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Some New Classes of Preinvex Functions and Directional Variational-Like Inequalities

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Dedicated to our respected Parents and Teachers

Abstract. In this paper, we introduce and study some new classes of invex sets and preinvex functions with respect to an arbitrary function k and the bifunction $\eta(.,.)$; which are called the generalized preinvex functions. These functions are nonconvex functions and include the preinvex function, convex functions and k-convex as special cases. We study some properties of generalized preinvex functions. It is shown that the minimum of generalized preinvex functions on the generalized invex sets can be characterized by a class of variational inequalities, which is called the directional variational-like inequalities. Using the auxiliary technique, several new inertial type methods for solving the directional variational-like inequalities are proposed and analyzed . Convergence analysis of the proposed methods is considered under suitable conditions. Some open problems are also suggested for future research.

1. Introduction

In recent years, several extensions and generalizations of the convex sets and convex functions have been considered and investigated. Hanson [6] introduced the concept of invex function for the differentiable functions, which played significant part in the mathematical programming. Ben-Israel and Mond [1] introduced the concept of invex set and preinvex functions. It is known that the differentiable preinvex function are invex functions. The converse also holds under certain conditions, see [11]. Noor [17] proved that the minimum of the differentiable preinvex functions on the invex set can be characterized by a class of variational inequalities, which is known as the variational-like inequality. It is worth mentioning that variational-like inequalities include variational inequalities, the origin of which can be traced back to Stampachia [36]. Variational inequalities can be viewed as a novel and significant extension of the variational principles. For the recent developments in variational-like inequalities and invex equilibrium problems, see [15, 17, 18, 32, 33] and the references therein. These results have inspired a great deal of subsequent work, which has expanded the role and applications of the convexity in nonlinear optimization and engineering sciences. Noor et al. [26–29] investigated the properties of the strongly preinvex functions and their variant forms.

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In a many problems, a set may not be a convex set. To overcome this drawback, the underlying set can be made a k-convex set with respect to an arbitrary function. Micherda et al [10] and Hazy [7] defined the so-called (h, k) convex function which is a natural generalization of the usual convexity, the s-convexity in the first and second sense. Noor [21, 25] introduced the k-convex functions and studied their characterizations. It is worth mentioning that for $k(t) = te^{i\varphi}$, the φ-convex functions were introduced and studied by Noor [21].

We would like to point out that invex sets, preinvex functions and k-convex sets, k-convex functions are distinctly different generalizations of convex sets and convex functions in various directions. These type of functions have played a leading role in the developments of various branches of pure and applied sciences. It is natural to unify these classes and investigate their characterizations. Motivated and inspired by the recent activities in these ares, we introduce some new classes of invex sets and preinvex functions which are called modified generalized invex sets and generalized preinvex functions. These new class of generalized invex sets and generalized preinvex functions include the φ-invex sets, φ-preinvex and Toader type k-convex sets and k-convex functions. The new class of generalized preinvex functions can be viewed as modified refinement of the (h, k) convex functions of Hazy. Several new concepts are defined and their properties have been studied. We prove that the minimum of the differential generalized preinvex functions on the generalized invex sets can be characterized by a class of variational-like inequalities, which are called directional variational-like inequalities. This results inspired us to consider the directional variational-like inequalities. It is well known that the projection methods, resolvent methods and their variant forms can not be used to solve the directional variational-like inequalities due to their nature. To overcome this drawback, one usually use the auxiliary principle technique, which is mainly due to Glowinski et al [5], which has been used [15–18, 20, 32, 33, 40]to suggest and analyze several new iterative methods for solving a wide class of unrelated problems arising in pure and applied sciences.. We again use the auxiliary principle technique to suggest some inertial proximal iterative methods for solving directional variational-like inequalities. Convergence analysis of the new proposed methods is considered under pseudomonontoncity and partially strongly monotonicity. Our method of convergence proof is very simple as compared with other methods. We have tried to convey the basic characterizations of these new classes of generalized preinvex functions and their applications in optimization theory along with some open problems.

2. Preliminaries

Let K_k be a nonempty closed set in a normed space H. We denote by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ the inner product and norm, respectively.

Definition 2.1. *The set $K_{k\eta}$ is said to be generalized invex set with respect to arbitrary function k and the bifunction $\eta(.,.)$,if*

$$u + k(t)\eta(v, u) \in K_{k\eta}, \qquad \forall u, v \in K_{k\eta}, \quad t \in [0, 1].$$

Clearly, for $k(t) = t$, the set $K_{k\eta}$ is an invex set K_η, which was introduced and studied by Ben-Israel and Mond[1]

If $k(t) = t^s, \quad s \in (0, 1]$ then the generalized invex set $K_{k\eta}$ reduces to:

$$u + t^s\eta(v, u) \in K_{k\eta}, \quad \forall u, v \in K_{k\eta}, \quad t \in [0, 1],$$

which is known as Toader type $k\eta$-invex set and appears to be a new one.
If $\eta(v, u) = v - u$, then the sets $K_{k\eta}$ reduces to the set k-convex sets K_k, which have been introduced and studied in [2, 7, 10].

From now onwards, the set $K_{k\eta}$ is a generalized invex set, unless otherwise specified.
We now introduce the concept of generalized preinvex function with respect to an arbitrary function k and bifunction $\eta(.,.)$.

