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Algorithmic Approach to the Solution of Pseudomonotone Equilibrium Problems and Generalized Variational Inequalities

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Abstract. In this paper, we consider pseudomonotone equilibrium problems and generalized variational inequalities in Hilbert spaces. We suggest an iterative procedure for solving pseudomonotone equilibrium problems and generalized variational inequalities. Strong convergence result is proved under some mild assumptions.

1. Introduction

Let H be a real Hilbert space. Denote its inner product and norm by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$, respectively. Let $C \subset H$ be a nonempty closed convex set.

Definition 1.1. *A bifunction* $g\colon C \times C \to \mathbb{R}$ *is said to be*

(i) *monotone if* $\forall u^\dagger, v^\dagger \in C$,

$$g(u^\dagger, v^\dagger) + g(v^\dagger, u^\dagger) \le 0.$$

(ii) *pseudomonotone if the following relation holds*

$$g(u^\dagger, v^\dagger) \ge 0 \Rightarrow g(v^\dagger, u^\dagger) \le 0, \forall u^\dagger, v^\dagger \in C.$$

Remark 1.2. *It is obviously that the monotonicity of g means the pseudomonotonicity of g. But the reverse is not necessarily correct.*

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Let $g\colon C\times C\to \mathbb{R}$ be a bifunction. Recall that the equilibrium problem is to find a point $x\in C$ such that

$$g(x,y)\geq 0,\ \forall y\in C. \tag{1}$$

Denote the solution set of (1) by $PME(C,g)$.

As an important tool, the equilibrium problem (1) incorporates in a useful way a great deal of problems, such as variational inequality problems ([2, 4, 6, 35, 38, 41, 45]), fixed point problems ([1, 3, 10, 14, 19, 23, 31, 33, 46]) and so on. The equilibrium problem (1) has been continuously investigated and extended in the literature, see e.g. [8, 9, 11, 15–17, 22, 26, 34, 39, 40, 47–49]).

Definition 1.3. *An operator $f: C\to H$ is said to be*

- *ω-strongly monotone if for all $u,v\in C$,*

$$\langle f(u)-f(v),u-v\rangle\geq \omega\|u-v\|^2, \tag{2}$$

  where $\omega>0$ is a constant.
- *τ-inverse strongly ϕ-monotone if for all $u,v\in C$,*

$$\langle f(u)-f(v),\phi(u)-\phi(v)\rangle\geq \tau\|f(u)-f(v)\|^2,$$

  where $\tau>0$ is a constant and $\phi: C\to C$ is an operator.

Let $f: C\to H$ and $\phi: C\to C$ be two operators. Recall that the generalized variational inequality ([21]) is to find a point $x^\dagger\in C$ such that

$$\langle f(x^\dagger),\phi(x)-\phi(x^\dagger)\rangle\geq 0,\ \forall x\in C. \tag{3}$$

Use $VI(C,f,\phi)$ to denote the solution set of problem (3).

If $\phi\equiv I$, the identity operator of C, then (3) reduces to find a point $x^\dagger\in C$ such that

$$\langle f(x^\dagger),x-x^\dagger\rangle\geq 0,\ \forall x\in C. \tag{4}$$

Variational inequality problems play important roles and provide a useful tool for studying numerous valuable problems coming from water resources, finance, economics, medical images and so on ([5, 7, 25, 29, 37, 43]). In order to solve (4), many iterative algorithms, such as projection methods, proximal point methods, extragradient methods, subgradient methods have been investigated, see, e.g., [12, 13, 18, 20, 24, 27, 30, 32, 36, 42, 44].

In the present paper, we are interested in the pseudomonotone equilibrium problems and generalized variational inequalities of finding a point $u^\dagger$ such that

$$u^\dagger\in VI(C,f,\phi) \text{ and } \phi(u^\dagger)\in PME(C,g). \tag{5}$$

Here, use Γ to denote the solution set of problem (5), that is,

$$\Gamma=\{x^*|x^*\in VI(C,f,\phi) \textit{ and } \phi(x^*)\in PME(C,g)\}.$$

In this paper, we construct an iterative algorithm for solving (5). We show that the presented algorithm strongly converges to en element in Γ.

2. Preliminaries

Let C be a nonempty closed convex subset of a real Hilbert space H. Recall that an operator $\psi: C\to H$ is called ρ-Lipschitz ($\rho\geq 0$) if

$$\|\psi(\tilde{u})-\psi(\tilde{v})\|\leq \rho\|\tilde{u}-\tilde{v}\|,\ \forall \tilde{u},\tilde{v}\in C.$$

When $\rho<1$, ψ is called ρ-contraction. When $\rho=1$, ψ is called nonexpansive.

A bifunction $g\colon C\times C\to\mathbb{R}$ is said to be jointly sequentially weakly continuous, if $s_n, t_n\in C$ satisfying $s_n\rightharpoonup x^\dagger$ and $t_n\rightharpoonup y^\dagger$ imply that $g(s_n,t_n)\to g(x^\dagger,y^\dagger)$.

An operator $F: H\to 2^H$ is called monotone iff $\langle u-v,\tilde{u}-\tilde{v}\rangle\geq 0$ for all $u,v\in dom(F)$, $\tilde{u}\in F(u)$, and $\tilde{v}\in F(v)$. A monotone operator F on H is called maximal iff its graph is not strictly contained in the graph of any other monotone operator on H.

