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Sufficient Condition for q-Starlike and q-Convex Functions Associated with Normalized Gauss Hypergeometric Function

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Abstract. The main object of this paper is to investigate and determine a sufficient condition for q-starlikeness and q-convexity for functions which are associated with normalized Gauss hypergeometric function.

1. Introduction and Preliminary

Let A denote the class of all functions of the form

$$f(z) = z + \sum_{m=0}^{\infty} a_m z^m,\ z \in \mathbb{U}, \tag{1}$$

which are analytic in open unit disk $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$ and satisfy the normalization conditions $f(0) = 0$ and $f'(0) = 1$. Let S^* be the subclass of functions $f \in A$ which are *starlike* in $\mathbb{U}$, that is f satisfy the following condition

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > 0,\ z \in \mathbb{U}.$$

Let C^* be the subclass of functions $f \in A$ which are *convex* in $\mathbb{U}$, that is f satisfy the inequality

$$\operatorname{Re}\left[1 + \frac{zf''(z)}{f'(z)}\right] > 0,\ z \in \mathbb{U}.$$

For two functions f and g which are analytic in $\mathbb{U}$ we say that the function f is *subordinate* to g, written $f(z) \prec g(z)$, if there exists a *Schwarz function* w, which is analytic in $\mathbb{U}$ and $w(0) = 0$, $|w(z)| < 1$, $z \in \mathbb{U}$, such that $f(z) = g(w(z))$, $z \in \mathbb{U}$.

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If g is a univalent function in $\mathbb{U}$, then it is well-known that

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0), \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

Let P denote the class of analytic functions with positive real part in $\mathbb{U}$ (*Carathéodory functions*), which have the power series expansion

$$p(z) = 1 + \sum_{m=1}^{\infty} p_m z^m, \ z \in \mathbb{U}.$$

Definition 1.1. *An analytic function h with $h(0) = 1$ belongs to the class $P[M, N]$, with $-1 \le N < M \le 1$, if and only of*

$$h(z) \prec \frac{1 + Mz}{1 + Nz}.$$

The class $P[M, N]$ of analytic functions was introduced and studied by Janowski [6], who showed that $h \in P[M, N]$ if and only if there exists a function $p \in P$, such that

$$h(z) = \frac{(M + 1)p(z) - (M - 1)}{(N + 1)p(z) - (N - 1)}, \ z \in \mathbb{U}.$$

Definition 1.2. *[6] (i) A function $f \in A$ is in the class $\mathcal{S}^*[M, N]$, with $-1 \le N < M \le 1$, if and only if*

$$\frac{zf'(z)}{f(z)} \prec \frac{1 + Mz}{1 + Nz}. \tag{2}$$

(ii) A function $f \in A$ is in the class $C^*[M, N]$, with $-1 \le N < M \le 1$, if and only if

$$1 + \frac{zf''(z)}{f'(z)} \prec \frac{1 + Mz}{1 + Nz}.$$

Definition 1.3. *The q-number $[m]_q$ defined in [21] for $q \in (0, 1)$, is given by*

$$[m]_q := \begin{cases} \dfrac{1 - q^m}{1 - q}, & \textit{if} \quad m \in \mathbb{C}, \\ \sum\limits_{k=0}^{m-1} q^k = 1 + q + q^2 + \cdots + q^{m-1}, & \textit{if} \quad m \in \mathbb{N} := \{1, 2, \dots\}. \end{cases}$$

Definition 1.4. *[21] The q-derivative $D_q f$ of a function f is defined as*

$$D_q f(z) := \begin{cases} \dfrac{f(z) - f(qz)}{(1 - q)z}, & \textit{if} \quad z \in \mathbb{C} \setminus \{0\}, \\ f'(0), & \textit{if} \quad z = 0, \end{cases}$$

provided that $f'(0)$ exists, and $0 < q < 1$.

From the Definition 1.4 it follows immediately that

$$\lim_{q \to 1} D_q f(z) = \lim_{q \to 1} \frac{f(z) - f(qz)}{(1 - q)z} = f'(z).$$

For a function $f \in A$ which has the power expansion series of the form (1), it is easy to check that

$$D_q f(z) = 1 + \sum_{m=2}^{\infty} [m]_q a_m z^{m-1}, \ z \in \mathbb{U},$$

as it was previously defined by Srivastava and Bansal [20], although in 1990, the q-derivative operator D_q was presumably first applied by Ismail et. al. in [5] to study a q-extension of the class S^* of starlike functions in $\mathbb{U}$, a firm footing of the usages of the q-calculus in the context of Geometric Function Theory was presented mainly and the basic (or q-) hypergeometric functions were first used in Geometric Function Theory in a 1989 book chapter by Srivastava (see, for details, [16]).

The study of the q-calculus has mesmerized the rigorous devotion of scholars and the theory of q-calculus in science and engineering has a very significant roll.

In the year 1989, Srivastava [16] defined the use of the generalized basic (or q-) hypergeometric functions in Geometric Function Theory, after that numerous scholars offered a significant work in the advancement of Geometric Function Theory.

Recently, Rehman et al. [14] investigated some subclasses of q-starlike functions including numerous coefficient inequalities and a sufficient condition. Further, Srivastava et al. [15, 18, 19, 21] issued a set of articles where they focused upon the classes of q-starlike functions associated with the Janowski functions from several different aspects. Moreover, a resent published survey-cum-expository review article by Srivastava [17] is very useful for researchers who are working on this area. The mathematical description and applications in geometrical function theory is systematically discussed for the fractional q-calculus and the fractional q-derivative operators. For other recent investigations involving the q-calculus, one may refer to [2, 8].

Definition 1.5. *A function $f \in A$ is in the class $\mathcal{S}_q^*$ if and and only if*

$$\left|\frac{z}{f(z)}D_q f(z) - \frac{1}{1-q}\right| < \frac{1}{1-q}, \; z \in \mathbb{U}. \tag{3}$$

It is observed that, as $q \to 1^-$ the closed disk

$$\left|w - \frac{1}{1-q}\right| < \frac{1}{1-q}$$

becomes the right-half plane and the class $\mathcal{S}_q^*$ of q-starlike functions diminishes to the acquainted class $\mathcal{S}^*$. Consistently, by with the principle of subordination among analytic functions, we can rewrite the inequality (3) as

$$\frac{z}{f(z)}D_q f(z) \prec \frac{1+z}{1-qz}. \tag{4}$$

One way to generalize the class $\mathcal{S}^*[M,N]$ of Definition 1.2 is to replace in (2) the function $(1+Mz)/(1+Nz)$ by the function $(1+z)/(1-qz)$ which is involved in (4). The appropriate definition of the corresponding q-extension $\mathcal{S}_q^*[M,N]$ is specified below.