Definition 2.2. *The function f on $K_{k\eta}$ is called generalized preinvex function, if there exists an arbitrary function k and bifunction $\eta(.,.)$, such that*

$$f(u+k(t)\eta(v,u)) \leq (1-k(t))f(u)+k(t)f(v), \quad \forall u,v \in K_{k\eta},\ t \in [0,1].$$

Obviously every preinvex function with $k(t) = t$ is a generalized preinvex function, but the converse may not be true. For the properties and applications of preinvex functions, see [11, 17, 18, 21–23, 26–28, 30, 31, 35, 38, 39] and the references therein
Also for $t = 1$, the generalized convex function reduces to:

$$f(u+k(1)\eta(v,u)) \leq f(v), \qquad \forall u,v \in K_{k\eta}. \tag{2.1}$$

If $k(t) = t^s, \quad s \in (0,1]$, then we have a new class of preinvex functions, which is called Toader's type preinvex functions.

Definition 2.3. *The function f on $K_{k\eta}$ is said to be quasi generalized preinvex function , if there exist a function k and the bifunction $\eta(.,.)$, such that*

$$f(u+k(t)\eta(v,u)) \leq \max\{f(u),f(v)\}, \qquad \forall u,v \in K_{k\eta}, \quad t \in [0,1].$$

Definition 2.4. *The function f on $K_{k\eta}$ is said to be logarithmic generalized preinvex function, if there exist a function k and the bifunction $\eta(.,.)$,such that*

$$f(u+k(t)\eta(v,u)) \leq (f(u))^{1-k(t)}(f(v))^{k(t)}, \quad \forall u,v \in K_{k\eta}, \quad t \in [0,1],$$

where $f(\cdot) > 0$.

From the above definitions, we have

$$\begin{aligned} f(u+k(t)\eta(v,u)) &\leq (f(u))^{1-k(t)}(f(v))^{k(t)} \\ &\leq (1-k(t))f(u)+k(t)f(v) \\ &\leq \max\{f(u),f(v)\}, \quad \forall u,v \in K_{k\eta}, \quad t \in [0,1], \end{aligned}$$

Logarithmic generalized preinvex function $\longrightarrow$ generalized preinvex functions and generalized preinvex functions $\longrightarrow$ quasi generalized preinvex functions, but the converse is not true.

We also need the following assumption regarding the bifunction $\eta(\cdot,\cdot)$ and the function $k(t)$.
Condition M. Let $\eta(\cdot,\cdot) : K_{k\eta} \times K_{k\eta} \to H$ satisfy assumptions

$$\begin{aligned} &\eta(u,u+k(t)\eta(v,u)) = -k(t)\eta(v,u) \\ &\eta(v,u+k(t)\eta(v,u)) = (1-k(t))\eta(v,u), \quad \forall u,v \in K_{k\eta}, t \in [0,1]. \end{aligned}$$

Clearly for $k(t) = 0$, we have $\eta(u,v) = 0$, if and only if $u = v, \forall u,v \in K_{k\eta}$. One can easily show [11, 12] that $\eta(u+k(t)\eta(v,u),u) = k(t)\eta(v,u), \forall u,v \in K_{k\eta}$.
For $k(t) = t$, Condition M reduces to the Condition C, which is mainly due to Mohan and Neogy [11].

3. Properties of generalized preinvex functions

In this section, we discuss the properties of generalized preinvex functions and their variant forms.

Lemma 3.1. *Let f be a generalized preinvex function. Then any local minimum of f on $K_{k\eta}$ is a global minimum.*

Proof. Let the generalized preinvex function f have a local minimum at $u \in K_{k\eta}$. Assume the contrary, that is, $f(v) < f(u)$ for some $v \in K_{k\eta}$. Since f is a k-preinvex function, so

$$f(u + k(t)\eta(v,u)) \leq f(u) + k(t)(f(v) - f(u)),$$

which implies that

$$f(u + k(t)\eta(v,u)) - f(u) < 0,$$

for arbitrary small $k(t) > 0$, contradicting the local minimum. □

Essentially using the technique and ideas of the classical convexity [12], one can easily prove the following results.

Theorem 3.2. *If f is a generalized preinvex function on the generalized invex set $K_{k\eta}$, then the level set $L_\alpha = \{u \in K_{k\eta} : f(u) \leq \alpha, \alpha \in \mathbb{R}\}$ is a generalized invex set with respect to the function k and bifunction $\eta(.,.)$.*

Theorem 3.3. *The function f is a generalized preinvex function, if and only if, $epi(f) = \{(u,\alpha) : u \in K_{k\eta}, \alpha \in \mathbb{R}, f(u) \leq \alpha\}$ is a generalized invex set with respect to the function k and bifunction $\eta(.,.)$.*

Theorem 3.4. *The function f is a quasi generalized preinvex function, if and only if, the level set $L_\alpha = \{u \in K_{k\eta} : f(u) \leq \alpha, \alpha \in \mathbb{R}\}$ is a generalized invex set with respect to the function k and the bifunction $\eta(.,.)$.*

Definition 3.5. *The function f is said to be a pseudo generalized preinvex function with respect to the function k and the bifunction $\eta(.,.)$, if there exists a strictly positive bifunction $W(\cdot,\cdot)$ such that*

$$\begin{aligned} f(v) &< f(u) \Rightarrow \\ f(u + k(t)\eta(v,u)) &\leq f(u) + k(t)(k(t) - 1)W(u,v), \qquad \forall u,v \in K_{k\eta}, \quad t \in (0,1). \end{aligned}$$