For given $u^\dagger\in H$, there exists a unique point in C, denoted by $proj_C[u^\dagger]$ such that

$$\|u^\dagger - proj_C[u^\dagger]\|\leq\|x-u^\dagger\|,\ \forall x\in C.$$

It is known that $proj_C$ is firmly nonexpansive, that is, $proj_C$ satisfies

$$\|proj_C[q^*]-proj_C[q^\dagger]\|^2\leq\langle proj_C[q^*]-proj_C[q^\dagger], q^*-q^\dagger\rangle,\ \forall q^*,q^\dagger\in H. \tag{6}$$

Moreover, $proj_C$ satisfies the following inequality

$$\langle q^*-proj_C[q^*], q^\dagger-proj_C[q^*]\rangle\leq 0,\ \forall q^*\in H, q^\dagger\in C. \tag{7}$$

Lemma 2.1 ([22]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $g\colon C\times C\to\mathbb{R}$ be a bifunction which satisfies condition (C4) stated in Sec. 3. Let $\{\lambda_n\}$ be a sequence satisfying $\lambda_n\in[\underline{\lambda},\overline{\lambda}]\subset(0,1]$. For given $r_n\in C$, set*

$$t_n=\arg\min_{u^\dagger\in C}\left\{g(r_n,u^\dagger)+\frac{1}{2\lambda_n}\|r_n-u^\dagger\|^2\right\}.$$

Then the boundedness of $\{r_n\}$ implies that $\{t_n\}$ is bounded.

Lemma 2.2 ([26]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $g\colon C\times C\to\mathbb{R}$ be a bifunction which satisfies condition (C4) stated in Sec. 3. For given two points $\bar{u},\bar{v}\in C$ and two sequences $\{a_n\}\subset C$ and $\{b_n\}\subset C$, if $a_n\rightharpoonup\bar{u}$ and $b_n\rightharpoonup\bar{v}$, respectively, then, for any $\epsilon>0$, there exist $\alpha>0$ and $N_\epsilon\in\mathbb{N}$ verifying*

$$\partial_2 g(b_n,a_n)\subset\partial_2 g(\bar{v},\bar{u})+\frac{\epsilon}{\alpha}B$$

for every $n\geq N_\epsilon$, where $B:=\{b\in H:\|b\|\leq 1\}$.

Lemma 2.3 ([28]). *Let $\{\tau_n\}\subset[0,\infty)$, $\{\zeta_n\}\subset(0,1)$ and $\{\eta_n\}$ be real number sequences. Suppose the following conditions are satisfied*

(i) $\tau_{n+1}\leq(1-\zeta_n)\tau_n+\eta_n, \forall n\geq 1;$

(ii) $\sum_{n=1}^\infty\zeta_n=\infty;$

(iii) $\limsup_{n\to\infty}\frac{\eta_n}{\zeta_n}\leq 0$ *or* $\sum_{n=1}^\infty|\eta_n|<\infty.$

Then $\lim_{n\to\infty}\tau_n=0.$

Lemma 2.4 ([19]). *Let $\{y_n\}$ be a real number sequence. Assume there exists at least a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that*

$$y_{n_k}\leq y_{n_k+1}$$

for all $k\geq 0$. For every $n\geq N_0$, define an integer sequence $\{\varphi(n)\}$ as

$$\varphi(n)=\max\{i\leq n: y_{n_i}<y_{n_i+1}\}.$$

Then $\varphi(n)\to\infty$ as $n\to\infty$ and for all $n\geq N_0$, $\max\{y_{\varphi(n)},y_n\}\leq y_{\varphi(n)+1}$.

3. Main results

In this section, we introduce an iterative algorithm for solving (5) and demonstrate its strong convergence. Firstly, we state some assumptions regarding to the involved operators and parameters. Let C be a nonempty closed convex subset of a real Hilbert space H. Assume that

(C1): $\psi : C \to C$ is a ρ-contractive operator;

(C2): $\phi : C \to C$ is a weakly continuous and ω-strongly monotone operator such that $Range(\phi) = C$;

(C3): $f : C \to H$ is a τ-inverse strongly ϕ-monotone operator;

(C4): $g : C \times C \to \mathbb{R}$ is a bifunction which satisfies the following conditions

(i): $g(x^\dagger, x^\dagger) = 0, \forall x^\dagger \in C$;

(ii): g is pseudomonotone on H;

(iii): g is jointly sequently weakly continuous on $C \times C$;

(iv): $\forall x^\dagger \in C$, $g(x^\dagger, x)$ is convex and subdifferentiable on the second variable $x \in C$.

Let $\{\tau_n\}$, $\{\lambda_n\}$ and $\{\zeta_n\}$ be three real number sequences in $[0, 1]$ and $\{\mu_n\}$ and $\{\beta_n\}$ be two real number sequences in $(0, \infty)$. Let $\gamma \in (0, 1)$ and $\alpha \in (0, 1)$ be two constants. Assume that

(A1): $\lim_{n\to\infty} \tau_n = 0$ and $\sum_{n=1}^{\infty} \tau_n = \infty$;

(A2): $\zeta_n \in [a_1, b_1] \subset (0, 1)$, $\lambda_n \in [a_2, b_2] \subset (0, 1]$, $\beta_n \in [a_3, b_3] \subset (0, 2)$ and $\mu_n \in [a_4, b_4] \subset (0, 2\tau)$;

(A3): $0 < \rho < \omega < 2\tau$.

In what follows, we suppose that $\Gamma \neq \emptyset$. Next, we present an iterative algorithm for solving problem (5).

Algorithm 3.1. *Let $x_0 \in C$ be a guess. Set $n = 0$.*

Step 1. For given x_n, compute

$$r_n = proj_C[\tau_n \psi(x_n) + (1 - \tau_n)(\phi(x_n) - \mu_n f(x_n))]. \tag{8}$$

Step 2. Compute

$$t_n = \arg\min_{u^\dagger \in C} \left\{ g(r_n, u^\dagger) + \frac{1}{2\lambda_n} \|r_n - u^\dagger\|^2 \right\}. \tag{9}$$

If $t_n = r_n$, then set $y_n = r_n$ and go to Step 5. Otherwise, continuous to the next step 3.