Definition 1.6. *A function $f \in A$ is said to be in the class $\mathcal{S}_q^*[M,N]$ if and only if*

$$\frac{zD_q f(z)}{f(z)} \prec \phi(z),$$

where

$$\phi(z) := \frac{(M+1)z + 2 + (M-1)qz}{(N+1)z + 2 + (N-1)qz}, \; -1 \le N < M \le 1, \; q \in (0,1).$$

Using the definition of the subordination it follows that $f \in A$ belongs to $\mathcal{S}_q^*[M,N]$ if and only if

$$\frac{zD_q f(z)}{f(z)} = \frac{(M+1)Q(w(z)) - (M-1)}{(N+1)Q(w(z)) - (N-1)}, \; z \in \mathbb{U}, \tag{5}$$

where $Q(z) = (1+z)/(1-qz)$, and w is a Schwarz function.

Remark 1.7. *(i) It is easy to check that*

$$\lim_{q\to 1^-} \mathcal{S}_q^*[M,N] = \mathcal{S}^*[M,N].$$

Also, $\mathcal{S}_q^*[1,-1] =: \mathcal{S}_q^*$*, where* $\mathcal{S}_q^*$ *is the class of functions introduced and studied by Ismail et. al [5].*

(ii) If w *is a Schwarz function, from the definition of* Q *we get that*

$$\left|Q(w(z)) - \frac{1}{1-q}\right| < \frac{1}{1-q},\ z \in \mathbb{U},$$

and from (5) *it follows*

$$Q(w(z)) = \frac{(N-1)\frac{zD_qf(z)}{f(z)} - (M-1)}{(N+1)\frac{zD_qf(z)}{f(z)} - (M+1)},\ z \in \mathbb{U}.$$

From these computations we conclude that a function $f \in A$ *is in the class* $\mathcal{S}_q^*[M,N]$*, if and only if*

$$\left|\frac{(N-1)\frac{zD_qf(z)}{f(z)} - (M-1)}{(N+1)\frac{zD_qf(z)}{f(z)} - (M+1)} - \frac{1}{1-q}\right| < \frac{1}{1-q},\ z \in \mathbb{U}.$$

In it's special case when $M = 1-2\beta$ *and* $N = -1$*, with* $0 \le \beta < 1$*, the function class* $\mathcal{S}_q^*[M,N]$ *reduces to the function class* $\mathcal{S}_q^*(\beta)$ *which was presented and deliberated by Agrawal and Sahoo [1].*

(iii) By means of the well-known Alexander's theorem [3], the class $C_q^*[M,N]$ *of* q*-convex functions can be defined in the following way:*

$$f \in C_q^*[M,N] \Leftrightarrow zD_qf(z) \in \mathcal{S}_q^*[M,N].$$

The well-known *Gaussian hypergeometric function* is given by the power series

$$F(\xi,\eta;\gamma;z) = \sum_{m=0}^{\infty} \frac{(\xi)_m(\eta)_m}{(\gamma)_m m!} z^m,\ z \in \mathbb{U},$$

where $(\lambda)_m$ is the *Pochhammer symbol* defined for $\lambda \in \mathbb{C} \setminus \{0,-1,-2,-3,\dots\}$ as

$$(\lambda)_m := \frac{\Gamma(\lambda+m)}{\Gamma(\lambda)} = \begin{cases} 1, & \text{if} \quad m = 0, \\ \lambda(\lambda+1)(\lambda+2)\dots(\lambda+m-1), & \text{if} \quad m \in \mathbb{N}. \end{cases}$$

It is assumed that ξ, η, and γ are real numbers with $\gamma \neq -m$ whenever $m \in \mathbb{N} \cup \{0\}$, and $\xi_0 = 1$ for $\xi \neq 0$. For more detail on the basic idea of Gauss hypergeometric functions we refer to [4, 7, 13, 22], and on the application related to Geometric Function Theory we refer to [9, 10, 12].

Throughout this work we frequently use the well-known formula (see [13])

$$F(\xi,\eta;\gamma;1) = \frac{\Gamma(\gamma)\Gamma(\gamma-\xi-\eta)}{\Gamma(\gamma-\xi)\Gamma(\gamma-\eta)}, \tag{6}$$

whenever $\mathrm{Re}(\gamma-\xi-\eta) > 0$ and $\mathrm{Re}\,\gamma > 0$, while the *normalized Gauss hypergeometric function* is given by

$$zF(\xi,\eta;\gamma;z) = \sum_{m=0}^{\infty} \frac{(\xi)_m(\eta)_m}{(\gamma)_m m!} z^{m+1} = z + \sum_{m=2}^{\infty} \frac{(\xi)_{m-1}(\eta)_{m-1}}{(\gamma)_{m-1}(m-1)!} z^m,\ z \in \mathbb{U}. \tag{7}$$

In this paper we determine sufficient conditions for q-starlike functions and q-convex functions associated with Gauss hypergeometric function by using following sufficient conditions obtained by Srivastava [21]:

Lemma 1.8. *[21] A function $f \in A$ is in the class $\mathcal{S}_q^*[M,N]$ if it satisfies the condition*

$$\sum_{m=2}^{\infty}\Big(2q[m-1]_q+\Big|(N+1)[m]_q-(M+1)\Big|\Big)|a_m|<|N-M|. \tag{8}$$

Lemma 1.9. *[21] A function $f \in A$ is in the class $C_q^*[M,N]$ if it satisfies the inequality*

$$\sum_{m=2}^{\infty}[m]_q\Big(2q[m-1]_q+\Big|(N+1)[m]_q-(M+1)\Big|\Big)|a_m|<|N-M|. \tag{9}$$

Lemma 1.10. *[11] Let $a,b \in \mathbb{C}\setminus\{0\}$ with $a \neq 1, b \neq 1$ and $c > max\{0; a+b-1\}$ with $c \neq 1$. Then, we have*

$$\sum_{n=0}^{\infty}\frac{(a)_n\,(b)_n}{(c)_n(1)_{n+1}}=\frac{1}{(a-1)(b-1)}\Big(\frac{\Gamma(c)\Gamma(c-a-b+1)}{\Gamma(c-a)\Gamma(c-b)}-(c-1)\Big).$$

2. Main Results

Theorem 2.1. *Let R_j, $j \in \{1,2,3\}$, be defined as follows:*
(i) If $\xi,\eta>0$ and $\gamma>\xi+\eta$, then R_1 is given by

$$R_1(\xi,\eta,\gamma,q):=\frac{1}{1-q}\Big\{\Big(q+N+2+M(1-q)\Big)\frac{\Gamma(\gamma)\Gamma(\gamma-\xi-\eta)}{\Gamma(\gamma-\xi)\Gamma(\gamma-\eta)} - (N+3)qF(\xi,\eta;\gamma;q)-(M+N+2)(1-q)\Big\}.$$