Theorem 3.6. *If the function f is a generalized preinvex function, then f is pseudo generalized preinvex function.*

Proof. Without loss of generality, we assume that $f(v) < f(u)$, $\forall u,v \in K_{k\eta}$. For every $t \in [0,1]$, we have

$$\begin{aligned} f(u + k(t)\eta(v,u)) &\leq (1 - k(t))f(u) + k(t)f(v) \\ &< f(u) + k(t)(k(t) - 1)\{f(u) - f(v)\} \\ &= f(u) + k(t)(k(t) - 1)W(u,v), \end{aligned}$$

where $W(u,v) = f(u) - f(v) > 0$. Thus, it follows the function f is a pseudo generalized preinvex function, which is the required result. □

Theorem 3.7. *Let f be a generalized preinvex function. If $g : L \to \mathbb{R}$ is a nondecreasing function, then $g \circ f$ is a generalized preinvex function.*

Proof. Since f is a generalized preinvex function and g is decreasing, we have, $\forall u,v \in K_{k\eta}, \quad t \in [0,1]$

$$\begin{aligned} g \circ f(u + k(t)\eta(v,u)) &\leq g[(1 - k(t))f(u) + k(t)f(v)] \\ &\leq (1 - k(t))g \circ f(u) + k(t)g \circ f(v), \end{aligned}$$

from which it follows that $g \circ f$ is a generalized preinvex function. □

We now introduce the concept of k-directional derivative.

Definition 3.8. *We define the k-directional derivative of f at a point $u \in K_{k\eta}$ in the direction $v \in K_{k\eta}$ by*

$$\begin{aligned} Df(u,\eta(v,u)) : &= f'_{k\eta}(u;\eta(v,u)) \\ &= \lim_{k(t)\to 0^+} \{\frac{f(u + k(t)\eta(v,u)) - f(u)}{k(t)}\}. \end{aligned}$$

Note that for $k(t) = t$ and $\eta(v, u) = v$, the k-directional derivative of f at $u \in K$ in the direction $v \in K$ coincides with the usual directional derivative of f at u in a direction v given by

$$Df(u, v) := f'(u; v) = \lim_{t \to 0^+} \frac{f(u + tv) - f(u)}{t}.$$

It is well known that the function $v \to f'_{k\eta}(u; \eta(v, u))$ is subadditive, positively homogeneous.

Definition 3.9. *The differentiable function f on $K_{k\eta}$ is said to be k-invex, if*

$$f(v) - f(u) \geq f'_{k\eta}(u; \eta(v, u)), \quad \forall u, v \in K_{k\eta},$$

where $f'_{k\eta}(u; \eta(v, u))$ is the k-directional derivative of f at $u \in K_{k\eta}$ in the direction of $v \in K_{k\eta}$.

Theorem 3.10. *Let f be a differential k-preinvex function on the k-invex set $K_{k\eta}$. Then the function $v \to f'_{k\eta}(u; \eta(v, u))$ is positively homogeneous and generalized preinvex function.*

Proof. It is follow from the definition of the k-directional derivative that $f'_{k\eta}(u; \lambda\eta(v, u)) = \lambda f'_{k\eta}(u; \eta(v, u))$, whenever $v \in K_{k\eta}$ and $\lambda \geq 0$. Thus the function $v \to f'_{k\eta}(u; \eta(v, u))$ is positively homogeneous.
To prove the generalized preinvexity of the function $v \to f'_{k\eta}(u; \eta(v, u))$, we consider $\forall u, v, z \in K_{k\eta}, \quad k(t) \geq 0$, $\lambda \in (0, 1)$,

$$\begin{aligned}
& \frac{1}{t}[f(u + k(t)(\lambda v + (1 - \lambda)\eta(v, z))) - f(u)] \\
= \ & \frac{1}{k(t)}[f(\lambda(u + k(t)\eta(v, u)) + (1 - \lambda)(u + k(t)\eta(z, u))) - f(u)] \\
\leq \ & \frac{1}{k(t)}[\lambda f(u + k(t)\eta(v, u)) + (1 - \lambda)f(u + k(t)\eta(z, u)) - f(u)] \\
= \ & \lambda \frac{f(u + k(t)\eta(v, u)) - f(u)}{k(t)} + (1 - \lambda)\frac{f(u + k(t)\eta(z, u)) - f(u)}{k(t)}.
\end{aligned} \tag{3.1}$$

Taking the limit as $k(t) \to 0^+$ in (3.1), we have

$$f'_{k\eta}(u; \lambda\eta(v, u) + (1 - \lambda)z) \leq \lambda f'_{k\eta}(u; \eta(v, u)) + (1 - \lambda)f'_{k\eta}(u; \eta(z, u)),$$

which shows that the function $v \to f'_{k\eta}(u; \eta(v, u))$ is k-preinvex, which is the required result. □

For $k(t) = t$, the generalized preinvex function f becomes the preinvex function and the generalized invex set K_k is an invex set.