Step 3. Find m_n as the smallest positive integer number m such that

$$2\lambda_n[g(w_{n,m}, r_n) - g(w_{n,m}, t_n)] \geq \gamma \|r_n - t_n\|^2 \tag{10}$$

where

$$w_{n,m} = (1 - \alpha^m) r_n + \alpha^m t_n. \tag{11}$$

Write $\alpha_n = \alpha^{m_n}$ and $w_n = w_{n,m_n}$.

Step 4. Compute

$$y_n = proj_C(r_n - \beta_n \theta_n \vartheta_n), \tag{12}$$

where $\vartheta_n \in \partial_2 g(w_n, r_n)$ and $\theta_n = \dfrac{g(w_n, r_n)}{\|\vartheta_n\|^2}$.

Step 5. Compute

$$\phi(x_{n+1}) = (1 - \zeta_n)\phi(x_n) + \zeta_n y_n. \tag{13}$$

Step 6. Set $n := n + 1$ and return to step 1.

Remark 3.2. *We have the following conclusions:*

(i) By the strong monotonicity of ϕ, *from* (2), *we have*

$$\|\phi(x) - \phi(y)\| \geq \omega\|x - y\|, \ \forall x, y \in C. \tag{14}$$

Thus, the following varaitional inequality has a unique solution denoted by q^*,

$$\langle \psi(x) - \phi(x), \phi(y) - \phi(x)\rangle \leq 0, \quad \forall y \in \Gamma.$$

So, we have

$$\langle \psi(q^*) - \phi(q^*), \phi(y) - \phi(q^*)\rangle \leq 0, \quad \forall y \in \Gamma. \tag{15}$$

(ii) Since f *is* τ*-inverse strongly* ϕ*-monotone, for any* $u \in C$, *we have*

$$\begin{aligned} &\|(\phi(u) - \mu f(u)) - (\phi(q^*) - \mu f(q^*))\|^2 \\ &\quad = \|\phi(u) - \phi(q^*)\|^2 - 2\mu\langle f(u) - f(q^*), \phi(u) - \phi(q^*)\rangle + \mu^2\|f(u) - f(q^*)\|^2 \\ &\quad \leq \|\phi(u) - \phi(q^*)\|^2 - 2\mu\tau\|f(u) - f(q^*)\|^2 + \mu^2\|f(u) - f(q^*)\|^2 \\ &\quad \leq \|\phi(u) - \phi(q^*)\|^2 + \mu(\mu - 2\tau)\|f(u) - f(q^*)\|^2. \end{aligned} \tag{16}$$

(iii) If $t_n = r_n$, *then* $t_n \in PME(C, g)$. *If* $t_n \neq r_n$, *then* $0 \notin \partial_2 g(w_n, r_n)$ *and* $\vartheta_n \neq 0$.

(iv) The linesearch rule (10) *is all well-defined.*

(v) $g(w_n, r_n) > 0$.

(vi) $\|y_n - x^*\|^2 \leq \|r_n - x^*\|^2 - \beta_n(2 - \beta_n)(\theta_n\|\vartheta_n\|)^2$ *for all* $x^* \in PME(C, g)$.

Proposition 3.3. *The sequences* $\{x_n\}$, $\{\phi(x_n)\}$, $\{r_n\}$ *and* $\{t_n\}$ *are bounded.*

Proof. Note that $q^* \in VI(C, f, \phi)$ and $\phi(q^*) \in PME(C, g)$. Then, $\phi(q^*) = proj_C[\phi(q^*) - \mu_n f(q^*)]$ for all $n \geq 0$. By virtue of (16), we get

$$\begin{aligned} \|(\phi(x_n) - \mu_n f(x_n)) - (\phi(q^*) - \mu_n f(q^*))\|^2 &\leq \|\phi(x_n) - \phi(q^*)\|^2 + \mu_n(\mu_n - 2\tau)\|f(x_n) - f(q^*)\|^2 \\ &\leq \|\phi(x_n) - \phi(q^*)\|^2, \end{aligned} \tag{17}$$

and

$$\|\phi(x_{n+1}) - \mu_{n+1}f(x_{n+1}) - (\phi(x_n) - \mu_{n+1}f(x_n))\|^2 \leq \|\phi(x_{n+1}) - \phi(x_n)\|^2 + \mu_{n+1}(\mu_{n+1} - 2\tau)\|f(x_{n+1}) - f(x_n)\|^2. \tag{18}$$

According to (8), (14) and (17), we have

$$\begin{aligned} \|r_n - \phi(q^*)\| &= \|proj_C[\tau_n\psi(x_n) + (1 - \tau_n)(\phi(x_n) - \mu_n f(x_n))] - proj_C[\phi(q^*) - \mu_n f(q^*)]\| \\ &\leq \|\tau_n(\psi(x_n) - \phi(q^*) + \mu_n f(q^*)) + (1 - \tau_n)((\phi(x_n) - \mu_n f(x_n)) - (\phi(q^*) - \mu_n f(q^*)))\| \\ &\leq \tau_n\|\psi(x_n) - \psi(q^*)\| + \tau_n\|\psi(q^*) - \phi(q^*) + \mu_n f(q^*)\| \\ &\quad + (1 - \tau_n)\|(\phi(x_n) - \mu_n f(x_n)) - (\phi(q^*) - \mu_n f(q^*))\| \\ &\leq \tau_n\rho\|x_n - q^*\| + \tau_n\|\psi(q^*) - \phi(q^*) + \mu_n f(q^*)\| + (1 - \tau_n)\|\phi(x_n) - \phi(q^*)\| \\ &\leq \tau_n\rho/\omega\|\phi(x_n) - \phi(q^*)\| + \tau_n\|\psi(q^*) - \phi(q^*) + \mu_n f(q^*)\| + (1 - \tau_n)\|\phi(x_n) - \phi(q^*)\| \\ &= [1 - (1 - \rho/\omega)\tau_n]\|\phi(x_n) - \phi(q^*)\| + \tau_n\|\psi(q^*) - \phi(q^*) + \mu_n f(q^*)\| \\ &\leq [1 - (1 - \rho/\omega)\tau_n]\|\phi(x_n) - \phi(q^*)\| + \tau_n(\|\psi(q^*) - \phi(q^*)\| + 2\tau\|f(q^*)\|). \end{aligned} \tag{19}$$