(ii) If $-1<\xi<0$, $\eta>0$ and $\gamma>\max\{0;\xi+\eta\}$, then R_2 is given by

$$R_2(\xi,\eta,\gamma,q):=\frac{1}{1-q}\Big\{-\Big(q+N+2+M(1-q)\Big)\frac{\Gamma(\gamma)\Gamma(\gamma-\xi-\eta)}{\Gamma(\gamma-\xi)\Gamma(\gamma-\eta)} - (N+3)qF(|\xi|,\eta;\gamma;q)+M(1-q)+N(1+q)+4q+2\Big\}.$$

(iii) If $\xi,\eta\in\mathbb{C}\setminus\{0\}$ and $\gamma>|\xi|+|\eta|$, then R_3 is given by

$$R_3(\xi,\eta,\gamma,q):=\frac{1}{1-q}\Big\{\Big(q+N+2+M(1-q)\Big)\frac{\Gamma(\gamma)\Gamma(\gamma-|\xi|-|\eta|)}{\Gamma(\gamma-|\xi|)\Gamma(\gamma-|\eta|)} - (N+3)qF(|\xi|,|\eta|;\gamma;q)-(M+N+2)(1-q)\Big\}.$$

If for any $j\in\{1,2,3\}$ the inequality

$$R_j(\xi,\eta,\gamma,q)<|N-M|$$

holds, then function $zF(\xi,\eta;\gamma;z)$ belongs to the class $\mathcal{S}_q^*[M,N]$.

Proof. Since

$$zF(\xi,\eta;\gamma;z)=z+\sum_{m=2}^{\infty}\frac{(\xi)_{m-1}(\eta)_{m-1}}{(\gamma)_{m-1}(m-1)!}z^m,\ z\in\mathbb{U},$$

according to Lemma 1.8, any function $f\in A$ is in the class $\mathcal{S}_q^*[M,N]$ if it satisfies the inequality (8). Then, for $f(z):=zF(\xi,\eta;\gamma;z)$ it is sufficient to show that (8) holds, for

$$a_m=\frac{(\xi)_{m-1}(\eta)_{m-1}}{(\gamma)_{m-1}(m-1)!},\quad\text{and}\quad [m]_q=\frac{1-q^m}{1-q}.$$

Using the triangle's inequality we get

$$\begin{aligned}
&\sum_{m=2}^{\infty}(2q[m-1]_q+|(N+1)[m]_q-(M+1)|)|a_m| \\
&\leq \sum_{m=2}^{\infty}2q\frac{1-q^{m-1}}{1-q}|a_m|+\sum_{m=2}^{\infty}(N+1)\frac{1-q^m}{1-q}|a_m|+\sum_{m=2}^{\infty}(M+1)|a_m| \\
&=\sum_{m=2}^{\infty}\left(\frac{2q+(N+1)}{1-q}+(M+1)\right)|a_m|-\sum_{m=2}^{\infty}\frac{(N+3)q^m}{1-q}|a_m|. \qquad (10)
\end{aligned}$$

Case (i) If $\xi, \eta > 0$ and $\gamma > \xi + \eta$, from (10) we get

$$\begin{aligned}
&\sum_{m=2}^{\infty}(2q[m-1]_q+|(N+1)[m]_q-(M+1)|)|a_m| \\
&\leq \left(\frac{2q+(N+1)}{1-q}+(M+1)\right)\sum_{m=2}^{\infty}\frac{(\xi)_{m-1}(\eta)_{m-1}}{(\gamma)_{m-1}(m-1)!}-\frac{N+3}{1-q}\sum_{m=2}^{\infty}\frac{(\xi)_{m-1}(\eta)_{m-1}q^m}{(\gamma)_{m-1}(m-1)!} \\
&=\frac{1}{1-q}\Big\{\big(q+N+2+M(1-q)\big)\big(F(\xi,\eta;\gamma;1)-1\big)-(N+3)q\big(F(\xi,\eta;\gamma;q)-1\big)\Big\} \\
&\leq \frac{1}{1-q}\Big\{\big(q+N+2+M(1-q)\big)F(\xi,\eta;\gamma;1)-(N+3)qF(\xi,\eta;\gamma;q)-(M+N+2)(1-q)\Big\}.
\end{aligned}$$

Using the identity (6), from the above inequality we have

$$\begin{aligned}
&\sum_{m=2}^{\infty}(2q[m-1]_q+|(N+1)[m]_q-(M+1)|)|a_m| \\
&\leq \frac{1}{1-q}\Big\{\big(q+N+2+M(1-q)\big)\frac{\Gamma(\gamma)\Gamma(\gamma-\xi-\eta)}{\Gamma(\gamma-\xi)\Gamma(\gamma-\eta)}-(N+3)qF(\xi,\eta;\gamma;q)-(M+N+2)(1-q)\Big\}=:R_1(\xi,\eta,\gamma,q),
\end{aligned}$$

and the assumption of the theorem implies (8), that is $zF(\xi,\eta;\gamma;z)\in\mathcal{S}_q^*[M,N]$.

Case (ii) If $-1 < \xi < 0$, $\eta > 0$ and $\gamma > 0$, then (10) leads to

$$\begin{aligned}
&\sum_{m=2}^{\infty}(2q[m-1]_q+|(N+1)[m]_q-(M+1)|)|a_m| \\
&\leq \left(\frac{2q+(N+1)}{1-q}+(M+1)\right)\sum_{m=2}^{\infty}\left|\frac{(\xi)_{m-1}(\eta)_{m-1}}{(\gamma)_{m-1}(m-1)!}\right|-\frac{N+3}{1-q}\sum_{m=2}^{\infty}\left|\frac{(\xi)_{m-1}(\eta)_{m-1}q^m}{(\gamma)_{m-1}(m-1)!}\right| \\
&=\left(\frac{2q+(N+1)}{1-q}+(M+1)\right)\sum_{m=2}^{\infty}\frac{|(\xi)_{m-1}|(\eta)_{m-1}}{(\gamma)_{m-1}(m-1)!}-\frac{N+3}{1-q}\sum_{m=2}^{\infty}\frac{|(\xi)_{m-1}|(\eta)_{m-1}q^m}{(\gamma)_{m-1}(m-1)!} \\
&=\left(\frac{2q+(N+1)}{1-q}+(M+1)\right)\sum_{m=0}^{\infty}\frac{|(\xi)_{m+1}|(\eta)_{m+1}}{(\gamma)_{m+1}(m+1)!}-\frac{N+3}{1-q}q\sum_{m=1}^{\infty}\frac{|(\xi)_m|(\eta)_m q^m}{(\gamma)_m m!} \\
&=\left(\frac{2q+(N+1)}{1-q}+(M+1)\right)\frac{|\xi|\eta}{\gamma}\sum_{m=o}^{\infty}\frac{|(\xi+1)_m|(\eta+1)_m}{(\gamma+1)_m(m+1)!}-\frac{N+3}{1-q}q\sum_{m=1}^{\infty}\frac{|(\xi)_m|(\eta)_m q^m}{(\gamma)_m m!}. \qquad (11)
\end{aligned}$$

