Filomat 36:12 (2022), 3941–3953
https://doi.org/10.2298/FIL2212941Z

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

An Iterative Approach to the Solution of Split Variational Inequalities

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Abstract. In this paper, we investigate the split variational inequality problem in Hilbert spaces, in which an operator is ϖ-inverse strongly ψ-monotone operator and another operator is pseudomonotone. We construct an iterative algorithm for solving the split variational inequality problem. We show the strong convergence of the suggested algorithm.

1. Introduction

The split problems have received much attention due to their applications in image denoising, signal processing and image reconstruction, see, [3, 5, 8–11, 17–20, 22, 24, 34, 47–49] and references therein. In this paper, we continue to investigate the split problems and relevant iterative algorithms. To begin with, let us first recall several concepts of the split problems and several popular algorithms in the literature. Recall that the split feasibility problem is to find a point $u^\dagger$ verifying

$$u^\dagger \in C \quad \text{and} \quad Au^\dagger \in Q \tag{1}$$

where C and Q are two closed convex subsets of two Hilbert spaces H and E, respectively, and $A : H \to E$ is a bounded linear operator.

A critical algorithm for solving (1) is Byrne's ([3]) CQ algorithm listed as follows

$$x_{n+1} = proj_C(x_n - \varpi A^*(I - proj_Q)Ax_n), n \geq 0, \tag{2}$$

where ϖ is step-size and $proj_C : H \to C$ is the orthogonal projection.

Consequently, CQ algorithm and its variant forms have been studied and developed, see, [4, 16, 28]. In the case where C and Q in (1) are the fixed point sets $Fix(S)$ and $Fix(T)$ of operators $S : H \to H$ and $T : E \to E$, respectively, problem (1) is called the split fixed point problem by Censor and Segal [10]. More precisely, the split fixed point problem is to find a point $u^\dagger \in H$ such that

$$u^\dagger \in Fix(S) \quad \text{and} \quad Au^\dagger \in Fix(T). \tag{3}$$

2020 *Mathematics Subject Classification*. 49J53; 49M37; 65K10; 90C25.
Keywords. Split variational inequality, Inverse strongly ψ-monotone operator, Pseudomonotone operator, Projection.
Received: 19 January 2021; Accepted: 16 February 2022
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There are a large number iterative algorithms for solving (3), see ([30, 32, 33, 35, 40, 44]). Among them, a basic algorithm has the following form

$$x_{n+1} = S(x_n - \varpi A^*(I - T)Ax_n),\ n \geq 0. \tag{4}$$

In the present paper, we are interested in the following split variational inequality problem of finding a point $u^\dagger$ such that

$$u^\dagger \in VI(C, f) \text{ and } Au^\dagger \in VI(C, g), \tag{5}$$

where $f : C \to H$ and $g : C \to H$ are two nonlinear operators, $VI(C, f)$ denotes the solution set of the variational inequality of finding a point $x^\dagger \in C$ such that

$$\langle f(x^\dagger), x - x^\dagger \rangle \geq 0,\ \forall x \in C, \tag{6}$$

and $VI(C, g)$ means the solution set of the variational inequality of finding a point $x^\dagger \in C$ such that

$$\langle g(x^\dagger), x - x^\dagger \rangle \geq 0,\ \forall x \in C. \tag{7}$$

Variational inequalities play critical roles and provide a valuable mathematical modelling for studying many important problems arising in water resources, finance, economics, medical images and so on ([1, 2, 6, 7, 9, 12, 25, 29, 38, 41–43, 46, 50]). A great deal of algorithms for solving (7) have been investigated, see, e.g., [14, 15, 26, 31, 36, 37, 39, 45]. An important technique for solving (7) is to use projection which has the following manner

$$x_{n+1} = proj_C[x_n - \zeta_n g(x_n)],\ n \geq 0. \tag{8}$$

By applying algorithms (4) and (8), Censor, Gibali and Reich [8] constructed the following iterative algorithm for solving (5):

$$x_{n+1} = proj_C(I - \zeta f)[x_n - \varpi A^*(I - proj_Q(I - \zeta g))Ax_n],\ n \geq 0. \tag{9}$$

Motivated by the work in this direction, in the present paper, we investigate the following split variational inequality problem of finding a point $u^\dagger$ such that

$$u^\dagger \in VI(C, f, \psi) \text{ and } \psi(u^\dagger) \in VI(C, g), \tag{10}$$

where $\psi : C \to C$ is a nonlinear operator, $f : C \to H$ is a ϖ-inverse strongly ψ-monotone operator, g is a pseudomonotone operator and $VI(C, f, \psi)$ denotes the solution set of the generalized variational inequality ([23]) of finding a point $x^\dagger \in C$ such that

$$\langle f(x^\dagger), \psi(x) - \psi(x^\dagger) \rangle \geq 0,\ \forall x \in C. \tag{11}$$