By (17) and (19), we obtain

$$\begin{aligned}\|r_n - \phi(q^*)\|^2 &\le \|\tau_n(\psi(x_n) - \phi(q^*) + \mu_n f(q^*)) + (1-\tau_n)((\phi(x_n) - \mu_n f(x_n)) - (\phi(q^*) - \mu_n f(q^*)))\|^2 \\ &\le \tau_n\|\psi(x_n) - \phi(q^*) + \mu_n f(q^*)\|^2 + (1-\tau_n)\|(\phi(x_n) - \mu_n f(x_n)) - (\phi(q^*) - \mu_n f(q^*))\|^2 \\ &\le \tau_n\|\psi(x_n) - \phi(q^*) + \mu_n f(q^*)\|^2 + (1-\tau_n)[\|\phi(x_n) - \phi(q^*)\|^2 + \mu_n(\mu_n - 2\tau)\|f(x_n) - f(q^*)\|^2].\end{aligned} \tag{20}$$

Note that

$$\begin{aligned}\|y_n - \phi(q^*)\|^2 &\le \|r_n - \phi(q^*)\|^2 - \beta_n(2-\beta_n)(\theta_n\|\vartheta_n\|)^2 \\ &\le \|r_n - \phi(q^*)\|^2.\end{aligned} \tag{21}$$

Combining (13), (19) and (21), we obtain

$$\begin{aligned}\|\phi(x_{n+1}) - \phi(q^*)\| &\le (1-\zeta_n)\|\phi(x_n) - \phi(q^*)\| + \zeta_n\|y_n - \phi(q^*)\| \\ &\le (1-\zeta_n)\|\phi(x_n) - \phi(q^*)\| + \zeta_n\|r_n - \phi(q^*)\| \\ &\le (1-\zeta_n)\|\phi(x_n) - \phi(q^*)\| + \zeta_n[1-(1-\rho/\omega)\tau_n]\|\phi(x_n) - \phi(q^*)\| \\ &\quad + \zeta_n\tau_n(\|\psi(q^*) - \phi(q^*)\| + 2\tau\|f(q^*)\|) \\ &= [1-(1-\rho/\omega)\zeta_n\tau_n]\|\phi(x_n) - \phi(q^*)\| + (1-\rho/\omega)\zeta_n\tau_n\frac{\|\psi(q^*) - \phi(q^*)\| + 2\tau\|f(q^*)\|}{1-\rho/\omega}.\end{aligned} \tag{22}$$

Hence,

$$\|\phi(x_n) - \phi(q^*)\| \le \max\Big\{\|\phi(x_0) - \phi(q^*)\|, \frac{\|\psi(q^*) - \phi(q^*)\| + 2\tau\|f(q^*)\|}{1-\rho/\omega}\Big\}.$$

It follows that

$$\|x_n - q^*\| \le \frac{1}{\omega}\|\phi(x_n) - \phi(q^*)\| \le \frac{1}{\omega}\max\Big\{\|\phi(x_0) - \phi(q^*)\|, \frac{\|\psi(q^*) - \phi(q^*)\| + 2\tau\|f(q^*)\|}{1-\rho/\omega}\Big\}.$$

Thus, $\{\phi(x_n)\}$, $\{x_n\}$ and $\{r_n\}$ are bounded. By Lemma 2.1, $\{t_n\}$ is bounded. □

Theorem 3.4. *The sequence $\{x_n\}$ converges strongly to $q^* \in \Gamma$ which solves VI* (15).

Proof. By (13) and (21), we derive

$$\begin{aligned}\|\phi(x_{n+1}) - \phi(q^*)\|^2 &= \|(1-\zeta_n)(\phi(x_n) - \phi(q^*)) + \zeta_n(y_n - \phi(q^*))\|^2 \\ &= (1-\zeta_n)\|\phi(x_n) - \phi(q^*)\|^2 + \zeta_n\|y_n - \phi(q^*)\|^2 - \zeta_n(1-\zeta_n)\|y_n - \phi(x_n)\|^2 \\ &\le (1-\zeta_n)\|\phi(x_n) - \phi(q^*)\|^2 + \zeta_n\|r_n - \phi(q^*)\|^2 - \zeta_n(1-\zeta_n)\|y_n - \phi(x_n)\|^2 \\ &\quad - \zeta_n\beta_n(2-\beta_n)(\theta_n\|\vartheta_n\|)^2.\end{aligned} \tag{23}$$

From (19), we get

$$\|r_n - \phi(q^*)\|^2 \le [1-(1-\rho/\omega)\tau_n]\|\phi(x_n) - \phi(q^*)\|^2 + (1-\rho/\omega)\tau_n\Big(\frac{\|\psi(q^*) - \phi(q^*)\| + 2\tau\|f(q^*)\|}{1-\rho/\omega}\Big)^2. \tag{24}$$