Since $|(a)_n| \le (|a|)_n$, from (11) we have

$$\sum_{m=2}^{\infty}(2q[m-1]_q + |(N+1)[m]_q - (M+1)|)|a_m|$$
$$\le \left(\frac{2q+(N+1)}{1-q} + (M+1)\right)\frac{|\xi|\eta}{\gamma}\sum_{m=o}^{\infty}\frac{(|\xi+1|)_m(\eta+1)_m}{(\gamma+1)_m(m+1)!} - \frac{N+3}{1-q}q\sum_{m=1}^{\infty}\frac{(|\xi|)_m(\eta)_m q^m}{(\gamma)_m m!}$$
$$= \left(\frac{2q+(N+1)}{1-q} + (M+1)\right)\frac{|\xi|\eta}{\gamma}\sum_{m=0}^{\infty}\frac{(\xi+1)_m(\eta+1)_m}{(\gamma+1)_m(m+1)!} - \frac{N+3}{1-q}q\sum_{m=1}^{\infty}\frac{(|\xi|)_m(\eta)_m q^m}{(\gamma)_m m!},$$

and using Lemma 1.10 and (6) we obtain

$$\sum_{m=2}^{\infty}(2q[m-1]_q + |(N+1)[m]_q - (M+1)|)|a_m|$$
$$\le \frac{1}{1-q}\left\{\Big(q+N+2+M(1-q)\Big)\left(1-\frac{\Gamma(\gamma)\Gamma(\gamma-\xi-\eta)}{\Gamma(\gamma-\xi)\Gamma(\gamma-\eta)}\right) - (N+3)q\Big(F(|\xi|,\eta;\gamma;q)-1\Big)\right\}$$
$$= \frac{1}{1-q}\left\{-\Big(q+N+2+M(1-q)\Big)\frac{\Gamma(\gamma)\Gamma(\gamma-\xi-\eta)}{\Gamma(\gamma-\xi)\Gamma(\gamma-\eta)} - (N+3)qF(|\xi|,\eta;\gamma;q)\right.$$
$$\left. + M(1-q) + N(1+q) + 4q + 2\right\} =: R_2(\xi,\eta,\gamma,q),$$

and the assumption of the theorem implies (8), that is $zF(\xi,\eta;\gamma;z) \in \mathcal{S}_q^*[M,N]$.

Case (iii) If $\xi,\eta \in \mathbb{C}\setminus\{0\}$, $\gamma > |\xi| + |\eta|$, from (10) we have

$$\sum_{m=2}^{\infty}(2q[m-1]_q + |(N+1)[m]_q - (M+1)|)|a_m|$$
$$\le \left(\frac{2q+(N+1)}{1-q} + (M+1)\right)\sum_{m=2}^{\infty}\left|\frac{(\xi)_{m-1}(\eta)_{m-1}}{(\gamma)_{m-1}(m-1)!}\right| - \frac{N+3}{1-q}\sum_{m=2}^{\infty}\left|\frac{(\xi)_{m-1}(\eta)_{m-1}q^m}{(\gamma)_{m-1}(m-1)!}\right|$$
$$= \left(\frac{2q+(N+1)}{1-q} + (M+1)\right)\sum_{m=1}^{\infty}\frac{|(\xi)_m||(\eta)_m|}{(\gamma)_m m!} - \frac{N+3}{1-q}q\sum_{m=1}^{\infty}\frac{|(\xi)_m||(\eta)_m|q^m}{(\gamma)_m m!}. \tag{12}$$

Since $|(a)_n| \le (|a|)_n$, from (12) by using (6) we deduce that

$$\sum_{m=2}^{\infty}(2q[m-1]_q + |(N+1)[m]_q - (M+1)|)|a_m|$$
$$\le \left(\frac{2q+(N+1)}{1-q} + (M+1)\right)\sum_{m=1}^{\infty}\frac{(|\xi|)_m(|\eta|)_m}{(\gamma)_m m!} - \frac{(N+3)q}{1-q}\sum_{m=1}^{\infty}\frac{(|\xi|)_m(|\eta|)_m q^m}{(\gamma)_m m!}$$
$$= \frac{1}{1-q}\left\{\Big(q+N+2+M(1-q)\Big)\left(\frac{\Gamma(\gamma)\Gamma(\gamma-|\xi|-|\eta|)}{\Gamma(\gamma-|\xi|)\Gamma(\gamma-|\eta|)}-1\right) - (N+3)q\Big(F(|\xi|,|\eta|;\gamma;q)-1\Big)\right\}$$
$$= \frac{1}{1-q}\left\{\Big(q+N+2+M(1-q)\Big)\frac{\Gamma(\gamma)\Gamma(\gamma-|\xi|-|\eta|)}{\Gamma(\gamma-|\xi|)\Gamma(\gamma-|\eta|)}\right.$$
$$\left. - (N+3)qF(|\xi|,|\eta|;\gamma;q) - (M+N+2)(1-q)\right\} =: R_3(\xi,\eta,\gamma,q).$$

and the assumption of the theorem implies (8), that is $zF(\xi,\eta;\gamma;z) \in \mathcal{S}_q^*[M,N]$. □

For the special case $M = 1-2\beta$, $0 \le \beta < 1$, and $N = -1$, we have $S_q^*[1-2\beta,-1] =: S_q^*(\beta)$ and Theorem 2.1 reduces to the following result:

Corollary 2.2. *Let R_j^*, $j \in \{1,2,3\}$, be defined as follows:*
(i) If $\xi, \eta > 0$ and $\gamma > \xi + \eta$, then R_1^* is given by

$$R_1^*(\xi,\eta,\gamma,q) := \frac{1}{1-q}\Big\{2\Big(1-\beta(1-q)\Big)\frac{\Gamma(\gamma)\Gamma(\gamma-\xi-\eta)}{\Gamma(\gamma-\xi)\Gamma(\gamma-\eta)} - 2qF(\xi,\eta;\gamma;q) - 2(1-\beta)(1-q)\Big\}.$$