Here, use Γ to denote the solution set of problem (11), that is,

$$\Gamma = VI(C, f, \psi) \bigcap \psi^{-1}(VI(C, g)).$$

In this paper, we construct an iterative algorithm for solving (10). We show that the presented algorithm strongly converges to en element in Γ.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. Let $C \subset H$ be a nonempty closed convex set. Recall that an operator $f : C \to H$ is said to be

- strongly monotone if

$$\langle f(u) - f(v), u - v \rangle \geq \tau \|u - v\|^2,\ \forall u, v \in C. \tag{12}$$

- ϖ-inverse strongly ψ-monotone if there exists a constant $\varpi > 0$ such that

$$\langle f(u) - f(v), \psi(u) - \psi(v)\rangle \geq \varpi\|f(u) - f(v)\|^2,\ \forall u, v \in C.$$

- f is pseudomonotone if

$$\langle f(v), u - v\rangle \geq 0 \Rightarrow \langle f(u), u - v\rangle \geq 0,\ \forall u, v \in C.$$

- L-Lipschitz ($L > 0$) if

$$\|f(u) - f(v)\| \leq L\|u - v\|,\ \forall u, v \in C.$$

If $L < 1$, then f is said to be L-contraction. If $L = 1$, then f is said to be nonexpansive.

An operator $T : H \to 2^H$ is said to be monotone iff $\langle x - y, u - v\rangle \geq 0$ for all $x, y \in dom(T)$, $u \in T(x)$, and $v \in T(y)$. A monotone operator T on H is said to be maximal iff its graph is not strictly contained in the graph of any other monotone operator on H.

For $\forall x^\dagger \in H$, there exists a unique point in C, denoted by $proj_C[x^\dagger]$ satisfying

$$\|x^\dagger - proj_C[x^\dagger]\| \leq \|x - x^\dagger\|,\ \forall x \in C.$$

Moreover, $proj_C$ is firmly nonexpansive, that is,

$$\|proj_C[q^*] - proj_C[v^\dagger]\|^2 \leq \langle proj_C[q^*] - proj_C[v^\dagger], q^* - v^\dagger\rangle,\ \forall q^*, v^\dagger \in H. \tag{13}$$

Further, $proj_C$ has the following property

$$\langle q^* - proj_C[q^*], x^\dagger - proj_C[q^*]\rangle \leq 0,\ \forall q^* \in H, x^\dagger \in C. \tag{14}$$

Lemma 2.1 ([13]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $g : H \to H$ be a continuous and pseudomonotone operator. Then $x^\dagger \in VI(C, g)$ iff $x^\dagger$ solves the following variational inequality*

$$\langle g(u^\dagger), u^\dagger - x^\dagger\rangle \geq 0,\ \forall u^\dagger \in H.$$

Lemma 2.2 ([27]). *Let $\{\varpi_n\} \subset [0, \infty)$, $\{\alpha_n\} \subset (0, 1)$ and $\{\eta_n\}$ be real number sequences. Suppose the following conditions are satisfied*

(i) $\varpi_{n+1} \leq (1 - \alpha_n)\varpi_n + \eta_n, \forall n \geq 1$;

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(iii) $\limsup_{n\to\infty} \frac{\eta_n}{\alpha_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\eta_n| < \infty$.

Then $\lim_{n\to\infty} \varpi_n = 0$.

Lemma 2.3 ([21]). *Let $\{x_n\}$ be a real number sequence. Assume there exists at least a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that*

$$x_{n_k} \leq x_{n_k+1}$$

for all $k \geq 0$. For every $n \geq N_0$, define an integer sequence $\{\beta(n)\}$ as

$$\beta(n) = \max\{i \leq n : x_{n_i} < x_{n_i+1}\}.$$

Then $\beta(n) \to \infty$ as $n \to \infty$ and for all $n \geq N_0$, $\max\{x_{\beta(n)}, x_n\} \leq x_{\beta(n)+1}$.

3. Main results

Let H be a real Hilbert space and C be a nonempty closed convex subset of H. Let $h : C \to C$ be a ρ-contractive operator. Let $\psi : C \to C$ be a weakly continuous and τ-strongly monotone operator with $R(\psi) = C$. Let $f : C \to H$ be a ϖ-inverse strongly ψ-monotone operator. Let the operator g be pseudomonotone on H, weakly sequentially continuous and L-Lipschitz continuous on C. Let $\{\varpi_n\}$ and $\{\alpha_n\}$ be two real number sequences in $[0, 1]$ and $\{\zeta_n\}$ be a real number sequence in $(0, \infty)$. Let $\varrho \in (0, 1)$, $\beta \in (0, 1)$, $\varsigma \in (0, 1)$ and $\alpha \in (0, 2)$ be four constants.