Now we consider two cases. Case 1. There exists some large enough $N_0 > 0$ such that $\{\|\phi(x_n) - \phi(q^*)\|\}$ is decreasing when $n \ge N_0$. Then, $\lim_{n\to\infty}\|\phi(x_n) - \phi(q^*)\|$ exists. In terms of (23), (24) and ($A1$), we have

$$\begin{aligned}&\zeta_n(1-\zeta_n)\|y_n - \phi(x_n)\|^2 + \zeta_n\beta_n(2-\beta_n)(\theta_n\|\vartheta_n\|)^2 \\ &\le \|\phi(x_n) - \phi(q^*)\|^2 - \|\phi(x_{n+1}) - \phi(q^*)\|^2 + \zeta_n[\|r_n - \phi(q^*)\|^2 - \|\phi(x_n) - \phi(q^*)\|^2] \\ &\le \|\phi(x_n) - \phi(q^*)\|^2 - \|\phi(x_{n+1}) - \phi(q^*)\|^2 + (1-\rho/\omega)\tau_n\Big(\frac{\|\psi(q^*) - \phi(q^*)\| + 2\tau\|f(q^*)\|}{1-\rho/\omega}\Big)^2 \\ &\to 0,\end{aligned}$$

which together with $(A2)$ implies that

$$\lim_{n\to\infty} \|y_n - \phi(x_n)\| = 0, \tag{25}$$

and

$$\lim_{n\to\infty} \theta_n \|\vartheta_n\| = 0. \tag{26}$$

Furthermore, by (13), we obtain

$$\lim_{n\to\infty} \|\phi(x_{n+1}) - \phi(x_n)\| = 0. \tag{27}$$

Based on (20) and (23), we get

$$\begin{aligned}
\|\phi(x_{n+1}) - \phi(q^*)\|^2 &\leq (1-\zeta_n)\|\phi(x_n) - \phi(q^*)\|^2 + \zeta_n\|r_n - \phi(q^*)\|^2 \\
&\leq (1-\zeta_n)\|\phi(x_n) - \phi(q^*)\|^2 + \zeta_n\tau_n\|\psi(x_n) - \phi(q^*) + \mu_n f(q^*)\|^2 \\
&\quad + \zeta_n(1-\tau_n)\mu_n(\mu_n - 2\tau)\|f(x_n) - f(q^*)\|^2 + \zeta_n(1-\tau_n)\|\phi(x_n) - \phi(q^*)\|^2 \\
&\leq \|\phi(x_n) - \phi(q^*)\|^2 + \zeta_n\tau_n\|\psi(x_n) - \phi(q^*) + \mu_n f(q^*)\|^2 \\
&\quad + \zeta_n(1-\tau_n)\mu_n(\mu_n - 2\tau)\|f(x_n) - f(q^*)\|^2.
\end{aligned} \tag{28}$$

It leads to

$$\begin{aligned}
\zeta_n(1-\tau_n)\mu_n(2\tau - \mu_n)\|f(x_n) - f(q^*)\|^2 &\leq \|\phi(x_n) - \phi(q^*)\|^2 - \|\phi(x_{n+1}) - \phi(q^*)\|^2 + \zeta_n\tau_n\|\psi(x_n) - \phi(q^*) + \mu_n f(q^*)\|^2 \\
&\leq (\|\phi(x_n) - \phi(q^*)\| + \|\phi(x_{n+1}) - \phi(q^*)\|)\|\phi(x_{n+1}) - \phi(x_n)\| \\
&\quad + \zeta_n\tau_n\|\psi(x_n) - \phi(q^*) + \mu_n f(q^*)\|^2 \\
&\to 0.
\end{aligned}$$

Therefore,

$$\lim_{n\to\infty} \|f(x_n) - f(q^*)\| = 0. \tag{29}$$

Set $d_n = \phi(x_n) - \mu_n f(x_n) - (\phi(q^*) - \mu_n f(q^*))$ for all $n \geq 0$. By (6) to (8), we get

$$\begin{aligned}
\|r_n - \phi(q^*)\|^2 &= \|proj_C[\tau_n\psi(x_n) + (1-\tau_n)(\phi(x_n) - \mu_n f(x_n))] - proj_C[\phi(q^*) - \mu_n f(q^*)]\|^2 \\
&\leq \langle \tau_n\psi(x_n) + (1-\tau_n)(\phi(x_n) - \mu_n f(x_n)) - \phi(q^*) - \mu_n f(q^*), r_n - \phi(q^*)\rangle \\
&= \tau_n\langle \psi(x_n) - \phi(q^*) + \mu_n f(q^*), r_n - \phi(q^*)\rangle + (1-\tau_n)\langle d_n, r_n - \phi(q^*)\rangle \\
&\leq \frac{1}{2}\Big\{\|d_n\|^2 + \|r_n - \phi(q^*)\|^2 - \|\phi(x_n) - r_n - \mu_n(f(x_n) - f(q^*))\|^2\Big\} \\
&\quad + \tau_n\langle \psi(x_n) - \phi(q^*) + \mu_n f(q^*), r_n - \phi(q^*)\rangle \\
&\leq \frac{1}{2}\Big\{\|\phi(x_n) - \phi(q^*)\|^2 + \|r_n - \phi(q^*)\|^2 - \|\phi(x_n) - r_n\|^2 - \mu_n^2\|f(x_n) - f(q^*)\| \\
&\quad + 2\mu_n\langle \phi(x_n) - r_n, f(x_n) - f(q^*)\rangle\Big\} + \tau_n\|\psi(x_n) - \phi(q^*) + \mu_n f(q^*)\|\|r_n - \phi(q^*)\|.
\end{aligned}$$

It results in that

$$\begin{aligned}
\|r_n - \phi(q^*)\|^2 &\leq \|\phi(x_n) - \phi(q^*)\|^2 - \|\phi(x_n) - r_n\|^2 + 2\mu_n\|\phi(x_n) - r_n\|\|f(x_n) - f(q^*)\| \\
&\quad + 2\tau_n\|\psi(x_n) - \phi(q^*) + \mu_n f(q^*)\|\|r_n - \phi(q^*)\|.
\end{aligned} \tag{30}$$

