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On Hermite-Hadamard Type Inequalities Associated with the Generalized Fractional Integrals

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Abstract. In this paper, we obtain new generalization of Hermite-Hadamard inequalities via generalized fractional integrals defined by Sarikaya and Ertuğral in [12]. We establish some midpoint and trapezoid type inequalities for functions whose first derivatives in absolute value are convex involving generalized fractional integrals.

1. Introduction

The inequalities discovered by C. Hermite and J. Hadamard for convex functions are considerable significant in the literature (see, e.g.,[1], [5], [11, p.137]). These inequalities state that if $f : I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

Both inequalities hold in the reversed direction if f is concave. We note that Hadamard's inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality.

The Hermite-Hadamard inequality, which is the first fundamental result for convex mappings with a natural geometrical interpretation and many applications, has drawn attention much interest in elementary mathematics. A number of mathematicians have devoted their efforts to generalise, refine, counterpart and extend it for different classes of functions such as using convex mappings. For some papers on Hermite-Hadamard type inequalities please refer to [2]-[4], [6], [9], [13]-[18].

The overall structure of the study takes the form of six sections including introduction. The remainder of this work is organized as follows: we first mention some works which focus on Hermite-Hadamard inequality. In Section 2, we summarize the generalized fractional integrals defined by Sarikaya and Ertuğral. In section 3 new Hermite-Hadamard type inequalities for generalized fractional integrals are proved. In Section 4 and Section 5 midpoint and trapezoid type inequalities for functions whose first derivatives in absolute value are convex via generalized fractional integrals are presented, respectively. Some conclusions and further directions of research are discussed in Section 6.

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2. New Generalized Fractional Integral Operators

In this section, we summarize the generalized fractional integrals defined by Sarikaya and Ertuğral in [12].

Let's define a function $\varphi : [0,\infty) \to [0,\infty)$ satisfying the following conditions :

$$\int_0^1 \frac{\varphi(t)}{t}dt < \infty.$$

We define the following left-sided and right-sided generalized fractional integral operators, respectively, as follows:

$$_{a+}I_{\varphi}f(x) = \int_a^x \frac{\varphi(x-t)}{x-t}f(t)dt, \quad x > a, \tag{2}$$

$$_{b-}I_{\varphi}f(x) = \int_x^b \frac{\varphi(t-x)}{t-x}f(t)dt, \quad x < b. \tag{3}$$

The most important feature of generalized fractional integrals is that they generalize some types of fractional integrals such as Riemann-Liouville fractional integral, k-Riemann-Liouville fractional integral, Katugampola fractional integrals, conformable fractional integral, Hadamard fractional integrals, etc. These important special cases of the integral operators (2) and (3) are mentioned below.

i) If we take $\varphi(t) = t$, the operator (2) and (3) reduce to the Riemann integral as follows:

$$I_{a+}f(x) = \int_a^x f(t)dt, \quad x > a,$$

$$I_{b-}f(x) = \int_x^b f(t)dt, \quad x < b.$$

ii) If we take $\varphi(t) = \frac{t^{\alpha}}{\Gamma(\alpha)}$, the operator (2) and (3) reduce to the Riemann-Liouville fractional integral [7] as follows:

$$J_{a+}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt, \quad x > a,$$

$$J_{b-}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt, \quad x < b.$$

iii) If we take $\varphi(t) = \frac{1}{k\Gamma_k(\alpha)}t^{\frac{\alpha}{k}}$, the operator (2) and (3) reduce to the k-Riemann-Liouville fractional integral as follows:

$$I_{a+,k}^{\alpha}f(x) = \frac{1}{k\Gamma_k(\alpha)}\int_a^x (x-t)^{\frac{\alpha}{k}-1}f(t)dt, \quad x > a,$$

$$I_{b-,k}^{\alpha}f(x) = \frac{1}{k\Gamma_k(\alpha)}\int_x^b (t-x)^{\frac{\alpha}{k}-1}f(t)dt, \quad x < b$$

where

$$\Gamma_k(\alpha) = \int_0^{\infty} t^{\alpha-1}e^{-\frac{t^k}{k}}dt, \quad \mathcal{R}(\alpha) > 0$$

and

$$\Gamma_k(\alpha) = k^{\frac{\alpha}{k}-1}\Gamma\left(\frac{\alpha}{k}\right), \quad \mathcal{R}(\alpha) > 0; k > 0$$

are given by Mubeen and Habibullah in [10].