In what follows, we suppose that $\Gamma \neq \emptyset$. Here, we present an iterative algorithm for solving problem (5).

Algorithm 3.1. *Let $x_0 \in C$ be a guess. Set $n = 0$.*

Step 1. For given x_n, compute

$$z_n = proj_C[\varpi_n h(x_n) + (1 - \varpi_n)(\psi(x_n) - \zeta_n f(x_n))]. \tag{15}$$

Step 2. Compute

$$y_n = proj_C[z_n - \beta\varrho^{n^\dagger} g(z_n)], \tag{16}$$

where $n^\dagger$ is chosen the smallest nonnegative integer number such that

$$\beta\varrho^{n^\dagger}\|g(y_n) - g(z_n)\| \leq \varsigma\|y_n - z_n\|. \tag{17}$$

Write $\varrho^{n^\dagger} = \varrho_n$. If $y_n = z_n$, then set $u_n = z_n$ and go to Step 3. Otherwise, compute

$$u_n = proj_C\Big[z_n - \alpha(1 - \varsigma)\|y_n - z_n\|^2 \frac{v_n}{\|v_n\|^2}\Big], \tag{18}$$

where $v_n = z_n - y_n + \beta\varrho_n g(y_n)$.

Step 3. Compute

$$\psi(x_{n+1}) = (1 - \alpha_n)\psi(x_n) + \alpha_n u_n. \tag{19}$$

Step 4. Set $n := n + 1$ and return to step 1.

Proposition 3.2. *(i)* (17) *is valid and $0 < \frac{\varrho\varsigma}{\beta L} < \varrho_n \leq 1 (\forall n \geq 0)$. (ii) If $y_n = z_n$, then $y_n \in VI(C, g)$. (iii) If $y_n \neq z_n$, then $v_n = z_n - y_n + \beta\varrho_n g(y_n) \neq 0$.*

Proof. (i) Since g is L-Lipschitz, $\beta\varrho^{n^\dagger}\|g(y_n) - g(z_n)\| \leq \beta\varrho^{n^\dagger} L\|y_n - z_n\|$. Thus, we can choose $n^\dagger$ such that $\varrho^{n^\dagger} \leq \frac{\varsigma}{\beta L}$. Hence, (17) holds. If $n^\dagger = 0$, then $\varrho^{n^\dagger} = 1$. If $n^\dagger > 0$, then $0 < \frac{\varrho\varsigma}{\beta L} < \varrho^{n^\dagger} < 1$.

(ii) is obvious. (iii) Let $z^\dagger \in \Gamma$. Since $y_n \in C$ and $z_n \in C$, we have $\langle g(z^\dagger), y_n - z^\dagger\rangle \geq 0$ and $\langle g(z^\dagger), z_n - z^\dagger\rangle \geq 0$. By the pseudomonotonicity of g, we deduce

$$\langle g(y_n), y_n - z^\dagger\rangle \geq 0, \tag{20}$$

and

$$\langle g(z_n), z_n - z^\dagger\rangle \geq 0. \tag{21}$$

In addition, from (14) and (16), we have

$$\langle z_n - \beta\varrho_n g(z_n) - y_n, y_n - z^\dagger\rangle \geq 0. \tag{22}$$

Thus, in accordance with (20)-(22), we obtain

$$\begin{aligned}\langle v_n, z_n - z^\dagger\rangle &= \langle z_n - y_n + \beta\varrho_n g(y_n), z_n - z^\dagger\rangle \\ &= \langle z_n - y_n - \beta\varrho_n g(z_n), z_n - z^\dagger\rangle + \beta\varrho_n\langle g(z_n), z_n - z^\dagger\rangle \\ &\quad + \beta\varrho_n\langle g(y_n), z_n - y_n\rangle + \beta\varrho_n\langle g(y_n), y_n - z^\dagger\rangle \\ &\geq \langle z_n - y_n - \beta\varrho_n g(z_n), z_n - z^\dagger\rangle + \beta\varrho_n\langle g(y_n), z_n - y_n\rangle \\ &= \langle z_n - y_n - \beta\varrho_n(g(z_n) - g(y_n)), z_n - y_n\rangle \\ &\quad + \langle z_n - y_n - \beta\varrho_n g(z_n), y_n - z^\dagger\rangle \\ &\geq \langle z_n - y_n - \beta\varrho_n(g(z_n) - g(y_n)), z_n - y_n\rangle \\ &\geq \|z_n - y_n\|^2 - \beta\varrho_n\|g(z_n) - g(y_n)\|\|z_n - y_n\| \\ &\geq (1-\varsigma)\|z_n - y_n\|^2 \\ &> 0.\end{aligned} \tag{23}$$