From (28) and (30), we have

$$\begin{aligned}
\|\phi(x_{n+1}) - \phi(q^*)\|^2 &\leq (1-\zeta_n)\|\phi(x_n) - \phi(q^*)\|^2 + \zeta_n\|r_n - \phi(q^*)\|^2 \\
&\leq \|\phi(x_n) - \phi(q^*)\|^2 - \zeta_n\|\phi(x_n) - r_n\|^2 + 2\mu_n\|\phi(x_n) - r_n\|\|f(x_n) - f(q^*)\| \\
&\quad + 2\tau_n\|\psi(x_n) - \phi(q^*) + \mu_n f(q^*)\|\|r_n - \phi(q^*)\|,
\end{aligned}$$

which implies that

$$\begin{aligned}\zeta_n\|\phi(x_n) - r_n\|^2 \leq\ & (\|\phi(x_n) - \phi(q^*)\| + \|\phi(x_{n+1}) - \phi(q^*)\|)\|\phi(x_{n+1}) - \phi(x_n)\| \\ & + 2\mu_n\|\phi(x_n) - r_n\|\|f(x_n) - f(q^*)\| \\ & + 2\tau_n\|\psi(x_n) - \phi(q^*) + \mu_n f(q^*)\|\|r_n - \phi(q^*)\|.\end{aligned} \tag{31}$$

According to $(A1)$, $(A2)$, (27), (29) and (31), we deduce

$$\lim_{n\to\infty} \|\phi(x_n) - r_n\| = 0. \tag{32}$$

By Lemma 2.2, $\{\vartheta_n\}$ is bounded. Based on (26), we derive

$$\lim_{n\to+\infty} g(w_n, r_n) = \lim_{n\to+\infty}(\theta_n\|\vartheta_n\|)\|\vartheta_n\| = 0. \tag{33}$$

With the help of the convexity of $g(w_n, \cdot)$, we have

$$0 = g(w_n, w_n) = g(w_n, (1 - \alpha_n)r_n + \alpha_n t_n) \leq (1 - \alpha_n)g(w_n, r_n) + \alpha_n g(w_n, t_n).$$

It follows from (10) that

$$g(w_n, r_n) \geq \alpha_n[g(w_n, r_n) - g(w_n, t_n)] \geq \frac{\gamma\alpha_n}{2\lambda_n}\|t_n - r_n\|^2.$$

Combining the above inequality with (33), we deduce

$$\lim_{n\to+\infty} \alpha_n\|t_n - r_n\|^2 = 0. \tag{34}$$

In addition, from (12), we have

$$\|y_n - r_n\| = \|proj_C(r_n - \beta_n\theta_n\vartheta_n) - proj_C(r_n)\| \leq \beta_n\theta_n\|\vartheta_n\|.$$

So, we get from (26) that

$$\lim_{n\to+\infty} \|y_n - r_n\| = 0. \tag{35}$$

Since $\{x_n\}$ and $\{r_n\}$ are bounded, we can choose a subsequence $\{n_i\}$ of $\{n\}$ such that $x_{n_i} \rightharpoonup p^\dagger$ and

$$\limsup_{n\to\infty}\langle\psi(q^*) - \phi(q^*), r_n - \phi(q^*)\rangle = \lim_{i\to\infty}\langle\psi(q^*) - \phi(q^*), r_{n_i} - \phi(q^*)\rangle. \tag{36}$$

Thus, $\phi(x_{n_i}) \rightharpoonup \phi(p^\dagger)$, $y_{n_i} \rightharpoonup \phi(p^\dagger)$ and $r_{n_i} \rightharpoonup \phi(p^\dagger)$.

Now, we show $p^\dagger \in VI(C, f, \phi)$. Define an operator A by

$$A(u^\dagger) = \begin{cases} f(u^\dagger) + N_C(u^\dagger), & u^\dagger \in C, \\ \emptyset, & u^\dagger \notin C. \end{cases}$$

It is known that A is maximal ϕ-monotone. Let $(u^\dagger, u) \in G(A)$. Then, $u - f(u^\dagger) \in N_C(u^\dagger)$ and $\langle\phi(u^\dagger) - \phi(x_n), u - f(u^\dagger)\rangle \geq 0$. Since

$$\langle\phi(u^\dagger) - r_n, r_n - [\tau_n\psi(x_n) + (1 - \tau_n)(\phi(x_n) - \mu_n f(x_n))]\rangle \geq 0,$$

we have

$$\langle\phi(u^\dagger) - r_n, \frac{r_n - \phi(x_n)}{\mu_n} + f(x_n)\rangle + \frac{\tau_n}{\mu_n}\langle\phi(u^\dagger) - r_n, \phi(x_n) - \mu_n f(x_n) - \psi(x_n)\rangle \geq 0.$$

Thus,

$$\begin{aligned}\langle \phi(u^\dagger) - \phi(x_{n_i}), u\rangle &\geq \langle \phi(u^\dagger) - \phi(x_{n_i}), f(u^\dagger)\rangle \\ &\geq \langle \phi(u^\dagger) - \phi(x_{n_i}), f(u^\dagger)\rangle - \langle \phi(u^\dagger) - r_{n_i}, \frac{r_{n_i} - \phi(x_{n_i})}{\mu_{n_i}} + f(x_{n_i})\rangle \\ &\quad - \frac{\tau_{n_i}}{\mu_{n_i}}\langle \phi(u^\dagger) - r_{n_i}, \phi(x_{n_i}) - \mu_{n_i} f(x_{n_i}) - \psi(x_{n_i})\rangle \\ &= \langle \phi(u^\dagger) - \phi(x_{n_i}), f(u^\dagger) - f(x_{n_i})\rangle + \langle r_{n_i} - \phi(x_{n_i}), f(x_{n_i})\rangle \\ &\quad - \frac{\tau_{n_i}}{\mu_{n_i}}\langle \phi(u^\dagger) - r_{n_i}, \phi(x_{n_i}) - \mu_{n_i} f(x_{n_i}) - \psi(x_{n_i})\rangle \\ &\quad - \langle \phi(u^\dagger) - r_{n_i}, \frac{r_n - \phi(x_{n_i})}{\mu_{n_i}}\rangle \\ &\geq \langle r_{n_i} - \phi(x_{n_i}), f(x_{n_i})\rangle - \langle \phi(u^\dagger) - r_{n_i}, \frac{r_n - \phi(x_{n_i})}{\mu_{n_i}}\rangle \\ &\quad - \frac{\tau_{n_i}}{\mu_{n_i}}\langle \phi(u^\dagger) - r_{n_i}, \phi(x_{n_i}) - \mu_{n_i} f(x_{n_i}) - \psi(x_{n_i})\rangle.\end{aligned} \tag{37}$$

Note that $\|r_{n_i} - \phi(x_{n_i})\| \to 0$, $\tau_{n_i} \to 0$ and $\phi(x_{n_i}) \rightharpoonup \phi(p^\dagger)$. Letting $i \to \infty$ in (37), we conclude that $\langle \phi(u^\dagger) - \phi(p^\dagger), u\rangle \geq 0$. Thus, $p^\dagger \in A^{-1}(0)$. So, $p^\dagger \in VI(C, f, \phi)$.