In [12], Sarıkaya and Ertuğral also establish the following Hermite-Hadamard inequality for the generalized fractional integral operators:

Theorem 2.1. *Let $f : [a,b] \to \mathbb{R}$ be a convex function on $[a,b]$ with $a < b$, then the following inequalities for fractional integral operators hold*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2\Psi(1)}\left[{}_{a+}I_{\varphi}f(b) + {}_{b-}I_{\varphi}f(a)\right] \leq \frac{f(a)+f(b)}{2} \tag{4}$$

where the mapping $\Psi : [0,1] \to \mathbb{R}$ is defined by

$$\Psi(x) = \int_0^x \frac{\varphi\left((b-a)\,t\right)}{t}dt.$$

3. Hermite-Hadamard Type Inequalities for GeneralizeD Fractional Integral Operators

In this section, we will present a new Hermite-Hadamard inequality associated with the generalized fractional integral operators.

Theorem 3.1. *Let $f : [a,b] \to \mathbb{R}$ be a function with $a < b$ and $f \in L_1[a,b]$. If f is a convex function on $[a,b]$, then we have the following inequalities for generalized fractional integral operators:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2\Lambda(1)}\left[{}_{a+}I_{\varphi}f\left(\frac{a+b}{2}\right) +_{b} I_{\varphi}f\left(\frac{a+b}{2}\right)\right] \leq \frac{f(a)+f(b)}{2} \tag{5}$$

where the mapping $\Lambda : [0,1] \to \mathbb{R}$ is defined by

$$\Lambda(x) = \int_0^x \frac{\varphi\left(\frac{b-a}{2}t\right)}{t}dt.$$

Proof. Since f is a convex function on $[a,b]$, we have

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2}$$

for $x, y \in [a,b]$ For $x = \frac{1-t}{2}a + \frac{1+t}{2}b$ and $y = \frac{1+t}{2}a + \frac{1-t}{2}b$, we obtain

$$2f\left(\frac{a+b}{2}\right) \leq f\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right) + f\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right). \tag{6}$$

Multiplying both sides of (6) by $\frac{\varphi\left(\frac{b-a}{2}t\right)}{t}$, then integrating the resulting inequality with respect to t over $[0,1]$, we get

$$\begin{aligned} &2f\left(\frac{a+b}{2}\right)\int_0^1 \frac{\varphi\left(\frac{b-a}{2}t\right)}{t}dt \\ \leq\ & \int_0^1 \frac{\varphi\left(\frac{b-a}{2}t\right)}{t}f\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right)dt + \int_0^1 \frac{\varphi\left(\frac{b-a}{2}t\right)}{t}f\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)dt. \end{aligned}$$

For $u = \frac{1-t}{2}a + \frac{1+t}{2}b$ and $v = \frac{1+t}{2}a + \frac{1-t}{2}b$, we obtain

$$2f\left(\frac{a+b}{2}\right)\Lambda(1)dt$$

$$\begin{aligned} &\leq \int_{\frac{a+b}{2}}^{b} \frac{\varphi\left(u-\frac{a+b}{2}\right)}{u-\frac{a+b}{2}} f(u)\,du + \int_{a}^{\frac{a+b}{2}} \frac{\varphi\left(\frac{a+b}{2}-v\right)}{\frac{a+b}{2}-v} f(v)\,dv \\ &= {}_{a+}I_{\varphi} f\left(\frac{a+b}{2}\right) + {}_{b-}I_{\varphi} f\left(\frac{a+b}{2}\right) \end{aligned}$$

and the first inequality is proved.

For the proof of the second inequality (5), we first note that if f is a convex function, it yields

$$f\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right) \leq \frac{1-t}{2}f(a)+\frac{1+t}{2}f(b)$$

and

$$f\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right) \leq \frac{1+t}{2}f(a)+\frac{1-t}{2}f(b).$$

By adding these inequalities together, one has the following inequality:

$$f\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)+f\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right) \leq f(a)+f(b). \tag{7}$$

Then multiplying both sides of (7) by $\frac{\varphi\left(\frac{b-a}{2}t\right)}{t}$ and integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$\int_0^1 \frac{\varphi\left(\frac{b-a}{2}t\right)}{t} f\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)dt + \int_0^1 \frac{\varphi\left(\frac{b-a}{2}t\right)}{t} f\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt \leq [f(a)+f(b)] \int_0^1 \frac{\varphi\left(\frac{b-a}{2}t\right)}{t}dt.$$

That is,

$${}_{a+}I_{\varphi} f(\frac{a+b}{2}) + {}_{b-}I_{\varphi} f(\frac{a+b}{2}) \leq \Lambda(1)\left[f(a)+f(b)\right].$$

Hence, the proof is completed. □

Remark 3.2. *Under assumption of Theorem 3.1 with $\varphi(t) = t$, then inequalities (5) reduce to inequalities (1).*

Corollary 3.3. *Under assumption of Theorem 3.1 with $\varphi(t) = \frac{t^{\alpha}}{\Gamma(\alpha)}$, then, we have the following inequalities*

$$f\left(\frac{a+b}{2}\right) \leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a+}^{\alpha} f\left(\frac{a+b}{2}\right) + J_{b-}^{\alpha} f\left(\frac{a+b}{2}\right)\right] \leq \frac{f(a)+f(b)}{2}.$$