Theorem 3.11. *Let $K_{k\eta}$ be a generalized invex set. If function $f : K_{k\eta} \to \mathbb{R}$ is differentiable generalized preinvex function such that $k(0) = 0$, and (2.1) holds, then the following statements are equivalent.*

1. *f is a generalized invex function.*
2. *$f(v) - f(u) \geq f'(u; \eta(v, u)), \quad \forall u, v \in K_{k\eta}$.*
3. *$k\eta$-directional derivative $f'_{k\eta}(\cdot, \cdot)$ of f is k-monotone, that is,*

$$f'_{k\eta}(u; \eta(v, u)) + f'_{k\eta}(v; \eta(u, v)) \leq 0, \quad \forall u, v \in K_{k\eta}.$$

Proof. Let f be a generalized preinvex function. Then

$$f(u + k(t)\eta(v, u)) \leq f(u) + k(t)\{f(v) - f(u)\} \quad \forall u, v \in K_{k\eta}, \quad t \in [0, 1],$$

which can be written as

$$(f(v) - f(u)) \geq \{\frac{f(u + k(t)\eta((v, u)) - f(u)}{k(t)}\}. \tag{3.2}$$

Taking the limit as $k(t) \to 0^+$ in (3.2), we have

$$f(v) - f(u) \geq f'_{k\eta}(u; \eta(v,u)), \quad \forall u, v \in K_{k\eta}, \tag{3.3}$$

showing that the generalized preinvex function f is a generalized invex function.

Changing the role of u and v in (3.3), we have

$$f(u) - f(v) \geq f'_{k\eta}(v; \eta(u,v)), \quad \forall u, v \in K_{k\eta}, \tag{3.4}$$

Adding (3.3) and (3.4), we have

$$f'_{k\eta}(u; \eta(v,u)) + f'_{k\eta}(v; \eta(u,v)) \leq 0, \quad \forall u, v \in K_{k\eta}, \tag{3.5}$$

which shows that the k-directional derivative $f'_{k\eta}(\cdot,\cdot)$ is k-monotone.

Conversely, let (3.5) hold. Since $K_{k\eta}$ is a k-invex set, so

$$\forall u, v \in K_{k\eta}, \quad t \in [0,1], \quad v_t = u + k(t)\eta(v,u) \in K_{k\eta}.$$

Replacing v by v_t in (3.5) and simplifying, we have

$$f'_{k\eta}(v_t; \eta(v,u)) \geq f'_{k\eta}(u; \eta(v,u)), \quad \forall u, v \in K_{k\eta}. \tag{3.6}$$

Consider the auxiliary function

$$\begin{aligned} \zeta(t) &= f(u + k(t)\eta(v,u))) - f(u) \\ &\quad + t f'_{k\eta}(u; \eta(v,u)), \forall u, v \in K_{k\eta}. \end{aligned} \tag{3.7}$$

Using $k(0) = 0$, we have

$$\begin{aligned} \zeta(0) &= 0 \quad , \\ \zeta(1) &= f(u + k(1)\eta(v,u)) - f(u) + f'_{k\eta}(u : \eta(v,u)). \end{aligned} \tag{3.8}$$

Since f is differentiable, so the function $\zeta(t)$ is also differentiable. Hence, using (3.6), we have

$$\begin{aligned} \zeta'(t) &= f'(u + k(t)\eta(v,u)), \eta(v,u)) \\ &\geq f'_{k\eta}(u; \eta(v,u)). \end{aligned} \tag{3.9}$$

Integrating the inequality (3.9) on the interval $[0,1]$ and using (3.8), we have

$$\begin{aligned} f(u + k(1)\eta(v,u)) - f(u) + f'_{k\eta}(u : \eta(v-u)) &= \zeta(1) - \zeta(0) \\ &\geq \int_0^1 f'_{k\eta}(u; \eta(v,u)) dt \\ &= f'_{k\eta}(u; \eta(v,u)), \end{aligned}$$

from which, using (2.1), we obtain

$$f'_{k\eta}(u; \eta(v,u)) \leq f(u + k(1)\eta(v,u)) - f(u) \leq f(v) - f(u).$$

which is the required(3.3).
Now from (3.3), we have

$$\begin{aligned} f(v) - f(u + k(t)\eta(v,u)) &\geq f'_{k\eta}(u + k(t)\eta(v,u)); \eta(v, u + k(t)\eta(v,u))) \\ &= (1 - k(t)) f'_{k\eta}(u + k(t)\eta(v,u)); \eta(v,u))) \end{aligned} \tag{3.10}$$

$$\begin{aligned} f(u) - f(u + k(t)\eta(v,u)) &\geq f'_{k\eta}(u + k(t)\eta(v,u)); \eta(u, u + k(t)\eta(v,u))) \\ &= -k(t) f'_{k\eta}(u + k(t)\eta(v,u)); \eta(v,u))).. \end{aligned} \tag{3.11}$$

Multiplying (3.10) by $k(t)$, (3.11) by $(1-k(t))$ and adding the resultant, we have

$$f(u+k(t)\eta(v,u)) \leq f(u)+k(t)\{f(v)-f(u)\} \quad \forall u,v \in K_{k\eta}, \quad \in [0,1],$$

which shows that the function f is a generalized preinvex function. □

Theorem 3.12. *Let the differential $f'_{k\eta}(.;.)$ of the generalized preinvex function f be Lipschitz continuous with constant $\beta \geq 0$. If $k(0)=0$, then*

$$\begin{aligned} & f(u+k(1)\eta(v,u))-f(u) \\ & \leq f'_{k\eta}(u;\eta(v,u))+\beta\|\eta(v,u)\|^2\int_0^{\prime} k(t)dt, \quad \forall u,v \in K_{k\eta}. \end{aligned} \tag{3.12}$$