Next, we show $\phi(p^\dagger) \in PME(C, g)$. From (34), we get

$$\lim_{i\to+\infty} \alpha_{n_i}\|t_{n_i} - r_{n_i}\|^2 = 0. \tag{38}$$

There exist two possibilities: $\limsup_{i\to+\infty} \alpha_{n_i} > 0$ and $\lim_{i\to+\infty} \alpha_{n_i} = 0$.

If $\limsup_{i\to+\infty} \alpha_{n_i} > 0$, there exists $\bar{\alpha} > 0$ and a subsequence of $\{\alpha_{n_i}\}$, still denoted by $\{\alpha_{n_i}\}$ such that for some $I_0 > 0$, $\alpha_{n_i} > \bar{\alpha}$ for all $i \geq I_0$. Consequently, by (38), we deduce

$$\lim_{i\to+\infty} \|t_{n_i} - r_{n_i}\| = 0. \tag{39}$$

Thus, $y^{k_i} \rightharpoonup \phi(p^\dagger)$. According to (9), we get

$$0 \in \partial_2 g(r_{n_i}, t_{n_i}) + \frac{1}{\lambda_{n_i}}(t_{n_i} - r_{n_i}) + N_C(t_{n_i}).$$

Then, there exists $\hat{\vartheta}_{n_i} \in \partial_2 g(r_{n_i}, t_{n_i})$ such that

$$\langle \hat{\vartheta}_{n_i}, y - t_{n_i}\rangle + \frac{1}{\lambda_{n_i}}\langle t_{n_i} - r_{n_i}, y - t_{n_i}\rangle \geq 0, \ \forall y \in C. \tag{40}$$

Thanks to the subdifferential inequality, we have

$$g(r_{n_i}, y) - g(r_{n_i}, t_{n_i}) \geq \langle \hat{\vartheta}_{n_i}, y - t_{n_i}\rangle, \ \forall y \in C. \tag{41}$$

Combining (40) and (41) to conclude

$$g(r_{n_i}, y) - g(r_{n_i}, t_{n_i}) + \frac{1}{\lambda_{n_i}}\langle t_{n_i} - r_{n_i}, y - t_{n_i}\rangle \geq 0, \ \forall y \in C. \tag{42}$$

Because of $\langle t_{n_i} - r_{n_i}, y - t_{n_i}\rangle \leq \|t_{n_i} - r_{n_i}\|\|y - t_{n_i}\|$, from (42), we get

$$g(r_{n_i}, y) - g(r_{n_i}, t_{n_i}) + \frac{1}{\lambda_{n_i}}\|t_{n_i} - r_{n_i}\|\|y - t_{n_i}\| \geq 0. \tag{43}$$

Letting $i \to +\infty$ in (43), by (39), we deduce

$$g(\phi(p^\dagger), y) \geq g(\phi(p^\dagger), \phi(p^\dagger)) = 0, \ \forall y \in C,$$

which means that $\phi(p^\dagger) \in PME(C, g)$.

For $\lim_{i\to+\infty} \alpha_{n_i} = 0$, since the sequence $\{t_{n_i}\}$ is bounded, without loss of generality, we may assume that $t_{n_i} \rightharpoonup \bar{y}$ as $i \to +\infty$. Replacing y by r_{n_i} in (43), we get

$$g(r_{n_i}, t_{n_i}) \le -\frac{1}{\lambda_{n_i}}\|t_{n_i} - r_{n_i}\|^2. \tag{44}$$

For $m_{n_i} - 1$, from (10), we have

$$g(w_{n_i,m_{n_i}-1}, r_{n_i}) - g(w_{n_i,m_{n_i}-1}, t_{n_i}) < \frac{\gamma}{2\lambda_{n_i}}\|t_{n_i} - r_{n_i}\|^2. \tag{45}$$

According to (44) and (45), we get

$$g(r_{n_i}, t_{n_i}) \le \frac{2}{\gamma}[g(w_{n_i,m_{n_i}-1}, t_{n_i}) - g(w_{n_i,m_{n_i}-1}, r_{n_i})]. \tag{46}$$

Letting $i \to +\infty$ in (46) and noting that $r_{n_i} \rightharpoonup \phi(p^\dagger)$, $t_{n_i} \rightharpoonup \bar{y}$ and $w_{n_i,m_{n_i}-1} \rightharpoonup \phi(p^\dagger)$ as $i \to +\infty$, we obtain

$$g(\phi(p^\dagger), \bar{y}) \le \frac{2}{\gamma} g(\phi(p^\dagger), \bar{y}).$$

Then, $g(\phi(p^\dagger), \bar{y}) \ge 0$ and hence $\lim_{i\to+\infty}\|t_{n_i} - r_{n_i}\| = 0$ by (44). Consequently, we can conclude that $\phi(p^\dagger) \in PME(C, g)$. Therefore, $p^\dagger \in VI(C, f, \phi) \cap \phi^{-1}(PME(C, g)) = \Gamma$.