Corollary 3.4. *Under assumption of Theorem 3.1 with $\varphi(t) = \frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, then, we have the following inequalities*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma_k(\alpha+k)\,2^{\frac{\alpha}{k}-1}}{(b-a)^{\frac{\alpha}{k}}}\left[I_{a+,k} f\left(\frac{a+b}{2}\right) + I_{b-,k} f\left(\frac{a+b}{2}\right)\right] \leq \frac{f(a)+f(b)}{2}.$$

4. Midpoint Type Inequalities for Differentiable Functions with Generalized Fractional Integral Operators

In this section, firstly we need to give a lemma for differentiable functions which will help us to prove our main theorems. Then, we present some midpoint type inequalities which are the generalization of those given in earlier works.

Lemma 4.1. *Let $f : [a,b] \to \mathbb{R}$ be differentiable function on (a,b) with $a < b$. If $f' \in L[a,b]$, then we have the following identity for generalized fractional integral operators:*

$$\begin{aligned} &\frac{1}{2\Lambda(1)}\left[{}_{a+}I_{\varphi}f\left(\frac{a+b}{2}\right)+{}_{b-}I_{\varphi}f\left(\frac{a+b}{2}\right)\right]-f\left(\frac{a+b}{2}\right) \qquad (8)\\ &= \frac{b-a}{4\Delta(0)}\left[\int_0^1 \Delta(t)f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)dt-\int_0^1 \Delta(t)f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt\right] \end{aligned}$$

where the mapping $\Delta(t)$ is defined by

$$\Delta(x) = \int_x^1 \frac{\varphi\left(\frac{b-a}{2}t\right)}{t}dt.$$

Proof. Integrating by parts, we have

$$\begin{aligned} I_1 &= \int_0^1 \Delta(t)f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)dt \qquad (9)\\ &= \frac{2}{b-a}\Delta(t)f\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\Big|_0^1+\frac{2}{b-a}\int_0^1 \frac{\varphi\left(\frac{b-a}{2}t\right)}{t}f\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)dt\\ &= -\frac{2}{b-a}\Delta(0)f\left(\frac{a+b}{2}\right)+\frac{2}{b-a}\,{}_{b-}I_{\varphi}f\left(\frac{a+b}{2}\right) \end{aligned}$$

and similarly we get

$$I_2 = \int_0^1 \Delta(t)f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt = \frac{2}{b-a}\Delta(0)f\left(\frac{a+b}{2}\right)-\frac{2}{b-a}\,{}_{a+}I_{\varphi}f\left(\frac{a+b}{2}\right). \qquad (10)$$

By subtracting equation (10) from (9), we have

$$\frac{b-a}{4\Delta(0)}(I_1-I_2) = 2\left[{}_{a+}I_{\varphi}f\left(\frac{a+b}{2}\right)+{}_{b-}I_{\varphi}f\left(\frac{a+b}{2}\right)\right]-4\Delta(0)f\left(\frac{a+b}{2}\right).$$

By re-arranging the last equality above, we get the desired result. □

Corollary 4.2. *Under assumption of Lemma (4.1) with $\varphi(t) = t$, then we have the following inequalities*

$$\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right) = \frac{b-a}{4}\left[\int_0^1 (1-t)f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)dt-\int_0^1 (1-t)f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt\right].$$

Corollary 4.3. *Under assumption of Lemma 4.1 with $\varphi(t) = \frac{t^{\alpha}}{\Gamma(\alpha)}$, then we have the following inequalities*

$$\begin{aligned} &\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f\left(\frac{a+b}{2}\right)+J_{b-}^{\alpha}\left(\frac{a+b}{2}\right)\right]-f\left(\frac{a+b}{2}\right)\\ &= \frac{b-a}{4}\int_0^1 (1-t^{\alpha})f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)dt-\int_0^1 (1-t^{\alpha})f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt. \end{aligned}$$

Corollary 4.4. *Under assumption of Lemma (4.1)with $\varphi(t) = \frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, then we have the following inequalities*

$$\begin{aligned}
&\frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}}\left[I^{\alpha}_{a+,k}f\left(\frac{a+b}{2}\right)+I^{\alpha}_{b-,k}\left(\frac{a+b}{2}\right)\right]-f\left(\frac{a+b}{2}\right)\\
=\ &\frac{b-a}{4}\int_0^1(1-t^{\frac{\alpha}{k}-1})f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)dt-\int_0^1(1-t^{\frac{\alpha}{k}-1})f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt.
\end{aligned}$$