(ii) If $-1 < \xi < 0$, $\eta > 0$ and $\gamma > \max\{0; \xi+\eta\}$, then R_2^* is given by

$$R_2^*(\xi,\eta,\gamma,q) := \frac{1}{1-q}\Big\{2\Big(\beta(1-q)-1\Big)\frac{\Gamma(\gamma)\Gamma(\gamma-\xi-\eta)}{\Gamma(\gamma-\xi)\Gamma(\gamma-\eta)} - 2qF(|\xi|,\eta;\gamma;q) + 2(1+q) - 2\beta(1-q)\Big\}.$$

(iii) If $\xi, \eta \in \mathbb{C} \setminus \{0\}$ and $\gamma > |\xi| + |\eta|$, then R_3^* is given by

$$R_3^*(\xi,\eta,\gamma,q) := \frac{1}{1-q}\Big\{2\Big(1-\beta(1-q)\Big)\frac{\Gamma(\gamma)\Gamma(\gamma-|\xi|-|\eta|)}{\Gamma(\gamma-|\xi|)\Gamma(\gamma-|\eta|)} - 2qF(|\xi|,|\eta|;\gamma;q) - 2(1-\beta)(1-q)\Big\}.$$

If for any $j \in \{1,2,3\}$ the inequality

$$R_j^*(\xi,\eta,\gamma,q) < 2(1-\beta)$$

holds for $0 \le \beta < 1$, then function $zF(\xi,\eta;\gamma;z)$ belongs to the class $\mathcal{S}_q^*(\beta)$.

For $\beta = 0$ the above corollary gives us the next special case:

Example 2.3. *Let $\widetilde{R}_j$, $j \in \{1,2,3\}$, be defined as follows:*
(i) If $\xi, \eta > 0$ and $\gamma > \xi + \eta$, then $\widetilde{R}_1$ is given by

$$\widetilde{R}_1(\xi,\eta,\gamma,q) = \frac{1}{1-q}\left\{\frac{2\Gamma(\gamma)\Gamma(\gamma-\xi-\eta)}{\Gamma(\gamma-\xi)\Gamma(\gamma-\eta)} - 2qF(\xi,\eta;\gamma;q) - 2(1-q)\right\}.$$

(ii) If $-1 < \xi < 0$, $\eta > 0$ and $\gamma > \max\{0; \xi+\eta\}$, then $\widetilde{R}_2$ is given by

$$\widetilde{R}_2(\xi,\eta,\gamma,q) := \frac{1}{1-q}\left\{-\frac{2\Gamma(\gamma)\Gamma(\gamma-\xi-\eta)}{\Gamma(\gamma-\xi)\Gamma(\gamma-\eta)} - 2qF(|\xi|,\eta;\gamma;q) + 2(1+q)\right\}.$$

(iii) If $\xi, \eta \in \mathbb{C} \setminus \{0\}$ and $\gamma > |\xi| + |\eta|$, then $\widetilde{R}_3$ is given by

$$\widetilde{R}_3(\xi,\eta,\gamma,q) := \frac{1}{1-q}\left\{\frac{2\Gamma(\gamma)\Gamma(\gamma-|\xi|-|\eta|)}{\Gamma(\gamma-|\xi|)\Gamma(\gamma-|\eta|)} - 2qF(|\xi|,|\eta|;\gamma;q) - 2(1-q)\right\}.$$

If for any $j \in \{1,2,3\}$ the inequality

$$\widetilde{R}_j(\xi,\eta,\gamma,q) < 2$$

holds, then function $zF(\xi,\eta;\gamma;z)$ belongs to the class $\mathcal{S}_q^*(0)$.

Theorem 2.4. *Let T_j, $j \in \{1,2,3\}$, be defined as follows:*
(i) If $\xi, \eta > 0$ and $\gamma > \xi + \eta$, then T_1 is given by

$$\begin{aligned} T_1(\xi,\eta,\gamma,q) := \frac{1}{(1-q)^2}\Big\{&\Big(N+2+q+M(1-q)\Big)\frac{\Gamma(\gamma)\Gamma(\gamma-\xi-\eta)}{\Gamma(\gamma-\xi)\Gamma(\gamma-\eta)} \\ &- \Big(M(1-q)+2N+5+q\Big)qF(\xi,\eta;\gamma;q) + (N+3)q^2F(\xi,\eta;\gamma;q^2) - (M+N+2)(1-q)^2\Big\}. \end{aligned}$$

(ii) If $-1 < \xi < 0$, $\eta > 0$ *and* $\gamma > \max\{0; \xi + \eta\}$, *then* T_2 *is given by*

$$T_2(\xi,\eta,\gamma,q) := \frac{1}{(1-q)^2}\bigg\{ -\Big(N+2+q+M(1-q)\Big)\frac{\Gamma(\gamma)\Gamma(\gamma-\xi-\eta)}{\Gamma(\gamma-\xi)\Gamma(\gamma-\eta)} - \Big(M(1-q)+2N+5+q\Big)qF(|\xi|,\eta;\gamma;q) + (N+3)q^2F(|\xi|,\eta;\gamma;q^2) + M(1-q^2) + 2(1+3q-q^2) + N(1+2q-q^2)\bigg\}.$$

(iii) If $\xi, \eta \in \mathbb{C}\setminus\{0\}$ *and* $\gamma > |\xi| + |\eta|$, *then* T_3 *is given by*

$$T_3(\xi,\eta,\gamma,q) := \frac{1}{(1-q)^2}\bigg\{\Big(N+2+q+M(1-q)\Big)\frac{\Gamma(\gamma)\Gamma(\gamma-|\xi|-|\eta|)}{\Gamma(\gamma-|\xi|)\Gamma(\gamma-|\eta|)} - \Big(M(1-q)+2N+5+q\Big)qF(|\xi|,|\eta|;\gamma;q) + (N+3)q^2F(|\xi|,|\eta|;\gamma;q^2) - (M+N+2)(1-q)^2\bigg\}.$$

If for any $j \in \{1,2,3\}$ *the inequality*

$$T_j(\xi,\eta,\gamma,q) < |N-M|$$

holds, then function $zF(\xi,\eta;\gamma;z)$ *belongs to the class* $C_q^*[M,N]$.

