho_{3}ka^{2HK}-k^{2}a^{2HK}c^{2HK}.
$$

By (2.9), we get

$$\begin{aligned}\lambda_3\rho_3-\mu_3^2 &\geq C_1ka^{2HK}c^{2HK}+C_1ka^{2HK}c^{2HK}-k^2a^{2HK}c^{2HK}\\ &=k(2C_1-k)a^{2HK}c^{2HK}, \end{aligned} \tag{A.10}$$

Thus (2.10) holds by replacing $k(2C_1-k)$ in $(A.10)$ with k.

In order to prove (2.11), let $e=r$, by (1.1) and $(A.2)$, we have

$$\begin{aligned}\mu_3 &= \Big\{\Big[(e+a)^{2H}+(e+a+b+c)^{2H}\Big]^K-\Big[(e+a)^{2H}+(e+a+b)^{2H}\Big]^K\\ &\qquad -\Big[e^{2H}+(e+a+b+c)^{2H}\Big]^K+\Big[e^{2H}+(e+a+b)^{2H}\Big]^K\Big\}\\ &\quad+\frac{1}{2}\Big[(2e+2a+b)^{2HK}-(2e+2a+b+c)^{2HK}+(2e+a+b+c)^{2HK}-(2e+a+b)^{2HK}\Big]\\ &\quad+\frac{1}{2}\Big[b^{2HK}-(b+c)^{2HK}+(a+b+c)^{2HK}-(a+b)^{2HK}\Big]\\ &:=\Delta_{3,1}+\Delta_{3,2}+\Delta_{3,3}.\end{aligned}$$

For $\Delta_{3,1}$, we obtain

$$\begin{aligned}\Delta_{3,1} &= \int_e^{e+a} d\Big\{\Big[x^{2H}+(e+a+b+c)^{2H}\Big]^K-\Big[x^{2H}+(e+a+b)^{2H}\Big]^K\Big\}\\ &=2HK\int_e^{e+a}x^{2H-1}\Big\{\Big[x^{2H}+(e+a+b+c)^{2H}\Big]^{K-1}-\Big[x^{2H}+(e+a+b)^{2H}\Big]^{K-1}\Big\}dx\\ &\leq 0, \end{aligned} \tag{A.11}$$

since $0<K\leq 1$.

For $\Delta_{3,2}$, and for $\alpha,\beta>0$ with $\alpha+\beta=1$, we get

$$\begin{aligned}\Delta_{3,2} &= -\frac{1}{2}\int_1^2 d\Big[(2e+b+c+ua)^{2HK}-(2e+b+ua)^{2HK}\Big]\\ &=-HKa\int_1^2\Big[(2e+b+c+ua)^{2HK-1}-(2e+b+ua)^{2HK-1}\Big]du\\ &=-HKa\int_1^2\Big[\int_0^1 d(2e+b+ua+vc)^{2HK-1}\Big]du\\ &=-HK(2HK-1)ac\int_1^2\int_0^1(2e+b+ua+vc)^{2HK-2}dvdu\\ &=HK(1-2HK)ac\int_1^2\int_0^1(2e+b+ua+vc)^{2HK-2}dvdu\\ &\leq HK(1-2HK)ac\int_1^2\int_0^1(b+ua+vc)^{2HK-2}dvdu\\ &\leq HK(1-2HK)ac\int_1^2\int_0^1\Big[b^{\alpha}(ua+vc)^{\beta}\Big]^{2HK-2}dvdu\\ &\leq kac\int_1^2\int_0^1\Big[b^{\alpha}(ua)^{\frac{\beta}{2}}(vc)^{\frac{\beta}{2}}\Big]^{2HK-2}dvdu\\ &\leq kb^{2\alpha(HK-1)}(ac)^{\beta(HK-1)+1}, \end{aligned} \tag{A.12}$$

since $0 < 2HK < 1$.