Theorem 4.5. *Let $f:[a,b]\to\mathbb{R}$ be differentiable function on (a,b) with $a<b$. If $|f'|$ is convex function, then we have the following inequality for generalized fractional integral operators:*

$$\begin{aligned}
&\left|\frac{1}{2\Lambda(1)}\left[{}_{a+}I_{\varphi}f\left(\frac{a+b}{2}\right)+{}_{b-}I_{\varphi}f\left(\frac{a+b}{2}\right)\right]-f\left(\frac{a+b}{2}\right)\right|\\
\leq\ &\frac{b-a}{4\Delta(0)}\left(\int_0^1|\Delta(t)|\,dt\right)\left[|f'(a)|+|f'(b)|\right].
\end{aligned}$$

Proof. From Lemma 4.1, by using the convexity of $|f'|$, we have

$$\begin{aligned}
&\left|\frac{1}{2\Lambda(1)}\left[{}_{a+}I_{\varphi}f\left(\frac{a+b}{2}\right)+{}_{b-}I_{\varphi}f\left(\frac{a+b}{2}\right)\right]-f\left(\frac{a+b}{2}\right)\right|\\
\leq\ &\frac{b-a}{4\Delta(0)}\left[\int_0^1|\Delta(t)|\left|f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right|dt+\int_0^1|\Delta(t)|\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|dt\right]\\
\leq\ &\frac{b-a}{4\Lambda(1)}\left[\int_0^1|\Delta(t)|\left[\frac{1-t}{2}|f'(a)|+\frac{1+t}{2}|f'(b)|\right]dt+\int_0^1|\Delta(t)|\left[\frac{1+t}{2}|f'(a)|+\frac{1-t}{2}|f'(b)|\right]dt\right]\\
=\ &\frac{b-a}{4\Delta(0)}\left(\int_0^1|\Delta(t)|\,dt\right)\left[|f'(a)|+|f'(b)|\right].
\end{aligned}$$

This completes the proof. □

Remark 4.6. *Under assumption of Theorem 4.5 with $\varphi(t)=t$, then we have the inequality*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx-f\left(\frac{a+b}{2}\right)\right|\leq\frac{b-a}{4}\left[\frac{|f'(a)|+|f'(b)|}{2}\right]$$

which was proved by Kırmacı in [8].

Corollary 4.7. *Under assumption of Theorem 4.5 with $\varphi(t)=\frac{t^{\alpha}}{\Gamma(\alpha)}$, then we have the following inequality*

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{a+}f\left(\frac{a+b}{2}\right)+J^{\alpha}_{b-}\left(\frac{a+b}{2}\right)\right]-f\left(\frac{a+b}{2}\right)\right|\leq\frac{\alpha(b-a)^{\alpha}}{2(\alpha+1)}\left[\frac{|f'(a)|+|f'(b)|}{2}\right]$$

Corollary 4.8. *Under assumption of Theorem 4.5 with $\varphi(t)=\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ then we have the following inequality*

$$\left|\frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}}\left[I^{\alpha}_{a+,k}f\left(\frac{a+b}{2}\right)+I^{\alpha}_{b-,k}\left(\frac{a+b}{2}\right)\right]-f\left(\frac{a+b}{2}\right)\right|\leq\frac{\alpha(b-a)^{1-\frac{\alpha}{k}}}{2(\alpha+k)}\left[\frac{|f'(a)|+|f'(b)|}{2}\right].$$

Theorem 4.9. *Let $f:[a,b]\to\mathbb{R}$ be differentiable function on (a,b) with $a<b$. If $\left|f'\right|^q$, $q>1$, is convex function, then we have the following inequalities for generalized fractional integral operators:*

$$\left|\frac{1}{2\Lambda(1)}\left[{}_{a+}I_{\varphi}f\left(\frac{a+b}{2}\right)+{}_{b-}I_{\varphi}f\left(\frac{a+b}{2}\right)\right]-f\left(\frac{a+b}{2}\right)\right| \tag{11}$$

$$\leq \frac{b-a}{4\Delta(0)}\left(\int_0^1|\Delta(t)|^p\,dt\right)^{\frac{1}{p}}\left[\left(\frac{\left|f'(a)\right|^q+3\left|f'(b)\right|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3\left|f'(a)\right|^q+\left|f'(b)\right|^q}{4}\right)^{\frac{1}{q}}\right]$$

$$\leq \frac{b-a}{2^{\frac{2}{q}}\Delta(0)}\left(\int_0^1|\Delta(t)|^p\,dt\right)^{\frac{1}{p}}\left[\left|f'(a)\right|+\left|f'(b)\right|\right]$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Taking modulus of (8) and using the well-known Hölder inequality, we obtain

$$\left|\frac{1}{2\Lambda(1)}\left[{}_{a+}I_{\varphi}f\left(\frac{a+b}{2}\right)+{}_{b-}I_{\varphi}f\left(\frac{a+b}{2}\right)\right]-f\left(\frac{a+b}{2}\right)\right| \tag{12}$$

$$\leq \frac{b-a}{4\Delta(0)}\left[\int_0^1|\Delta(t)|\left|f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right|dt+\int_0^1|\Delta(t)|\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|dt\right]$$

$$\leq \frac{b-a}{4\Delta(0)}\left(\int_0^1|\Delta(t)|^p\,dt\right)^{\frac{1}{p}}\left[\left(\int_0^1\left|f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}+\left(\int_0^1\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right].$$