Proof. Since the function $f(z) := zF(\xi;\eta;\gamma;z)$ is given by (7), according to Lemma 1.9 any function $f \in A$ belongs to the class $C_q^*[M,N]$ if it satisfies the inequality (9) for

$$a_m = \frac{(\xi)_{m-1}(\eta)_{m-1}}{(\gamma)_{m-1}(m-1)!}, \quad \text{and} \quad [m]_q = \frac{1-q^m}{1-q}.$$

Using first the triangle's inequality, we have

$$\begin{aligned}
&\sum_{m=2}^{\infty}[m]_q\Big(2q[m-1]_q + \big|(N+1)[m]_q-(M+1)\big|\Big)|a_m| \\
&\le \sum_{m=2}^{\infty} 2q[m]_q[m-1]_q|a_m| + \sum_{m=2}^{\infty}(N+1)[m]_q|[m]_q|a_m| + \sum_{m=2}^{\infty}(M+1)[m]_q|a_m| \\
&= \sum_{m=2}^{\infty} 2q\ \frac{1-q^m}{1-q}\ \frac{1-q^{m-1}}{1-q}\ |a_m| + \sum_{m=2}^{\infty}(N+1)\ \frac{1-q^m}{1-q}\ \frac{1-q^m}{1-q}\ |a_m| + \sum_{m=2}^{\infty}(M+1)\ \frac{1-q^m}{1-q}\ |a_m| \\
&= \sum_{m=2}^{\infty}\left(\frac{2q+(N+1)+(M+1)(1-q)}{(1-q)^2}\right)|a_m| \\
&- \sum_{m=2}^{\infty}\left(\frac{(M+1)(1-q)+2(N+1)+2q+2}{(1-q)^2}\right)q^m|a_m| + \sum_{m=2}^{\infty}\left(\frac{2+(N+1)}{(1-q)^2}\right)q^{2m}|a_m| \\
&= \frac{q+N+2+M(1-q)}{(1-q)^2}\sum_{m=2}^{\infty}|a_m| - \frac{M(1-q)+2N+5+q}{(1-q)^2}\sum_{m=2}^{\infty}q^m|a_m| + \frac{N+3}{(1-q)^2}\sum_{m=2}^{\infty}q^{2m}|a_m|. \qquad (13)
\end{aligned}$$

Case (i) If $\xi,\eta>0$ and $\gamma>\xi+\eta$, from (13) we obtain

$$\begin{aligned}
&\sum_{m=2}^{\infty}[m]_q\Big(2q[m-1]_q+\Big|(N+1)[m]_q-(M+1)\Big|\Big)|a_m|\\
&\le \frac{q+N+2+M(1-q)}{(1-q)^2}\sum_{m=2}^{\infty}\frac{(\xi)_{m-1}(\eta)_{m-1}}{(\gamma)_{m-1}(m-1)!}+\frac{N+3}{(1-q)^2}\sum_{m=2}^{\infty}\frac{(\xi)_{m-1}(\eta)_{m-1}}{(\gamma)_{m-1}(m-1)!}q^{2m}\\
&-\frac{M(1-q)+2N+5+q}{(1-q)^2}\sum_{m=2}^{\infty}\frac{(\xi)_{m-1}(\eta)_{m-1}}{(\gamma)_{m-1}(m-1)!}q^{m}\\
&=\frac{q+N+2+M(1-q)}{(1-q)^2}\sum_{m=1}^{\infty}\frac{(\xi)_{m}(\eta)_{m}}{(\gamma)_{m}m!}+\frac{N+3}{(1-q)^2}q^2\sum_{m=1}^{\infty}\frac{(\xi)_{m}(\eta)_{m}}{(\gamma)_{m}m!}q^{2m}\\
&-\frac{M(1-q)+2N+5+q}{(1-q)^2}q\sum_{m=1}^{\infty}\frac{(\xi)_{m}(\eta)_{m}}{(\gamma)_{m}m!}q^{m}\\
&=\frac{q+N+2+M(1-q)}{(1-q)^2}(F(\xi,\eta,\gamma,1)-1)+\frac{(N+3)q^2}{(1-q)^2}(F(\xi,\eta;\gamma;q^2)-1)\\
&-\frac{M(1-q)+2N+5+q}{(1-q)^2}q(F(\xi,\eta;\gamma;q)-1),
\end{aligned}$$

and using the formula (6) we get

$$\begin{aligned}
&\sum_{m=2}^{\infty}[m]_q\Big(2q[m-1]_q+\Big|(N+1)[m]_q-(M+1)\Big|\Big)|a_m|\\
&\le\frac{1}{(1-q)^2}\Big\{\Big(N+2+q+M(1-q)\Big)\frac{\Gamma(\gamma)\Gamma(\gamma-\xi-\eta)}{\Gamma(\gamma-\xi)\Gamma(\gamma-\eta)}\\
&-\Big(M(1-q)+2N+5+q\Big)qF(\xi,\eta;\gamma;q)+(N+3)q^2F(\xi,\eta;\gamma;q^2)\\
&-(M+N+2)(1-q)^2\Big\}=:T_1(\xi,\eta,\gamma,q).
\end{aligned}$$

Therefore, the assumption of the theorem implies (9), hence $zF(\xi,\eta;\gamma;z)\in C_q^*[M,N]$.

Case (ii) If $-1<\xi<0$, $\eta>0$ and $\gamma>\max\{0;\xi+\eta\}$, then the inequality (13) leads to

$$\begin{aligned}
&\sum_{m=2}^{\infty}[m]_q\Big(2q[m-1]_q+\Big|(N+1)[m]_q-(M+1)\Big|\Big)|a_m|\\
&\le \frac{q+N+2+M(1-q)}{(1-q)^2}\sum_{m=2}^{\infty}\left|\frac{(\xi)_{m-1}(\eta)_{m-1}}{(\gamma)_{m-1}(m-1)!}\right|+\frac{N+3}{(1-q)^2}\sum_{m=2}^{\infty}\left|\frac{(\xi)_{m-1}(\eta)_{m-1}}{(\gamma)_{m-1}(m-1)!}q^{2m}\right|\\
&-\frac{M(1-q)+2N+5+q}{(1-q)^2}\sum_{m=2}^{\infty}\left|\frac{(\xi)_{m-1}(\eta)_{m-1}}{(\gamma)_{m-1}(m-1)!}q^{m}\right|\\
&=\frac{q+N+2+M(1-q)}{(1-q)^2}\sum_{m=2}^{\infty}\frac{|(\xi)_{m-1}|(\eta)_{m-1}}{(\gamma)_{m-1}(m-1)!}+\frac{N+3}{(1-q)^2}\sum_{m=2}^{\infty}\frac{|(\xi)_{m-1}|(\eta)_{m-1}}{(\gamma)_{m-1}(m-1)!}q^{2m}\\
&-\frac{M(1-q)+2N+5+q}{(1-q)^2}\sum_{m=2}^{\infty}\frac{|(\xi)_{m-1}|(\eta)_{m-1}}{(\gamma)_{m-1}(m-1)!}q^{m}
\end{aligned}$$