Proof. Since $K_{k\eta}$ is a generalized invex set, $\forall u,v \in K_{k\eta}, \quad t \in [0,1]$, we consider the function

$$\varphi(t)=f(u+k(t)\eta(v,u))-f(u)-tf'_{k\eta}(u;\eta(v,u)).$$

Using $k(0)=0$, we obtain

$$\varphi(0)=0, \quad \varphi(1)=f(u+k(1)\eta(v,u))-f(u)-f'_{k\eta}(u;\eta(v,u)).$$

Also

$$\varphi'(t)=f'_k(u+k(t)\eta(v,u);\eta(v,u))-f'{}_k(u;\eta(v,u)). \tag{3.13}$$

Integrating (3.13) on the interval $[0,1]$ and using the Lipschitz continuity of $f'_k(.;.)$ with constant $\beta \geq 0$, we have

$$\begin{aligned} \varphi(1) &= f(u+k(1)\eta(v,u))-f(u)-f'_k(u;\eta(v,u)) \\ &\leq \int_0^1 |\varphi'(t)|dt \\ &= \int_0^{\prime} |f'_k(u+k(t)\eta(v,u));\eta(v,u))-f'{}_k(u;\eta(v,u))|dt \\ &\leq \beta\int_0^{\prime} k(t)\|\eta(v,u)\|^2dt=\beta\|\eta(v,u)\|^2\int_0^{\prime} k(t)dt, \end{aligned}$$

□

4. Directional variational-like inequalities

In this section, we introduce and consider a new class of variational-like inequalities, which is called directional variational-like inequality.
For given bifunctions $D(.,.),\eta(.,.): K_{k\eta}\times K_{k\eta} \longrightarrow R$, we consider the problem of finding $u \in K_{k\eta}$ such that

$$D(u,\eta(v,u)) \geq 0, \quad \forall v \in K_{k\eta}, \tag{4.1}$$

which is called the *directional variational-like inequality.*
We now show that the inequality (4.1) naturally arises as a minimum of the differentiable generalized preinvex functions on the generalized invex sets. This is the main motivation of our next result.

Theorem 4.1. *Let f be a differentiable generalized preinvex function on the generalizedinvex $K_{k\eta}$. Then the $u \in K_{k\eta}$ is the minimum of the differentiable generalized preinvex function f on the generalized invex set $K_{k\eta}$, if and only if, $u \in K_{k\eta}$ satisfies the inequality*

$$f'_{k\eta}(u;\eta(v,u)) \geq 0, \quad \forall u,v \in K_{k\eta}. \tag{4.2}$$

Proof. Let $u \in K_{k\eta}$ be a minimum of the generalized preinvex function f. Then

$$f(u) \leq f(v), \quad \forall v \in K_{k\eta}. \tag{4.3}$$

Since K is generalized invex set, so, $\forall u, v, \in K_{k\eta}, \quad t \in [0,1]$,
$v_t = u + k(t)\eta(v,u) \in K_{k\eta}$. Taking $v = v_t$ in (4.3), we have

$$f(u) \leq f(v_t) = f(u + k(t)\eta(v,u)),$$

which implies that

$$\frac{f(u + k(t)\eta(v,u)) - f(u)}{k(t)} \geq 0.$$

Taking the limit as $t \to 0^+$ in the above inequality, we have

$$f'_k(u;\eta(v,u)) \geq 0 \quad \forall v \in K_{k\eta},$$

the required (4.2).
Conversely, let $u \in K_{k\eta}$ be a solution of (4.2). Since f is a generalized preinvex function, it follows that

$$f(v) - f(u) \geq f'_{k\eta}(u;\eta(v,u)) \geq 0,$$

which implies that

$$f(u) \leq f(v), \quad \forall v \in K_{k\eta},$$

showing that $u \in K_{k\eta}$ is the minimum of the generalized preinvex function f, the required result. □

The inequality of the type (4.2) is called the directional variational-like inequality, which is a special case of directional variational-like inequality (4.1).

For $k(t) = t$, the generalized preinvex functions reduces to preinvex function, then the problem (4.2) coincides with classical directional variational-like inequalities. It is worth mentioning that even the directional variational-like inequalities have not been studied in the literature.

We now discuss some important special cases of directional variational-like inequalities.

Special Cases

(I). We note that, if $K_{k\eta} \equiv K_\eta$, the invex set in H, then problem (4.1) is equivalent to finding $u \in K_\eta$ such that

$$D(u, \eta(v,u)) \geq 0, \qquad \forall v \in K_\eta. \tag{4.4}$$

Inequality of type (4.4) is called the *directional variational-like inequality,* which appears to be a new one.

(II). If $D(u, \eta(v,u)) = \langle Tu, \eta(v,u)\rangle, \quad K_{k\eta} = K_\eta$, where T is a nonlinear operator, then problem (4.1) is equivalent to finding $u \in K_\eta$ such that

$$\langle Tu, \eta(v,u)\rangle \geq 0, \quad \forall v \in K_\eta, \tag{4.5}$$

which is called the variational-like inequality, see Noor [16–21].