Since $\left|f'\right|^q$, $q>1$, is convex, we have

$$\int_0^1\left|f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right|^q dt\leq\int_0^1\left[\frac{1-t}{2}\left|f'(a)\right|^q+\frac{1+t}{2}\left|f'(b)\right|^q\right]dt=\frac{\left|f'(a)\right|^q+3\left|f'(b)\right|^q}{4} \tag{13}$$

and similarly

$$\int_0^1\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt\leq\frac{3\left|f'(a)\right|^q+\left|f'(b)\right|^q}{4}. \tag{14}$$

By substituting inequalities (13) and (14) in (12), we obtain the first inequalty in (11).

For the proof of second inequality, let $a_1=\left|f'(a)\right|^q$, $b_1=3\left|f'(b)\right|^q$, $a_2=3\left|f'(a)\right|^q$ and $b_2=\left|f'(b)\right|^q$. Using the fact that

$$\sum_{k=1}^n(a_k+b_k)^s\leq\sum_{k=1}^n a_k^s+\sum_{k=1}^n b_k^s,\ 0\leq s<1 \tag{15}$$

and $1+3^{\frac{1}{q}}\leq 4$, then the desired result can be obtained straightforwardly. □

Corollary 4.10. *Under assumption of Lemma (4.9)with $\varphi(t)=t$, then we hve the following inequalities*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx-f\left(\frac{a+b}{2}\right)\right|$$

$$\leq \frac{b-a}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{\left|f'(a)\right|^{q}+3\left|f'(b)\right|^{q}}{4}\right)^{\frac{1}{q}}+\left(\frac{3\left|f'(a)\right|^{q}+\left|f'(b)\right|^{q}}{4}\right)^{\frac{1}{q}}\right]$$

$$\leq \frac{b-a}{2^{\frac{2}{q}}(p+1)^{\frac{1}{p}}}\left[\left|f'(a)\right|+\left|f'(b)\right|\right].$$

Corollary 4.11. *Under assumption of Lemma (4.9)with $\varphi(t)=\frac{t^{\alpha}}{\Gamma(\alpha)}$, then we hve the following inequalities*

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f\left(\frac{a+b}{2}\right)+J_{b-}^{\alpha}\left(\frac{a+b}{2}\right)\right]-f\left(\frac{a+b}{2}\right)\right|$$

$$\leq \frac{b-a}{4}\left(\int_{0}^{1}(1-x^{\alpha})^{p}\,dx\right)^{\frac{1}{p}}\left[\left(\frac{\left|f'(a)\right|^{q}+3\left|f'(b)\right|^{q}}{4}\right)^{\frac{1}{q}}+\left(\frac{3\left|f'(a)\right|^{q}+\left|f'(b)\right|^{q}}{4}\right)^{\frac{1}{q}}\right]$$

$$\leq \frac{b-a}{2^{\frac{2}{q}}}\left(\int_{0}^{1}(1-x^{\alpha})^{p}\,dx\right)^{\frac{1}{p}}\left[\left|f'(a)\right|+\left|f'(b)\right|\right].$$

Theorem 4.12. *Let $f:[a,b]\to\mathbb{R}$ be differentiable function on (a,b) with $a<b$. If $\left|f'\right|^{q}$, $q\geq 1$, is a convex function, then we have the following inequality for generalized fractional integral operators:*

$$\left|\frac{1}{2\Lambda(1)}\left[{}_{a+}I_{\varphi}f\left(\frac{a+b}{2}\right)+{}_{b-}I_{\varphi}f\left(\frac{a+b}{2}\right)\right]-f\left(\frac{a+b}{2}\right)\right|$$

$$\leq \frac{b-a}{2^{2+\frac{1}{q}}\Lambda(1)}\left(\int_{0}^{1}|\Delta(t)|\,dt\right)^{1-\frac{1}{q}}\left[\left(B_1\left|f'(a)\right|^{q}+B_2\left|f'(b)\right|^{q}\right)^{\frac{1}{q}}+\left(B_2\left|f'(a)\right|^{q}+B_1\left|f'(b)\right|^{q}\right)^{\frac{1}{q}}\right]$$

where the constants B_1 and B_2 are defined by

$$B_1=\int_{0}^{1}|\Delta(t)|(1-t)\,dt \text{ and } B_2=\int_{0}^{1}|\Delta(t)|(1+t)\,dt.$$

Proof. The case of $q=1$ is obvious from Theorem 4.5.