$$= \frac{q+N+2+M(1-q)}{(1-q)^2}\sum_{m=0}^{\infty}\frac{|(\xi)_{m+1}|(\eta)_{m+1}}{(\gamma)_{m+1}(m+1)!}+\frac{N+3}{(1-q)^2}q^2\sum_{m=1}^{\infty}\frac{|(\xi)_m|(\eta)_m}{(\gamma)_m m!}q^{2m}$$
$$-\frac{M(1-q)+2N+5+q}{(1-q)^2}q\sum_{m=1}^{\infty}\frac{|(\xi)_m|(\eta)_m}{(\gamma)_m m!}q^m$$
$$= \frac{q+N+2+M(1-q)}{(1-q)^2}\frac{|\xi|\eta}{\gamma}\sum_{m=0}^{\infty}\frac{|(\xi+1)_m|(\eta+1)_m}{(\gamma+1)_m(m+1)!}+\frac{(N+3)q^2}{(1-q)^2}\sum_{m=1}^{\infty}\frac{|(\xi)_m|(\eta)_m}{(\gamma)_m m!}q^{2m}$$
$$-\frac{M(1-q)+2N+5+q}{(1-q)^2}q\sum_{m=1}^{\infty}\frac{|(\xi)_m|(\eta)_m}{(\gamma)_m m!}q^m.$$

Since $|(a)_n| \leq (|a|)_n$, from the above inequality it follows that

$$\sum_{m=2}^{\infty}[m]_q\Big(2q[m-1]_q+\Big|(N+1)[m]_q-(M+1)\Big|\Big)|a_m|$$
$$\leq \frac{q+N+2+M(1-q)}{(1-q)^2}\frac{|\xi|\eta}{\gamma}\sum_{m=0}^{\infty}\frac{(\xi+1)_m(\eta+1)_m}{(\gamma+1)_m(m+1)!}$$
$$+\frac{(N+3)q^2}{(1-q)^2}\sum_{m=1}^{\infty}\frac{(|\xi|)_m|(\eta)_m}{(\gamma)_m m!}q^{2m}-\frac{M(1-q)+2N+5+q}{(1-q)^2}q\sum_{m=1}^{\infty}\frac{(|\xi|)_m(\eta)_m}{(\gamma)_m m!}q^m,$$

and using Lemma 1.10 and (6) we get

$$\sum_{m=2}^{\infty}[m]_q\Big(2q[m-1]_q+\Big|(N+1)[m]_q-(M+1)\Big|\Big)|a_m|$$
$$\leq \frac{1}{(1-q)^2}\Big\{\Big(N+2+q+M(1-q)\Big)\Big(1-\frac{\Gamma(\gamma)\Gamma(\gamma-\xi-\eta)}{\Gamma(\gamma-\xi)\Gamma(\gamma-\eta)}\Big)$$
$$+(N+3)q^2\Big(F(|\xi|,\eta;\gamma;q^2)-1\Big)-\Big(M(1-q)+2N+5+q\Big)q\Big(F(|\xi|,\eta;\gamma;q)-1\Big)\Big\}$$
$$=\frac{1}{(1-q)^2}\Big\{-\Big(N+2+q+M(1-q)\Big)\frac{\Gamma(\gamma)\Gamma(\gamma-\xi-\eta)}{\Gamma(\gamma-\xi)\Gamma(\gamma-\eta)}+(N+3)q^2F(|\xi|,\eta;\gamma;q^2)$$
$$-(M\Big(1-q)+2N+5+q\Big)qF(|\xi|,\eta;\gamma;q)+M(1-q^2)+2(1+3q-q^2)$$
$$+N(1+2q-q^2)\Big\} =: T_2(\xi,\eta,\gamma,q).$$

Since the assumption of the theorem implies (9), it follows $zF(\xi,\eta;\gamma;z)\in C_q^*[M,N]$.

Case (iii) If $\xi,\eta\in\mathbb{C}\setminus\{0\}$, $\gamma>|\xi|+|\eta|$, then the inequality (13) leads to

$$\sum_{m=2}^{\infty}[m]_q\Big(2q[m-1]_q+\Big|(N+1)[m]_q-(M+1)\Big|\Big)|a_m|$$
$$\leq \frac{q+N+2+M(1-q)}{(1-q)^2}\sum_{m=2}^{\infty}\left|\frac{(\xi)_{m-1}(\eta)_{m-1}}{(\gamma)_{m-1}(m-1)!}\right|+\frac{N+3}{(1-q)^2}\sum_{m=2}^{\infty}\left|\frac{(\xi)_{m-1}(\eta)_{m-1}}{(\gamma)_{m-1}(m-1)!}q^{2m}\right|$$
$$-\frac{M(1-q)+2N+5+q}{(1-q)^2}\sum_{m=2}^{\infty}\left|\frac{(\xi)_{m-1}(\eta)_{m-1}}{(\gamma)_{m-1}(m-1)!}q^m\right|$$

$$= \frac{q+N+2+M(1-q)}{(1-q)^2}\sum_{m=2}^{\infty}\frac{|(\xi)_{m-1}||(\eta)_{m-1}|}{(\gamma)_{m-1}(m-1)!} + \frac{N+3}{(1-q)^2}\sum_{m=2}^{\infty}\frac{|(\xi)_{m-1}||(\eta)_{m-1}|}{(\gamma)_{m-1}(m-1)!}q^{2m}$$
$$- \frac{M(1-q)+2N+5+q}{(1-q)^2}\sum_{m=2}^{\infty}\frac{|(\xi)_{m-1}||(\eta)_{m-1}|}{(\gamma)_{m-1}(m-1)!}q^{m}.$$

Since $(a)_n| \le (|a|)_n$, the above inequality implies

$$\sum_{m=2}^{\infty}[m]_q\Big(2q[m-1]_q + \Big|(N+1)[m]_q-(M+1)\Big|\Big)|a_m|$$
$$\le \frac{q+N+2+M(1-q)}{(1-q)^2}\sum_{m=1}^{\infty}\frac{(|\xi|)_m|(\eta)_m|}{(\gamma)_m m!} + \frac{N+3}{(1-q)^2}q^2\sum_{m=1}^{\infty}\frac{(|\xi|)_m(|\eta|)_m}{(\gamma)_m m!}q^{2m}$$
$$- \frac{M(1-q)+2N+5+q}{(1-q)^2}q\sum_{m=1}^{\infty}\frac{(|\xi|)_m(|\eta|)_m}{(\gamma)_m m!}q^{m}$$
$$= \frac{q+N+2+M(1-q)}{(1-q)^2}\left(F(|\xi|,|\eta|;\gamma,1)-1\right) + \frac{(N+3)q^2}{(1-q)^2}\left(F(|\xi|,|\eta|;\gamma;q^2)-1\right)$$
$$- \frac{M(1-q)+2N+5+q}{(1-q)^2}q\left(F(|\xi|,|\eta|;\gamma;q)-1\right),$$

and using the formula (6) we conclude that

$$\sum_{m=2}^{\infty}[m]_q\Big(2q[m-1]_q + \Big|(N+1)[m]_q-(M+1)\Big|\Big)|a_m|$$
$$\le \frac{1}{(1-q)^2}\Big\{\Big(N+2+q+M(1-q)\Big)\frac{\Gamma(\gamma)\Gamma(\gamma-|\xi|-|\eta|)}{\Gamma(\gamma-|\xi|)\Gamma(\gamma-|\eta|)} + (N+3)q^2F(|\xi|,|\eta|;\gamma;q^2)$$
$$- \Big(M(1-q)+2N+5+q\Big)qF(|\xi|,|\eta|;\gamma;q) - (M+N+2)(1-q)^2\Big\} =: T_3(\xi,\eta,\gamma,q).$$