(III). If $D(u, \eta(v,u)) = \langle Tu, v-u\rangle$, where T is a nonlinear operator and $K_{k\eta} = K$, the convex set, then problem (4.3) is equivalent to finding $u \in H : g(u) \in K_r$ such that

$$\langle Tu, v-u\rangle \geq 0, \quad \forall v \in K, \tag{4.6}$$

which is called variational inequality, introduced and studied by Stampachia [36]. It has been shown a wide class of obstacle boundary value and initial value problems which arise in various branches of pure and applied sciences can be studied in the general framework of variational inequalities (4.11). For the applications, numerical methods, sensitivity analysis, dynamical system, merit functions and other aspects of variational inequalities, see [5, 8, 13, 16–18, 20, 32, 33, 36, 40] and the references therein.

It is worth mentioning that for suitable and appropriate choice of the operators, generalized invex sets and spaces, one can obtain a wide class of variational inequalities and optimization programming. This shows that the directional variational-like inequalities are quite flexible and unified ones.

We now recall the following concepts and results.

Definition 4.2. *A bifunction $D(.,.) : H \times H \to H$ is said to be:*
(i) k-monotone, if and only if,

$$D(u, \eta(v, u)) + D(v, \eta(u, v)) \leq 0, \quad \forall u, v \in H.$$

(ii) k-pseudomonotone, if and only if,

$$D(u, \eta(v.u))) \geq 0 \quad \textit{implies that} \quad - D(v, \eta(u, v)), \geq 0, \quad \forall u, v \in H.$$

(iii) partially relaxed strongly k-monotone, if and only if, there exists a constant $\alpha > 0$ such that

$$D(u, \eta(v, u))) + D(v, \eta(z, v)) \leq \alpha \|\eta(z, u)\|^2, \quad \forall u, v, z \in H.$$

Note that for $z = u$, partially relaxed strongly k-monotonicity reduces to η-monotonicity. It is known that η-monotonicity implies k-pseudomonotonicity; but the converse is not true.

We also recall the well-known result.

$$2\langle u, v\rangle = \|u + v\|^2 - \|u\|^2 - \|v\|^2, \quad \forall u, v \in H. \tag{4.7}$$

Theorem 4.3. *Let the bifucntion $D(.,.)$ be k-pseudo-monotone, hemicontinuous and $\lim_{t \longrightarrow 0} k(t) = 0$. If Condition M holds, then the directional variational-like inequality is equivalent to finding $u \in K_{k\eta}$ such that*

$$D(v, \eta(u, v)) \geq 0, \quad \forall v \in K_{k\eta}. \tag{4.8}$$

Proof. Let $u \in K_{k\eta}$ be a solution of inequality (4.1). Then, using the k-pseudo monotonicity of the bifunction $D(.,.)$, we have

$$-D(v, \eta(u, v)) \geq 0, \quad \forall v \in K_{k\eta}. \tag{4.9}$$

Since $K_{k\eta}$ is a generalized invex set, so, $\forall u, v \in K_{k\eta}, \quad t \in [0, 1]$,
$v_t = u + k(t)\eta(v, u) \in K_{k\eta}$.
Replacing v by v_t in (4.9) and using Condition M, we obtain

$$\begin{aligned} -D(v_t, \eta(u, v_t)) &= -D(u + k(t)\eta(v, u); \eta(u, u + k(t)\eta(v, u))) \\ &= k(t)D(u + k(t)\eta(v, u); \eta(v, u)) \geq 0, \end{aligned}$$

which implies that

$$D(u + k(t)\eta(v, u), \eta(v, u)) \geq 0, \quad \forall v \in K_{k\eta}$$

Using the hemicontinuity of the bifunction $D(.,.)$ and taking the limit, we obtain the inequality (4.1), since $\lim_{t \longrightarrow 0} k(t) = 0$. □

Remark 4.4. *We would like to mention that the inequality of the type (4.9) is known as the Minty directional variational-like inequality or dual directional variational-like inequality. Using this equivalent result, one can show that the solution set of the directional variational-like inequalities is a closed generalized invex set.*

Due to the inherent nonlinearity, the projection method and its variant form can not be used to suggest the iterative methods for solving these directional variational-like inequalities. To overcome these drawback, we now use the auxiliary principle technique of Glowinski et al.[10] as developed in [15–18, 20, 32, 33, 40] to suggest and analyze some iterative methods for solving the directional k-variational-like inequality (4.1). This technique does not involve the concept of the projection, which is the main advantage of this technique.

For a given $u \in K_{k\eta}$ satisfying (4.1), consider the problem of finding $w \in\in K_{k\eta}$ such that

$$\rho D(w, \eta(v, w)) + \langle w - u, v - w\rangle \geq 0, \forall v \in K_{k\eta}, \tag{4.10}$$

where $\rho > 0$ is a constant. Inequality of type (4.10) is called the auxiliary directional variational-like inequality. Note that if $w = u$, then w is a solution of (4.1). This simple observation enables us to suggest the following iterative method for solving the directional k-variational-like inequalities (4.1).

Algorithm 4.5. *For a given $u_0 \in K_{k\eta}$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\rho D(u_{n+1}, \eta(u, u_{n+1})) + \langle u_{n+1} - u_n, v - u_{n+1}\rangle \geq 0, \forall v \in K_{k\eta}. \tag{4.11}$$

Algorithm 4.5 is called the proximal point algorithm for solving directional k-variational-like inequality (4.1).
If $K_{k\eta} = K_\eta$, then the k-invex set $K_{k\eta}$ becomes the standard invex set K_η, and consequently Algorithm 4.5 reduces to:

Algorithm 4.6. *For a given $u_0 \in K_\eta$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\rho D(u_{n+1}, \eta(v, u_{n+1})) + \langle u_{n+1} - u_n, v - u_{n+1}\rangle \geq 0, \quad \forall v \in K_\eta,$$

which is known as the proximal point algorithm for solving directional variational-like inequalities (4.2) , which appears to be a new one.