For $\Delta_{3,3}$, we deduce,

$$\begin{aligned}\Delta_{3,3} &= \frac{1}{2}\left[(a+b+c)^{2HK} - (b+c)^{2HK} - (a+b)^{2HK} + b^{2HK}\right] \\ &= \frac{1}{2}\int_0^1 d\left[(b+a+vc)^{2HK} - (b+vc)^{2HK}\right] \\ &= HKc\int_0^1 \left[(b+a+vc)^{2HK-1} - (b+vc)^{2HK-1}\right]dv \\ &= HKc\int_0^1 \left[\int_0^1 d(b+ua+vc)^{2HK-1}\right]dv \\ &= HK(2HK-1)ac\int_0^1\int_0^1 (b+ua+vc)^{2HK-2}dudv \\ &\leq 0, \end{aligned} \tag{A.13}$$

since $0 < 2HK < 1$. Therefore (2.11) holds from $(A.11), (A.12)$ and $(A.13)$. Thus we have finished the proof of Lemma 2.2.

Proof of Lemma 2.3 Since $(r, s, r', s') \in D_1 = \{(r, s, r', s')|0 < r < r' < s < s' < t\}$, denoting $a = r' - r, b = s - r', c = s' - s$, we have

$$\int_{D_1}\Theta_1 drdsdr'ds' \leq k\int_{[0,t]^3}\Theta_1 dadbdc = k\int_{[0,t]^3}\left[(\lambda_1\rho_1 - \mu_1^2)^{-\frac{d}{2}} - (\lambda_1\rho_1)^{-\frac{d}{2}}\right]dadbdc.$$

On one hand, by (2.4), we get

$$\begin{aligned}\lambda_1\rho_1 - \mu_1^2 &\geq k\left[(a+b)^{2HK}c^{2HK} + (b+c)^{2HK}a^{2HK}\right] \\ &\geq k(a+b)^{HK}(b+c)^{HK}a^{HK}c^{HK} \\ &= k\left(a+\frac{b}{2}+\frac{b}{2}\right)^{HK}\left(\frac{b}{2}+\frac{b}{2}+c\right)^{HK}a^{HK}c^{HK} \\ &\geq k(abc)^{\frac{4HK}{3}}. \end{aligned} \tag{A.14}$$

On the other hand, by (2.3), we deduce

$$\lambda_1\rho_1 \geq k(a+b)^{2HK}(b+c)^{2HK} = k\left(\frac{a}{2}+\frac{a}{2}+b\right)^{2HK}\left(b+\frac{c}{2}+\frac{c}{2}\right)^{2HK} \geq k(abc)^{\frac{4HK}{3}}. \tag{A.15}$$

Thus (2.12) holds from $(A.14), (A.15)$ and $HKd < \frac{3}{2}$. The proof of Lemma 2.3 is completed.

Proof of Lemma 2.4 Since $(r, s, r', s') \in D_2 = \{(r, s, r', s')|0 < r < r' < s' < s < t\}$, denoting $a = r' - r, b = s' - r', c = s - s'$, we decompose the region $D_2 = I_1 \cup I_2 \cup I_3$, where $I_1 = \{b \geq \eta_1 a\}, I_2 = \{b \geq \eta_2 c\}, I_3 = \{b < \eta_1 a, b < \eta_2 c\}$ for some fixed but arbitrary $\eta_1 > 0$ and $\eta_2 > 0$.

For $I_1 = \{b \geq \eta_1 a\}$, by (2.7), we have, if $HKd < 1$, then

$$\begin{aligned}\int_{I_1}(\lambda_2\rho_2 - \mu_2^2)^{-\frac{d}{2}}drdsdr'ds' &\leq k\int_{b\geq\eta_1 a} b^{-HKd}(a^{2HK} + c^{2HK})^{-\frac{d}{2}}dadbdc \\ &= k\int_0^t\int_0^t (a^{2HK} + c^{2HK})^{-\frac{d}{2}}\left(\int_{\eta_1 a}^t b^{-HKd}db\right)dadc\end{aligned}$$

$$
\begin{aligned}
&= \frac{k}{1-HKd}\int_0^t\int_0^t (a^{2HK}+c^{2HK})^{-\frac{d}{2}}\left(t^{1-HKd}-(\eta_1 a)^{1-HKd}\right)dadc\\
&\le \frac{kt^{1-HKd}}{1-HKd}\int_0^t\int_0^t (a^{2HK}+c^{2HK})^{-\frac{d}{2}}dadc\\
&\le k\int_0^t\int_0^t (ac)^{-\frac{HKd}{2}}dadc\\
&< +\infty, \qquad (A.16)
\end{aligned}
$$