It follows that the assumption of the theorem implies (9), hence $zF(\xi,\eta;\gamma;z) \in C_q^*[M,N]$. □

For the special case $M = 1-2\beta$, $0 \le \beta < 1$ and $N = -1$, we have $C_q^*[1-2\beta,-1] =: C_q^*(\beta)$, and Theorem 2.4 reduces to the following result:

Corollary 2.5. *Let T_j^*, $j \in \{1,2,3\}$, be defined as follows:*
(i) If $\xi,\eta > 0$ and $\gamma > \xi+\eta$, then T_1^* is given by

$$T_1^*(\xi,\eta,\gamma,q) := \frac{1}{(1-q)^2}\Big\{2\Big(1-\beta(1-q)\Big)\frac{\Gamma(\gamma)\Gamma(\gamma-\xi-\eta)}{\Gamma(\gamma-\xi)\Gamma(\gamma-\eta)}$$
$$- 2\Big(2-\beta(1-q)\Big)qF(\xi,\eta;\gamma;q) + 2q^2F(\xi,\eta;\gamma;q^2) - 2(1-\beta)(1-q)^2\Big\}.$$

(ii) If $-1 < \xi < 0$, $\eta > 0$ and $\gamma > \max\{0;\xi+\eta\}$, then T_2^* is given by

$$T_2^*(\xi,\eta,\gamma,q) := \frac{1}{(1-q)^2}\Big\{2\Big(\beta(1-q)-1\Big)\frac{\Gamma(\gamma)\Gamma(\gamma-\xi-\eta)}{\Gamma(\gamma-\xi)\Gamma(\gamma-\eta)} - 2\Big(2-\beta(1-q)\Big)qF(|\xi|,\eta;\gamma;q) + 2q^2F(|\xi|,\eta;\gamma;q^2)$$
$$+ 2(1+2q-q^2) - 2\beta(1-q^2)\Big\}.$$

(iii) If $\xi,\eta\in\mathbb{C}\setminus\{0\}$ and $\gamma>|\xi|+|\eta|$, then T_3^* is given by

$$T_3^*(\xi,\eta,\gamma,q):=\frac{1}{(1-q)^2}\Big\{2\Big(1-\beta(1-q)\Big)\frac{\Gamma(\gamma)\Gamma(\gamma-|\xi|-|\eta|)}{\Gamma(\gamma-|\xi|)\Gamma(\gamma-|\eta|)}-2\Big(2-\beta(1-q)\Big)qF(|\xi|,|\eta|;\gamma;q)$$
$$+2q^2F(|\xi|,|\eta|;\gamma;q^2)-2(1-\beta)(1-q)^2\Big\}.$$

If for any $j\in\{1,2,3\}$ the inequality

$$T_j^*(\xi,\eta,\gamma,q)<2(1-\beta)$$

holds for $0\le\beta<1$, then function $zF(\xi,\eta;\gamma;z)$ belongs to the class $C_q^*(\beta)$.

For $\beta=0$ the above corollary gives us the next example:

Example 2.6. *Let $\widetilde{T}_j$, $j\in\{1,2,3\}$, be defined as follows:*
(i) If $\xi,\eta>0$ and $\gamma>\xi+\eta$, then $\widetilde{T}_1$ is given by

$$\widetilde{T}_1(\xi,\eta,\gamma,q):=\frac{1}{(1-q)^2}\Big\{\frac{2\Gamma(\gamma)\Gamma(\gamma-\xi-\eta)}{\Gamma(\gamma-\xi)\Gamma(\gamma-\eta)}-4qF(\xi,\eta;\gamma;q)2q^2F(\xi,\eta;\gamma;q^2)-2(1-q)^2\Big\}.$$

(ii) If $-1<\xi<0$, $\eta>0$ and $\gamma>\max\{0;\xi+\eta\}$, then $\widetilde{T}_2$ is given by

$$\widetilde{T}_2(\xi,\eta,\gamma,q):=\frac{1}{(1-q)^2}\Big\{-\frac{2\Gamma(\gamma)\Gamma(\gamma-\xi-\eta)}{\Gamma(\gamma-\xi)\Gamma(\gamma-\eta)}-4qF(|\xi|,\eta;\gamma;q)+2q^2F(|\xi|,\eta;\gamma;q^2)+2(1+2q-q^2)\Big\}.$$

(iii) If $\xi,\eta\in\mathbb{C}\setminus\{0\}$ and $\gamma>|\xi|+|\eta|$, then $\widetilde{T}_3$ is given by

$$\widetilde{T}_3(\xi,\eta,\gamma,q):=\frac{1}{(1-q)^2}\Big\{\frac{2\Gamma(\gamma)\Gamma(\gamma-|\xi|-|\eta|)}{\Gamma(\gamma-|\xi|)\Gamma(\gamma-|\eta|)}-4qF(|\xi|,|\eta|;\gamma;q)+2q^2F(|\xi|,|\eta|;\gamma;q^2)-2(1-q)^2\Big\}.$$

If for any $j\in\{1,2,3\}$ the inequality

$$\widetilde{T}_j(\xi,\eta,\gamma,q)<2$$

holds, then function $zF(\xi,\eta;\gamma;z)$ belongs to the class $C_q^*(0)$.

3. Conclusion

In this paper, sufficient conditions for q-starlikenss and q-convexity for a function asscoaitated with normalized Gauss hypergeometric functions are obtained. Motivated by a recently-published survey-cum-expository review article by Srivastava [b], the interested reader's attention is drawn toward the possibility of investigating the basic (or q-) extensions of the results which are presented in this paper. However, as already pointed out by Srivastava [17], their further extensions using the so-called (p,q)-calculus will be rather trivial and inconsequential variations of the suggested extensions which are based upon the classical q-calculus, the additional parameter p being redundant or superfluous (see, for details, ([17], p. 340))

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