From (36), we obtain

$$\begin{aligned}\limsup_{n\to\infty}\langle\psi(q^*) - \phi(q^*), r_n - \phi(q^*)\rangle &= \lim_{i\to\infty}\langle\psi(q^*) - \phi(q^*), r_{n_i} - \phi(q^*)\rangle \\ &= \langle\psi(q^*) - \phi(q^*), \phi(p^\dagger) - \phi(q^*)\rangle \le 0.\end{aligned} \tag{47}$$

By (8), we have

$$\begin{aligned}\|r_n - \phi(q^*)\|^2 &= \|proj_C[\tau_n\psi(x_n) + (1-\tau_n)(\phi(x_n) - \mu_n f(x_n))] - proj_C[\phi(q^*) - (1-\tau_n)\mu_n f(q^*)]\|^2 \\ &\le \langle\tau_n(\psi(x_n) - \phi(q^*)) + (1-\tau_n)w_n, r_n - \phi(q^*)\rangle \\ &= \tau_n\langle\psi(x_n) - \psi(q^*), r_n - \phi(q^*)\rangle + \tau_n\langle\psi(q^*) - \phi(q^*), r_n - \phi(q^*)\rangle + (1-\tau_n)\langle w_n, r_n - \phi(q^*)\rangle \\ &\le [1 - (1-\rho/\omega)\tau_n]\|\phi(x_n) - \phi(q^*)\|\|r_n - \phi(q^*)\| + \tau_n\langle\psi(q^*) - \phi(q^*), r_n - \phi(q^*)\rangle \\ &\le \frac{1 - (1-\rho/\omega)\tau_n}{2}\|\phi(x_n) - \phi(q^*)\|^2 + \frac{1}{2}\|r_n - \phi(q^*)\|^2 + \tau_n\langle\psi(q^*) - \phi(q^*), r_n - \phi(q^*)\rangle.\end{aligned}$$

It follows that

$$\|r_n - \phi(q^*)\|^2 \le [1 - (1-\rho/\omega)\tau_n]\|\phi(x_n) - \phi(q^*)\|^2 + 2\tau_n\langle\psi(q^*) - \phi(q^*), r_n - \phi(q^*)\rangle.$$

Therefore,

$$\begin{aligned}\|\phi(x_{n+1}) - \phi(q^*)\|^2 &\le (1-\zeta_n)\|\phi(x_n) - \phi(q^*)\|^2 + \zeta_n\|r_n - \phi(q^*)\|^2 \\ &\le [1 - (1-\rho/\omega)\zeta_n\tau_n]\|\phi(x_n) - \phi(q^*)\|^2 + 2\zeta_n\tau_n\langle\psi(q^*) - \phi(q^*), r_n - \phi(q^*)\rangle.\end{aligned} \tag{48}$$

By Lemma 2.3 and (48), we conclude that $\phi(x_n) \to \phi(q^*)$ and $x_n \to q^*$.

Case 2. There exists an integer $n_0 > N$ such that $\|\phi(x_{n_0}) - \phi(q^*)\| \le \|\phi(x_{n_0+1}) - \phi(q^*)\|$. Let $\phi_n = \{\|\phi(x_n) - \phi(q^*)\|^2\}$. Then, we have $\phi_{n_0} \le \phi_{n_0+1}$. Let $\{\varphi_n\}$ be an integer sequence defined by, for all $n \ge n_0$,

$$\varphi(n) = \max\{l \in \mathbb{N} | n_0 \le l \le n, \phi_l \le \phi_{l+1}\}.$$

Note that $\varphi(n)$ is non-decreasing and satisfies $\lim_{n\to\infty}\varphi(n) = \infty$ and $\phi_{\varphi(n)} \le \phi_{\varphi(n)+1}$, $\forall n \ge n_0$.

Similarly, we can deduce

$$\limsup_{n\to\infty}\langle\psi(q^*) - \phi(q^*), r_{\varphi(n)} - \phi(q^*)\rangle \le 0 \tag{49}$$

and

$$\phi_{\varphi(n)+1} \leq [1 - \frac{2(1-\rho/\omega)\tau_{\varphi}(n)\zeta_{\varphi}(n)}{1-\tau_{\varphi}(n)\rho/\omega}]\phi_{\varphi(n)} + \frac{2(1-\rho/\omega)\tau_{\varphi}(n)\zeta_{\varphi}(n)}{1-\tau_{\varphi}(n)\rho/\omega} \times \left\{\frac{\tau_{\varphi}(n)}{2(1-\rho/\omega)}M + \frac{1}{1-\rho/\omega}\langle\psi(q^*) - \phi(q^*), r_{\varphi}(n) - \phi(q^*)\rangle\right\}. \tag{50}$$

Note that $\phi_{\varphi(n)} \leq \phi_{\varphi(n)+1}$. By (50), we have

$$\phi_{\varphi(n)} \leq \frac{\tau_{\varphi}(n)}{2(1-\rho/\omega)}M + \frac{1}{1-\rho/\omega}\langle\psi(q^*) - \phi(q^*), r_{\varphi}(n) - \phi(q^*)\rangle. \tag{51}$$

Based on (49) and (51), we derive

$$\limsup_{n\to\infty} \phi_{\varphi(n)} \leq 0,$$

and thus

$$\lim_{n\to\infty} \phi_{\varphi(n)} = 0. \tag{52}$$

From (50), we can deduce

$$\limsup_{n\to\infty} \phi_{\varphi(n)+1} \leq \limsup_{n\to\infty} \phi_{\varphi(n)}.$$

This together with (52) implies that

$$\lim_{n\to\infty} \phi_{\varphi(n)+1} = 0.$$

By Lemma 2.4, we obtain

$$0 \leq \phi_n \leq \max\{\phi_{\varphi(n)}, \phi_{\varphi(n)+1}\}.$$

Therefore, $\phi_n \to 0$. That is, $\phi(x_n) \to \phi(q^*)$ and thus $x_n \to q^*$. This completes the proof. □

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