since $HKd < 1 < 2$, and by (2.6), we get

$$
\begin{aligned}
\int_{I_1}(\lambda_2\rho_2)^{-\frac{d}{2}}drdsdr^{'}ds^{'} &\le k\int_{b\ge\eta_1 a} b^{-HKd}(a+b+c)^{-HKd}dadbdc\\
&\le k\int_{b\ge\eta_1 a} b^{-HKd}(a+c)^{-HKd}dadbdc\\
&= \frac{k}{1-HKd}\int_0^t\int_0^t (a+c)^{-HKd}\left(t^{1-HKd}-(\eta_1 a)^{1-HKd}\right)dadc\\
&\le k\int_0^t\int_0^t (ac)^{-\frac{HKd}{2}}dadc\\
&< +\infty, \qquad (A.17)
\end{aligned}
$$

since $HKd < 1 < 2$. Thus, by $(A.16)$ and $(A.17)$, we have, if $HKd < 1$, then

$$
\int_{I_1}\Theta_2 drdsdr^{'}ds^{'} = \int_{I_1}\left[(\lambda_2\rho_2-\mu_2^2)^{-\frac{d}{2}}-(\lambda_2\rho_2)^{-\frac{d}{2}}\right]drdsdr^{'}ds^{'} < +\infty. \qquad (A.18)
$$

If $1 < HKd < \frac{3}{2}$, we obtain

$$
\begin{aligned}
\int_{I_1}(\lambda_2\rho_2-\mu_2^2)^{-\frac{d}{2}}drdsdr^{'}ds^{'} &\le k\int_{b\ge\eta_1 a} b^{-HKd}(a^{2HK}+c^{2HK})^{-\frac{d}{2}}dadbdc\\
&= k\int_0^t\int_0^t (a^{2HK}+c^{2HK})^{-\frac{d}{2}}\left(\int_{\eta_1 a}^t b^{-HKd}db\right)dadc\\
&= \frac{k}{HKd-1}\int_0^t\int_0^t (a^{2HK}+c^{2HK})^{-\frac{d}{2}}\left((\eta_1 a)^{1-HKd}-t^{1-HKd}\right)dadc\\
&\le \frac{k}{HKd-1}\int_0^t\int_0^t \left(a^{2HK}+\frac{1}{2}c^{2HK}+\frac{1}{2}c^{2HK}\right)^{-\frac{d}{2}}a^{1-HKd}dadc\\
&\le k\int_0^t\int_0^t a^{1-\frac{4HKd}{3}}c^{-\frac{2HKd}{3}}dadc\\
&< +\infty. \qquad (A.19)
\end{aligned}
$$

We also have, if $1 < HKd < \frac{3}{2}$, then

$$
\begin{aligned}
\int_{I_1}(\lambda_2\rho_2)^{-\frac{d}{2}}drdsdr^{'}ds^{'} &\le k\int_{b\ge\eta_1 a} b^{-HKd}(a+b+c)^{-HKd}dadbdc\\
&\le k\int_{b\ge\eta_1 a} b^{-HKd}(a+c)^{-HKd}dadbdc
\end{aligned}
$$

$$= \frac{k}{HKd-1}\int_0^t\int_0^t (a+c)^{-HKd}\left((\eta_1 a)^{1-HKd} - t^{1-HKd}\right)dadc$$
$$\leq k\int_0^t\int_0^t \left(a+\frac{c}{2}+\frac{c}{2}\right)^{-HKd} a^{1-HKd}dadc$$
$$\leq k\int_0^t\int_0^t a^{1-\frac{4HKd}{3}}c^{-\frac{2HKd}{3}}dadc$$
$$< +\infty. \qquad (A.20)$$

Thus, by (A.19) and (A.20), we have, if $1 < HKd < \frac{3}{2}$, then

$$\int_{I_1}\Theta_2 drdsdr^{'}ds^{'} = \int_{I_1}\left[(\lambda_2\rho_2-\mu_2^2)^{-\frac{d}{2}} - (\lambda_2\rho_2)^{-\frac{d}{2}}\right]drdsdr^{'}ds^{'} < +\infty. \qquad (A.21)$$