For $q>1$ we proceed as follows. Taking modulus of (8) and using well-known power mean inequality, we obtain

$$\left|\frac{1}{2\Lambda(1)}\left[{}_{a+}I_{\varphi}f\left(\frac{a+b}{2}\right)+{}_{b-}I_{\varphi}f\left(\frac{a+b}{2}\right)\right]-f\left(\frac{a+b}{2}\right)\right|$$

$$\leq \frac{b-a}{4\Lambda(1)}\left[\int_{0}^{1}|\Delta(t)|\left|f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right|dt+\int_{0}^{1}|\Delta(t)|\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|dt\right]$$

$$\leq \frac{b-a}{4\Lambda(1)}\left(\int_{0}^{1}|\Delta(t)|\,dt\right)^{1-\frac{1}{q}}\left[\left(\int_{0}^{1}|\Delta(t)|\left|f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right|^{q}dt\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_{0}^{1}|\Delta(t)|\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^{q}dt\right)^{\frac{1}{q}}\right].$$

Since $\left|f'\right|^q$ is convex, we have

$$\begin{aligned}
&\left|\frac{1}{2\Lambda(1)}\left[{}_{a+}I_{\varphi}f\left(\frac{a+b}{2}\right)+{}_{b-}I_{\varphi}f\left(\frac{a+b}{2}\right)\right]-f\left(\frac{a+b}{2}\right)\right| \\
\leq\ & \frac{b-a}{4\Lambda(1)}\left(\int_0^1|\Delta(t)|\,dt\right)^{1-\frac{1}{q}}\left[\left(\int_0^1|\Delta(t)|\left[\frac{1-t}{2}\left|f'(a)\right|^q+\frac{1+t}{2}\left|f'(b)\right|^q\right]dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_0^1|\Delta(t)|\left[\frac{1+t}{2}\left|f'(a)\right|^q+\frac{1-t}{2}\left|f'(b)\right|^q\right]dt\right)^{\frac{1}{q}}\right] \\
=\ & \frac{b-a}{2^{2+\frac{1}{q}}\Lambda(1)}\left(\int_0^1|\Delta(t)|\,dt\right)^{1-\frac{1}{q}}\left[\left(B_1\left|f'(a)\right|^q+B_2\left|f'(b)\right|^q\right)^{\frac{1}{q}}+\left(B_2\left|f'(a)\right|^q+B_1\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right]
\end{aligned}$$

which completes the proof. □

5. Trapezoid Type Inequalities for Differentiable Functions with Generalized Fractional Integral Operators

In this section, firstly we need to give a lemma for differentiable functions which will help us to prove our main theorems. Then, we present some trapezoid type inequalities which are the generalization of those given in earlier studies.

Lemma 5.1. *Let $f:[a,b]\to\mathbb{R}$ be differentiable function on (a,b) with $a<b$. If $f'\in L[a,b]$, then we have the following identity for generalized fractional integral operators:*

$$\begin{aligned}
&\frac{f(a)+f(b)}{2}-\frac{1}{2\Lambda(1)}\left[{}_{a+}I_{\varphi}f\left(\frac{a+b}{2}\right)+{}_{b-}I_{\varphi}f\left(\frac{a+b}{2}\right)\right] \qquad (16)\\
=\ & \frac{b-a}{4\Lambda(1)}\left[\int_0^1\Lambda(t)f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)dt-\int_0^1\Lambda(t)f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt\right].
\end{aligned}$$

Proof. Integrating by parts, we have

$$\begin{aligned}
I_3 &= \int_0^1\Lambda(t)f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)dt \qquad (17)\\
&= -\frac{2}{b-a}\Lambda(t)f\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\Big|_0^1-\frac{2}{b-a}\int_0^1\frac{\varphi\left(\frac{b-a}{2}t\right)}{t}f\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)dt \\
&= \frac{2}{b-a}\Lambda(1)f(b)-\frac{2}{b-a}\,{}_{b-}I_{\varphi}f\left(\frac{a+b}{2}\right)
\end{aligned}$$

and similarly we get

$$I_4=\int_0^1\Lambda(t)f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt=-\frac{2}{b-a}\Lambda(1)f(a)+\frac{2}{b-a}\,{}_{a+}I_{\varphi}f\left(\frac{a+b}{2}\right). \qquad (18)$$

Thus, we have

$$\frac{b-a}{4\Lambda(1)}(I_3-I_4)=\frac{f(a)+f(b)}{2}-\frac{1}{2\Lambda(1)}\left[{}_{a+}I_{\varphi}f\left(\frac{a+b}{2}\right)+{}_{b-}I_{\varphi}f\left(\frac{a+b}{2}\right)\right].$$

This completes the proof. □

Corollary 5.2. *Under assumption of Lemma (5.1)with $\varphi(t)=t$, then we have the following inequalities*

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx=\frac{b-a}{4}\left[\int_0^1 tf'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)dt-\int_0^1 tf'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt\right].$$