We now consider the convergence criteria of Algorithm 4.5 and this is the main motivation of our next result.

Theorem 4.7. *Let the operator $D(.,.) : K_{k\eta} \times K_{k\eta} \longrightarrow H$ be k-pseudomonotone. If u_{n+1} is the approximate solution obtained from Algorithm 4.5 and $u \in K_{k\eta}$ is a solution of (4.1), then*

$$\|u - u_{n+1}\|^2 \leq \|u - u_n\|^2 - \|u_n - u_{n+1}\|^2. \tag{4.12}$$

Proof. Let $u \in K_{k\eta}$ be a solution of (4.1). Then

$$-D(g(v, \eta(u, v)) \geq 0, \quad \forall v \in K_{k\eta}, \tag{4.13}$$

since $D(.,.)$ is k-pseudomonotone.
Taking $v = u_{n+1}$ in (4.13), we have

$$-D(g(u_{n+1}), \eta(u, u_{n+1})) \geq 0. \tag{4.14}$$

Setting $v = u$ in (4.2), and using (4.11), we have

$$\langle u_{n+1} - u_n, u - u_{n+1}\rangle \geq -\rho D(u_{n+1}, \eta(u, u_{n+1})) \geq 0. \tag{4.15}$$

Setting $v = u - u_{n+1}$ and $u = u_{n+1} - u_n$ in (4.7), we obtain

$$\begin{aligned} 2\langle u_{n+1} - u_n, u - u_{n+1}\rangle &= \|u - u_n\|^2 - \|u_n - u_{n+1}\|^2 \\ &\quad -\|u - u_{n+1}\|^2. \end{aligned} \tag{4.16}$$

From (4.15) and (4.16), we obtain (4.12), which is the required result. □

Theorem 4.8. *Let H be a finite dimension subspace and let u_{n+1} be the approximate solution obtained from Algorithm 4.5. If $u \in K_{k\eta}$ is a solution of (4.1), then* $\lim_{n\longrightarrow\infty} u_n = u$.

Proof. Let $u \in K_{k\eta}$ be a solution of(4.1). Then it follows from (4.12) that the sequence $\{u_n\}$ is bounded and

$$\sum_{n=0}^{\infty} \|u_n - u_{n+1}\|^2 \leq \|u_0 - u\|^2,$$

which implies that

$$\lim_{n\longrightarrow\infty} \|u_n - u_{n+1}\| = 0. \tag{4.17}$$

Let $\hat{u}$ be a cluster point of the sequence $\{u_n\}$ and let the subsequence $\{u_j\}$ of the sequence $\{u_n\}$ converge to $\hat{u} \in K_{k\eta}$. replacing u_n by u_{n_j} in (4.17) and taking the limit $n_j \longrightarrow \infty$ and using (4.11), we have

$$D(\hat{u}, \eta(v, \hat{u})) \geq 0, \quad \forall v \in K_{k\eta},$$

which implies that $\hat{u}$ solves the directional variational-like inequality (4.1) and

$$\|u_n - u_{n+1}\|^2 \leq \|\hat{u} - u_n\|^2.$$

Thus it follows from the above inequality that the sequence $\{u_n\}$ has exactly one cluster point $\hat{u}$ and $\lim_{n\longrightarrow\infty} u_n = \hat{u}$. the required result. □

It is well-known that to implement the proximal point methods, one has to calculate the approximate solution implicitly, which is in itself a difficult problem. To overcome this drawback, we suggest another iterative method, the convergence of which requires only partially relaxed strongly monotonicity, which is a weaker condition that monotonicity.

For a given $u \in K_{k\eta}$ satisfying (4.1), consider the problem of finding $w \in K_{k\eta}$ such that

$$\rho D(u, \eta(v, w)) + \langle w - u, v - w\rangle \geq 0, \quad \forall \quad v \in K_{k\eta}, \tag{4.18}$$

which is also called the auxiliary directional variational-like inequality. Note that problems (4.3) and (4.18) are quite different. If $w = u$, then clearly w is a solution of the directional variational-like inequality (4.1). This fact enables us to suggest and analyze the following iterative method for solving the directional variational-like inequality (4.1).

Algorithm 4.9. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\rho D(u_n, \eta(v, u_{n+1})) + \langle u_{n+1} - u_n, v - u_{n+1}\rangle \geq 0, \forall v \in K_{k\eta}. \tag{4.19}$$

Note that, for $K_{k\eta} = K_\eta$, the generalizedinvex set $K_{k\eta}$ becomes an invex set K_η and Algorithm 4.9 reduces to:

Algorithm 4.10. *For a given $u_0 \in K$, calculate the approximate solution u_{n+1} by the iterative scheme*

$$\rho D(u_n, \eta(v, u_{n+1})) + \langle u_{n+1} - u_n, v - u_{n+1}\rangle \geq 0, \quad \forall v \in K_\eta,$$

which is known as the iterative method for solving the directional variational-like inequalities (4.2).