If $HKd = 1$, we get

$$\int_{I_1}(\lambda_2\rho_2-\mu_2^2)^{-\frac{d}{2}}drdsdr^{'}ds^{'} \leq k\int_{b\geq\eta_1 a} b^{-HKd}(a^{2HK}+c^{2HK})^{-\frac{d}{2}}dadbdc$$
$$= k\int_0^t\int_0^t (a^{2HK}+c^{2HK})^{-\frac{d}{2}}\left(\int_{\eta_1 a}^t b^{-1}db\right)dadc$$
$$= k\int_0^t\int_0^t (a^{2HK}+c^{2HK})^{-\frac{d}{2}}\left(\ln t - \ln(\eta_1 a)\right)dadc$$
$$\leq k\int_0^t\int_0^t (ac)^{-\frac{1}{2}}dadc$$
$$< +\infty. \qquad (A.22)$$

Similarly, we have, if $HKd = 1$,

$$\int_{I_1}(\lambda_2\rho_2)^{-\frac{d}{2}}drdsdr^{'}ds^{'} < +\infty. \qquad (A.23)$$

By (A.22) and (A.23), we deduce, if $HKd = 1$, then

$$\int_{I_1}\Theta_2 drdsdr^{'}ds^{'} = \int_{I_1}\left[(\lambda_2\rho_2-\mu_2^2)^{-\frac{d}{2}} - (\lambda_2\rho_2)^{-\frac{d}{2}}\right]drdsdr^{'}ds^{'} < +\infty. \qquad (A.24)$$

Combining (A.18), (A.21) and (A.24), we get, as $HKd < \frac{3}{2}$,

$$\int_{I_1}\Theta_2 drdsdr^{'}ds^{'} < +\infty. \qquad (A.25)$$

For $I_2 = \{b \geq \eta_2 c\}$, we similarly obtain, as $HKd < \frac{3}{2}$,

$$\int_{I_2}\Theta_2 drdsdr^{'}ds^{'} < +\infty. \qquad (A.26)$$

Therefore (2.13) holds from (A.25) and (A.26).

For $I_3 = \{b < \eta_1 a, b < \eta_2 c\}$, we have, if $HKd < 1$, then

$$\int_{I_3}(\lambda_2\rho_2-\mu_2^2)^{-\frac{d}{2}}drdsdr^{'}ds^{'}$$

$$
\begin{aligned}
&\leq k \int_{b<\eta_1 a, b<\eta_2 c} b^{-HKd}(a^{2HK}+c^{2HK})^{-\frac{d}{2}} dadbdc \\
&\leq k \int_{b<\eta_1 a, b<\eta_2 c} b^{-HKd} a^{-\frac{HKd}{2}} c^{-\frac{HKd}{2}} dadbdc \\
&= k \int_0^t b^{-HKd} \left(\int_{\frac{b}{\eta_1}}^t a^{-\frac{HKd}{2}} da \right) \left(\int_{\frac{b}{\eta_2}}^t c^{-\frac{HKd}{2}} dc \right) db \\
&= \frac{k}{\left(1-\frac{HKd}{2}\right)^2} \int_0^t b^{-HKd} \left(t^{1-\frac{HKd}{2}} - (\frac{b}{\eta_1})^{1-\frac{HKd}{2}} \right) \left(t^{1-\frac{HKd}{2}} - (\frac{b}{\eta_2})^{1-\frac{HKd}{2}} \right) db \\
&= \frac{k}{\left(1-\frac{HKd}{2}\right)^2} \left\{ t^{2-HKd} \int_0^t b^{-HKd} db - \left[(\frac{t}{\eta_1})^{1-\frac{HKd}{2}} + (\frac{t}{\eta_2})^{1-\frac{HKd}{2}} \right] \int_0^t b^{1-\frac{3HKd}{2}} db \right. \\
&\qquad \left. + (\frac{1}{\eta_1 \eta_2})^{1-\frac{HKd}{2}} \int_0^t b^{2-2HKd} db \right\} \\
&< +\infty, \qquad (A.27)
\end{aligned}
$$

since $HKd < 1 < \frac{4}{3} < \frac{3}{2}$.