Corollary 5.3. *Under assumption of Lemma (5.1) with $\varphi(t)=\frac{t^{\alpha}}{\Gamma(\alpha)}$, then we have the following inequalities*

$$\begin{aligned}&\frac{f(a)+f(b)}{2}-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f\left(\frac{a+b}{2}\right)+J_{b-}^{\alpha}\left(\frac{a+b}{2}\right)\right]\\ =\;&\frac{b-a}{4}\left[\int_0^1 t^{\alpha}f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)dt-\int_0^1 t^{\alpha}f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt\right].\end{aligned}$$

Corollary 5.4. *Under assumption of Lemma (5.1) with $\varphi(t)=\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, then we have the following inequalities*

$$\begin{aligned}&\frac{f(a)+f(b)}{2}-\frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}}\left[I_{a+,k}^{\alpha}f\left(\frac{a+b}{2}\right)+I_{b-,k}^{\alpha}\left(\frac{a+b}{2}\right)\right]\\ =\;&\frac{b-a}{4}\left[\int_0^1 t^{\frac{\alpha}{k}-1}f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)dt-\int_0^1 t^{\frac{\alpha}{k}-1}f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt\right].\end{aligned}$$

Theorem 5.5. *Let $f:[a,b]\to\mathbb{R}$ be differentiable function on (a,b) with $a<b$. If $\left|f'\right|$ is a convex function, then we have the following inequality for generalized fractional integral operators:*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{2\Lambda(1)}\left[{}_{a+}I_{\varphi}f\left(\frac{a+b}{2}\right)+{}_{b-}I_{\varphi}f\left(\frac{a+b}{2}\right)\right]\right|\\ \leq\;&\frac{b-a}{4\Lambda(1)}\left(\int_0^1|\Lambda(t)|\,dt\right)\left[\left|f'(a)\right|+\left|f'(b)\right|\right].\end{aligned}$$

Proof. From Lemma 5.1, by the using convexity of $\left|f'\right|$,we have

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{2\Lambda(1)}\left[{}_{a+}I_{\varphi}f\left(\frac{a+b}{2}\right)+{}_{b-}I_{\varphi}f\left(\frac{a+b}{2}\right)\right]\right|\\ \leq\;&\frac{b-a}{4\Lambda(1)}\left[\int_0^1|\Lambda(t)|\left|f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right|dt+\int_0^1|\Lambda(t)|\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|dt\right]\\ \leq\;&\frac{b-a}{4\Lambda(1)}\left[\int_0^1|\Lambda(t)|\left[\frac{1-t}{2}\left|f'(a)\right|+\frac{1+t}{2}\left|f'(b)\right|\right]dt+\int_0^1|\Lambda(t)|\left[\frac{1+t}{2}\left|f'(a)\right|+\frac{1-t}{2}\left|f'(b)\right|\right]dt\right]\end{aligned}$$

$$= \frac{b-a}{4\Lambda(1)}\left(\int_0^1 |\Lambda(t)|\,dt\right)\left[\left|f'(a)\right| + \left|f'(b)\right|\right]$$

which completes the proof. □

Theorem 5.6. *Let $f : [a,b] \to \mathbb{R}$ be differentiable function on (a,b) with $a < b$. If $\left|f'\right|^q$, $q > 1$, is a convex function, then we have the following inequality for generalized fractional integral operators:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{2\Lambda(1)}\left[{}_{a+}I_{\varphi}f\left(\frac{a+b}{2}\right) + {}_{b-}I_{\varphi}f\left(\frac{a+b}{2}\right)\right]\right| \tag{19}$$

$$\leq \frac{b-a}{4\Lambda(1)}\left(\int_0^1 |\Lambda(t)|^p\,dt\right)^{\frac{1}{p}}\left[\left(\frac{\left|f'(a)\right|^q + 3\left|f'(b)\right|^q}{4}\right)^{\frac{1}{q}} + \left(\frac{3\left|f'(a)\right|^q + \left|f'(b)\right|^q}{4}\right)^{\frac{1}{q}}\right]$$

$$\leq \frac{b-a}{2^{\frac{2}{q}}\Lambda(1)}\left(\int_0^1 |\Lambda(t)|^p\,dt\right)^{\frac{1}{p}}\left[\left|f'(a)\right| + \left|f'(b)\right|\right]$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Similar to proof of Theorem 4.9, by using the well-known Hölder inequality and convexity of $\left|f'\right|^q$, we obtain