We now study the convergence of Algorithm 4.9 and this is the main motivation of our next result.

Theorem 4.11. *Let the operator $D(.,.)$ be partially relaxed strongly monotone with constant $\alpha > 0$. If u_{n+1} is the approximate solution obtained from Algorithm 4.9 and $u \in K_{k\eta}$ is a solution of (4.1), then*

$$\|u - u_{n+1}\|^2 \leq \|u - u_n\|^2 - \{1 - 2\rho\alpha\}\|u_n - u_{n+1}\|^2. \tag{4.20}$$

Proof. Let $u \in K_{k\eta}$ be a solution of (4.1). Then

$$D(u, \eta(v, u)) \geq 0, \quad \forall v \in K_{k\eta}. \tag{4.21}$$

Taking $v = u_{n+1}$ in (4.21), we have

$$D(u, \eta(u_{n+1}, u)) \geq 0. \tag{4.22}$$

Letting $v = u$ in (4.19), we obtain

$$\rho D(u_n, \eta(u, u_{n+1}) + \langle u_{n+1} - u_n, u - u_{n+1}\rangle \geq 0,$$

which implies that

$$\begin{aligned} \langle u_{n+1} - u_n, u - u_{n+1}\rangle &\geq -\rho D(u_n, \eta(u, u_{n+1})) \\ &\geq -\rho\{D(u_n, \eta(u, u_{n+1})) \\ &\quad +D(u, \eta(u_{n+1}, u)\} \\ &\geq -\alpha\rho\|u_n - u_{n+1}\|^2. \end{aligned} \tag{4.23}$$

since $D(.,.)$ is partially relaxed strongly monotone with constant $\alpha > 0$.
Combining (4.22) and (4.23), we obtain the required result (4.20). □

Using essentially the technique of Theorem 4.8, one can study the convergence analysis of Algorithm 4.9.

Using again the auxiliary principle technique, we can consider the following the following problems

For a given $u \in K_{k\eta}$ satisfying (4.1), consider the problem of finding $w \in K_{k\eta}$ such that

$$\rho D(w, \eta(v, w)) + \langle w - u + \alpha(u - u), v - w\rangle \geq 0, \forall v \in K_{k\eta}, \tag{4.24}$$

where $\rho > 0$ and α are constants. Note that, if $w = u$, then w is a solution of (4.1). Consequently, one can suggest and analyze the following iterative method for solving the directional variational-like inequality (4.1).

Algorithm 4.12. *For a given $u_0 \in K_{k\eta}$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\begin{aligned} \rho D(u_{n+1}, \eta(v, u_{n+1})) \quad + \quad &\langle u_{n+1} - u_n + \alpha(u_n - u_{n-1}), v - u_{n+1}\rangle \\ &\geq 0, \quad \forall v \in K_{k\eta}. \end{aligned} \tag{4.25}$$

Algorithm 4.12 is called the inertial proximal point algorithm for solving directional variational-like inequality (4.1).

Using again the auxiliary principle technique, we can consider the following the following problems.

For a given $u \in K_{k\eta}$ satisfying (4.1), consider the problem of finding $w \in K_{k\eta}$ such that

$$\rho D(u, \eta(v, w)) + \langle w - u + \alpha(u - u), v - w\rangle \geq 0, \forall v \in K_{k\eta}, \tag{4.26}$$

where $\rho > 0$ and α are constants. Note that, if $w = u$, then w is a solution of (4.1). Consequently, one can suggest and analyze the following iterative method for solving the directional k-variational-like inequality (4.1).

Algorithm 4.13. *For a given $u_0 \in K_{k\eta}$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\begin{aligned} \rho D(u_n, \eta(v, u_{n+1})) \quad + \quad &\langle u_{n+1} - u_n + \alpha(u_n - u_{n-1}), v - u_{n+1}\rangle \\ &\geq 0, \quad \forall v \in K_{k\eta}. \end{aligned} \tag{4.27}$$

Algorithm 4.13 is called the inertial explicit algorithm for solving directional variational-like inequality (4.1).

Convergence analysis of Algorithm 4.12 and Algorithm 4.13 can be studied using the above ideas and techniques of Noor [17] and Noor at al. [22, 23].

Remark 4.14. *For $k(t) = k$, the directional variational-like inequalities reduce to variational-like inequalities. For the applications, numerical methods and other aspects of variational-like inequalities, see [15, 17, 18, 32, 33] and the references therein. Interested readers may explore the applications and other aspects such as gap functions, error bounds, sensitivity of directional variational-like inequalities in various branches of pure and applied sciences.*

Conclusion

In this paper, we have introduced and studied some new classes preinvex functions, which is called the generalized preinvex functions. These concepts are more general and unifying ones than the previous ones. Several new properties of these generalized preinvex functions are discussed and their relations with previously known results are highlighted. It is shown that the optimality conditions of the differentiable generalized preinvex functions can be characterised by a class of directional variational-like inequalities. This result is used to introduce some new classes of directional variational-like inequalities (4.1). Some new inertial type proximal methods are proposed using the auxiliary principle technique and their convergence analysis is considered under some suitable conditions. It is itself an interesting problem to develop some efficient numerical methods for solving directional variational-like inequalities along with their applications in pure and applied sciences. Despite the current activity, much clearly remains to be done in these fields. It is expected that the ideas and techniques of this paper may be starting point for future research activities.

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