We also have, if $HKd < 1$, then

$$
\begin{aligned}
&\int_{I_3} (\lambda_2 \rho_2)^{-\frac{d}{2}} drdsdr'ds' \\
&\leq k \int_{b<\eta_1 a, b<\eta_2 c} b^{-HKd}(a+b+c)^{-HKd} dadbdc \\
&= k \int_0^t b^{-HKd} \left[\int_{\frac{b}{\eta_2}}^t \left(\int_{\frac{b}{\eta_1}}^t (a+b+c)^{-HKd} da \right) dc \right] db \\
&= \frac{k}{1-HKd} \int_0^t b^{-HKd} \left[\int_{\frac{b}{\eta_2}}^t \left((t+b+c)^{1-HKd} - \left(\frac{b}{\eta_1} + b + c \right)^{1-HKd} \right) dc \right] db \\
&= \frac{k}{(1-HKd)(2-HKd)} \int_0^t b^{-HKd} \left\{ \left[(2t+b)^{2-HKd} - \left(t + (1+\frac{1}{\eta_2})b \right)^{2-HKd} \right] \right. \\
&\qquad \left. - \left[\left((\frac{1}{\eta_1}+1)b + t \right)^{2-HKd} - \left((\frac{1}{\eta_1} + 1 + \frac{1}{\eta_2})b \right)^{2-HKd} \right] \right\} db \\
&\leq \frac{k}{(1-HKd)(2-HKd)} \int_0^t b^{-HKd} \left\{ (3t)^{2-HKd} - t^{2-HKd} \right. \\
&\qquad \left. - t^{2-HKd} + \left((\frac{1}{\eta_1} + 1 + \frac{1}{\eta_2})b \right)^{2-HKd} \right\} db \\
&= \frac{k\left[(3t)^{2-HKd} - 2t^{2-HKd}\right]}{(1-HKd)(2-HKd)} \int_0^t b^{-HKd} db \\
&\qquad + \frac{k}{(1-HKd)(2-HKd)} \left(\frac{1}{\eta_1} + 1 + \frac{1}{\eta_2} \right)^{2-HKd} \int_0^t b^{2-2HKd} db
\end{aligned}
$$

$$< +\infty, \tag{A.28}$$

since $HKd < 1 < \frac{3}{2}$. Thus (2.14) holds from (A.27) and (A.28). The proof of Lemma 2.4 is finished.

Remark 2. Since $\int_0^t b^{-HKd}db$ appears in (A.27) and (A.28), we obtain $\int_0^t b^{-HKd}db = +\infty$ as $HKd \geq 1$. The holding condition for (2.14) is $HKd < 1$ using our proving method. It is similar to the case of (2.16) in Lemma 2.5.

Proof of Lemma 2.5. Since $(r,s,r',s') \in D_3 = \{(r,s,r',s')|0 < r < s < r' < s' < t\}$, denoting $a = s-r, b = r'-s, c = s'-r'$, we decompose the region $D_3 := J_1 + J_2 + J_3 + J_4$, where $J_1 = \{a \geq \eta_1 b, c \geq \eta_2 b\}, J_2 = \{a < \eta_1 b, c < \eta_2 b\}, J_3 = \{a \geq \eta_1 b, c < \eta_2 b\}, J_4 = \{a < \eta_1 b, c \geq \eta_2 b\}$ for some fixed but arbitrary $\eta_1 > 0$ and $\eta_2 > 0$.

For $J_1 = \{a \geq \eta_1 b, c \geq \eta_2 b\}$, by (2.10), we have, if $HKd < \frac{3}{2}$ and $HKd \neq 1$, then

$$\begin{aligned}
&\int_{J_1} (\lambda_3\rho_3 - \mu_3^2)^{-\frac{d}{2}} drdsdr'ds' \\
&\leq k \int_{a\geq\eta_1 b, c\geq\eta_2 b} (ac)^{-HKd} dadbdc \\
&= k \int_0^t db \int_{\eta_1 b}^t a^{-HKd} da \int_{\eta_2 b}^t c^{-HKd} dc \\
&= \frac{k}{(1-HKd)^2}\Bigg[t^{3-2HKd} - \left(\eta_1^{1-HKd} + \eta_2^{1-HKd}\right) t^{1-HKd} \int_0^t b^{1-HKd} db \\
&\qquad + (\eta_1\eta_2)^{1-HKd} \int_0^t b^{2-2HKd} db\Bigg] \\
&< +\infty,
\end{aligned} \tag{A.29}$$