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{2\Lambda(1)}\left[{}_{a+}I_{\varphi}f\left(\frac{a+b}{2}\right) + {}_{b-}I_{\varphi}f\left(\frac{a+b}{2}\right)\right]\right|$$

$$\leq \frac{b-a}{4\Lambda(1)}\left[\int_0^1 |\Lambda(t)|\left|f'\left(\frac{t}{2}a + \frac{2-t}{2}b\right)\right|dt + \int_0^1 |\Lambda(t)|\left|f'\left(\frac{2-t}{2}a + \frac{t}{2}b\right)\right|dt\right]$$

$$\leq \frac{b-a}{4\Lambda(1)}\left(\int_0^1 |\Lambda(t)|^p\,dt\right)^{\frac{1}{p}}\left[\left(\frac{\left|f'(a)\right|^q + 3\left|f'(b)\right|^q}{4}\right)^{\frac{1}{q}} + \left(\frac{3\left|f'(a)\right|^q + \left|f'(b)\right|^q}{4}\right)^{\frac{1}{q}}\right].$$

This completes the proof of first inequality in (19)

The proof of second inequality in (19) is obvious from the inequality (15). □

Theorem 5.7. *Let $f : [a,b] \to \mathbb{R}$ be differentiable function on (a,b) with $a < b$. If $\left|f'\right|^q$, $q \geq 1$, is convex function, then we have the following inequality for generalized fractional integral operators:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{2\Lambda(1)}\left[{}_{a+}I_{\varphi}f\left(\frac{a+b}{2}\right) + {}_{b-}I_{\varphi}f\left(\frac{a+b}{2}\right)\right]\right|$$

$$\leq \frac{b-a}{2^{2+\frac{1}{q}}\Lambda(1)}\left(\int_0^1 |\Lambda(t)|\,dt\right)^{1-\frac{1}{q}}\left[\left(B_3\left|f'(a)\right|^q + B_4\left|f'(b)\right|^q\right)^{\frac{1}{q}} + \left(B_4\left|f'(a)\right|^q + B_3\left|f'(b)\right|\right)^{\frac{1}{q}}\right]$$

where the constants B_3 and B_4 are defined by

$$B_3 = \int_0^1 |\Lambda(t)|(1-t)\,dt \text{ and } B_4 = \int_0^1 |\Lambda(t)|(1+t)\,dt.$$

Proof. The case of the $q = 1$ is obvious from the Theorem 5.5.

For $q > 1$, using well-known power mean inequality in Lemma 5.1, we obtain

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{2\Lambda(1)}\left[{}_{a+}I_{\varphi}f\left(\frac{a+b}{2}\right)+{}_{b-}I_{\varphi}f\left(\frac{a+b}{2}\right)\right]\right| \\
\leq\ & \frac{b-a}{4\Lambda(1)}\left[\int_0^1|\Lambda(t)|\left|f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right|dt+\int_0^1|\Lambda(t)|\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|dt\right] \\
\leq\ & \frac{b-a}{4\Lambda(1)}\left(\int_0^1|\Lambda(t)|^q\,dt\right)^{1-\frac{1}{q}} \\
&\times\left[\left(\int_0^1|\Lambda(t)|\left|f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}+\left(\int_0^1|\Lambda(t)|\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right].
\end{aligned}$$

By the using convexity of $\left|f'\right|^q$, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{2\Lambda(1)}\left[{}_{a+}I_{\varphi}f\left(\frac{a+b}{2}\right)+{}_{b-}I_{\varphi}f\left(\frac{a+b}{2}\right)\right]\right| \\
\leq\ & \frac{b-a}{4\Lambda(1)}\left(\int_0^1|\Lambda(t)|^q\,dt\right)^{1-\frac{1}{q}} \\
&\times\left[\left(\int_0^1|\Lambda(t)|\left[\frac{1-t}{2}\left|f'(a)\right|^q+\frac{1+t}{2}\left|f'(b)\right|^q\right]dt\right)^{\frac{1}{q}}+\left(\int_0^1|\Lambda(t)|\left[\frac{1+t}{2}\left|f'(a)\right|^q+\left|f'(b)\right|^q\frac{1-t}{2}\right]dt\right)^{\frac{1}{q}}\right] \\
=\ & \frac{b-a}{2^{2+\frac{1}{q}}\Lambda(1)}\left(\int_0^1|\Lambda(t)|^q\,dt\right)^{1-\frac{1}{q}}\left[\left(B_3\left|f'(a)\right|^q+B_4\left|f'(b)\right|^q\right)^{\frac{1}{q}}+\left(B_4\left|f'(a)\right|^q+B_3\left|f'(b)\right|\right)^{\frac{1}{q}}\right].
\end{aligned}$$

The proof is completely completed. □

Remark 5.8. *By special choice of the function φ in Theorem 5.5-Theorem 5.7, it can be written some remarks and corollaries. We left them to interested readers.*

6. Concluding Remarks

In this study, we consider the Hermite-Hadamard for convex function involving generalized fractional integrals defined by Sarikaya and Ertuğral in [12]. We also focus on midpoint and trapezoid type inequalities for functions whose first derivatives in absolute value are convex via generalized fractional integrals. The results presented in this study would provide generalizations of those given in earlier works.

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