since $HKd < \frac{3}{2} < 2$. If $HKd = 1$, we also get

$$\begin{aligned}
&\int_{J_1} (\lambda_3\rho_3 - \mu_3^2)^{-\frac{d}{2}} drdsdr'ds' \\
&\leq k \int_{a\geq\eta_1 b, c\geq\eta_2 b} (ac)^{-1} dadbdc \\
&= k \int_0^t db \int_{\eta_1 b}^t \frac{1}{a} da \int_{\eta_2 b}^t \frac{1}{c} dc \\
&= k\left(t\ln^2 t - \ln t(\ln(\eta_1) + \ln(\eta_2)) \int_0^t \ln b db + \int_0^t \ln(\eta_1 b)\ln(\eta_2 b) db\right) \\
&< +\infty,
\end{aligned} \tag{A.30}$$

since $\int_0^t \ln b db < +\infty$ and $\int_0^t \ln^2 b db < +\infty$.

Similarly, we obtain, if $HKd < \frac{3}{2}$, then

$$\int_{J_1} (\lambda_3\rho_3)^{-\frac{d}{2}} drdsdr'ds' < +\infty. \tag{A.31}$$

Hence (2.15) holds from (A.29), (A.30) and (A.31).

For $J_2 = \{a < \eta_1 b, c < \eta_2 b\}$, we have, if $HKd < 1$, then

$$\int_{J_2} (\lambda_3\rho_3 - \mu_3^2)^{-\frac{d}{2}} drdsdr'ds'$$

$$\leq k\int_{a<\eta_1 b, c<\eta_2 b}(ac)^{-HKd}dadbdc$$
$$= k\int_0^t db\int_0^{\eta_1 b}a^{-HKd}da\int_0^{\eta_2 b}c^{-HKd}dc$$
$$= \frac{k(\eta_1\eta_2)^{1-HKd}}{(1-HKd)^2}\int_0^t b^{2-2HKd}db$$
$$< +\infty, \tag{A.32}$$

since $HKd < 1 < \frac{3}{2}$. Similarly, we get, if $HKd < 1$, then

$$\int_{J_2}(\lambda_3\rho_3)^{-\frac{d}{2}}drdsdr^{'}ds^{'} < +\infty. \tag{A.33}$$

By $(A.32)$ and $(A.33)$, we deduce, as $HKd < 1$,

$$\int_{J_2}\Theta_3 drdsdr^{'}ds^{'} < +\infty. \tag{A.34}$$

For $J_3 = \{a \geq \eta_1 b, c < \eta_2 b\}$, we have, if $HKd < 1$, then

$$\int_{J_3}(\lambda_3\rho_3-\mu_3^2)^{-\frac{d}{2}}drdsdr^{'}ds^{'}$$
$$\leq k\int_{a\geq\eta_1 b, c<\eta_2 b}(ac)^{-HKd}dadbdc$$
$$= k\int_0^t db\int_{\eta_1 b}^{t}a^{-HKd}da\int_0^{\eta_2 b}c^{-HKd}dc$$
$$= \frac{k(\eta_2)^{1-HKd}}{(1-HKd)^2}\int_0^t\left[t^{1-HKd}-(\eta_1 b)^{1-HKd}\right]b^{1-HKd}db$$
$$< +\infty, \tag{A.35}$$

since $HKd < 1 < \frac{3}{2} < 2$. Similarly, we get, if $HKd < 1$, then

$$\int_{J_3}(\lambda_3\rho_3)^{-\frac{d}{2}}drdsdr^{'}ds^{'} < +\infty. \tag{A.36}$$

By $(A.35)$ and $(A.36)$, we deduce, as $HKd < 1$,

$$\int_{J_3}\Theta_3 drdsdr^{'}ds^{'} < +\infty. \tag{A.37}$$

For $J_4 = \{a < \eta_1 b, c \geq \eta_2 b\}$, we similarly obtain, if $HKd < 1$, then

$$\int_{J_4}\Theta_3 drdsdr^{'}ds^{'} < +\infty. \tag{A.38}$$

Therefore (2.16) holds from $(A.34)$, $(A.37)$ and $(A.38)$. Consequently we have finished the proof of Lemma 2.5.

References

[1] S. Albeverio, Y. Hu, X. Zhou, A remark on non smoothness of self intersection local time of planar Brownian motion, Statistics and Probability Letters 32 (1997) 57–65.

[2] S. M. Berman, Local nondeterminism and local times of Gaussian processes, Bulletin of the American Mathematical Society 79 (1973) 475–477.

[3] Z. Chen, L. Sang, X. Hao, Renormalized self-intersection local time of bifractional Brownian motion, Journal of Inequalities and Applications (2018) 326.

[4] Z. Chen, D. Wu, Y. Xiao, Smoothness of local times and self-intersection local times of Gaussian random fields, Frontiers of Mathematics in China 10 (2015) 777–805.

[5] C. El-Nouty, J. L. Journé, The sub-bifractional Brownian motion, Studia Scientiarum Mathematicarum Hungarica 50 (2013) 67–121.

[6] S. He, W. Yang, R. Yao, J. Wang, Local times of self-intersection for multidimensional Brownian motion, Nagoya Mathematical Journal 138 (1995) 51–64.

[7] Y. Hu, On the self-intersection local time of Brownian motion-via chaos expansion, Publicacions Matematiques 40 (1996) 337–350.

[8] Y. Hu, Self-intersection local time of fractional Brownian motions-via chaos expansion, Journal of Mathematics of Kyoto University 41 (2001) 233–250.

[9] Y. Hu, D. Nualart, Renormalized self-intersection local time for fractional Brownian motion, Annals of Probability 33 (2005) 948–983.

[10] Y. Jiang, Y. Wang, Self-intersection local times and collision local times of bifractional Brownian motions, Science in China Series A: Mathematics 52(2009) 1905–1919.

[11] P. Jung, G. Markowsky, On the Tanaka formula for the derivative of self-intersection local time of fractional Brownian motion, Stochastic Processes and their Applications 124 (2014) 3846–3868.

[12] P. Jung, G. Markowsky, Hölder continuity and occupation-time formulas for fBm self-intersection local time and its derivative, Journal of Theoretical Probability 28 (2015) 299–312.

[13] N. Kuang, On the collision local time of sub-bifractional Brownian motions, Advances in Mathematics (China) 48(2019) 627–640.

[14] N. Kuang, Y. Li, Berry-Esséen bounds and almost sure CLT for the quadratic variation of the sub-bifractional Brownian motion, Communications in Statistics-Simulation and Computation 51(2022) 4257–4275.

[15] N. Kuang, B. Liu, Parameter estimations for the sub-fractional Brownian motion with drift at discrete observation, Brazilian Journal of Probability and Statistics 29 (2015), 778–789.

[16] N. Kuang, B. Liu, Least squares estimator for α-sub-fractional bridges, Statistical Papers 59(2018) 893–912.

[17] N. Kuang, H. Xie, Maximum likelihood estimator for the sub-fractional Brownian motion approximated by a random walk, Annals of the Institute of Statistical Mathematics 67 (2015) 75–91.

[18] N. Kuang, H. Xie, Asymptotic behavior of weighted cubic variation of sub-fractional brownian motion, Communications In Statistics-Simulation and Computation 46 (2017) 215–229.

[19] N.Kuang, H.Xie, Derivative of self-intersection local time for the sub-bifractional Brownian motion. AIMS Mathematics 7(2022) 10286–10302.

[20] J. Rosen, The intersection local time of fractional Brownian motion in the plane, Journal of Multivariate Analysis 23(1987) 37–46.

[21] S. Varadhan, Appendix to "Euclidean quantum field theory" by K. Symanzik. In Local Quantum Theory (R. Jost, ed.). Academic Press, New York, 1969.

[22] Y. Xiao, Strong local nondeterminism and the sample path properties of Gaussian random fields, In: Lai, Tze Leung, Shao, Qiman, Qian, Lianfen (Eds.), Asymptotic Theory in Probability and Statistics with Applications, Higher Education Press, Beijing, 2007.

[23] H.Xie, N.Kuang, Least squares type estimations for discretely observed nonergodic Gaussian Ornstein-Uhlenbeck processes of the second kind, AIMS Mathematics 7(2022) 1095–1114.