Therefore, $v_n = z_n - y_n + \beta\varrho_n g(y_n) \neq 0$. □

Remark 3.3. *(i) By the strong monotonicity of ψ, we conclude from* (12) *that*

$$\|\psi(x) - \psi(y)\| \geq \tau\|x - y\|, \ \forall x, y \in C. \tag{24}$$

Thus, the following varaitional inequality has a unique solution denoted by q^*,

$$\langle h(x) - \psi(x), \psi(y) - \psi(x)\rangle \leq 0, \quad \forall y \in \Gamma.$$

So, we have

$$\langle h(q^*) - \psi(q^*), \psi(y) - \psi(q^*)\rangle \leq 0, \quad \forall y \in \Gamma. \tag{25}$$

(ii) Since f is ϖ-inverse strongly ψ-monotone, for any $u \in C$, we have

$$\begin{aligned}&\|(\psi(u) - \zeta f(u)) - (\psi(q^*) - \zeta f(q^*))\|^2 \\ &\quad = \|\psi(u) - \psi(q^*)\|^2 - 2\zeta\langle f(u) - f(q^*), \psi(u) - \psi(q^*)\rangle + \zeta^2\|f(u) - f(q^*)\|^2 \\ &\quad \leq \|\psi(u) - \psi(q^*)\|^2 - 2\zeta\varpi\|f(u) - f(q^*)\|^2 + \zeta^2\|f(u) - f(q^*)\|^2 \\ &\quad \leq \|\psi(u) - \psi(q^*)\|^2 + \zeta(\zeta - 2\varpi)\|f(u) - f(q^*)\|^2.\end{aligned} \tag{26}$$

Theorem 3.4. *Suppose that the following assumptions hold:*

(C1): $\lim_{n\to\infty}\varpi_n = 0$ *and* $\sum_{n=1}^{\infty}\varpi_n = \infty$;

(C2): $0 < \liminf_{n\to\infty}\alpha_n \leq \limsup_{n\to\infty}\alpha_n < 1$;

(C3): $0 < \rho < \tau < 2\varpi$ *and* $0 < \liminf_{n\to\infty}\zeta_n \leq \limsup_{n\to\infty}\zeta_n < 2\varpi$.

Then the sequence $\{x_n\}$ generated by Algorithm 3.1 converges strongly to $q^* \in \Gamma$ which solves VI (25).

Proof. Note that $q^* \in VI(C, f, \psi)$ and $\psi(q^*) \in VI(C, g)$. Then, $\psi(q^*) = proj_C[\psi(q^*) - \zeta_n f(q^*)]$ for all $n \geq 0$. By virtue of (26), we get

$$\begin{aligned}\|(\psi(x_n) - \zeta_n f(x_n)) - (\psi(q^*) - \zeta_n f(q^*))\|^2 &\leq \|\psi(x_n) - \psi(q^*)\|^2 + \zeta_n(\zeta_n - 2\varpi)\|f(x_n) - f(q^*)\|^2 \\ &\leq \|\psi(x_n) - \psi(q^*)\|^2,\end{aligned} \tag{27}$$

and

$$\|\psi(x_{n+1}) - \zeta_{n+1}f(x_{n+1}) - (\psi(x_n) - \zeta_{n+1}f(x_n))\|^2 \leq \|\psi(x_{n+1}) - \psi(x_n)\|^2 + \zeta_{n+1}(\zeta_{n+1} - 2\varpi)\|f(x_{n+1}) - f(x_n)\|^2. \tag{28}$$

Based on (15), (24) and (27), we obtain

$$\begin{aligned}
\|z_n - \psi(q^*)\| &= \|proj_C[\varpi_n h(x_n) + (1-\varpi_n)(\psi(x_n) - \zeta_n f(x_n))] - proj_C[\psi(q^*) - \zeta_n f(q^*)]\| \\
&\le \|\varpi_n(h(x_n) - \psi(q^*) + \zeta_n f(q^*)) + (1-\varpi_n)((\psi(x_n) - \zeta_n f(x_n)) - (\psi(q^*) - \zeta_n f(q^*)))\| \\
&\le \varpi_n\|h(x_n) - h(q^*)\| + \varpi_n\|h(q^*) - \psi(q^*) + \zeta_n f(q^*)\| \\
&\quad + (1-\varpi_n)\|(\psi(x_n) - \zeta_n f(x_n)) - (\psi(q^*) - \zeta_n f(q^*))\| \\
&\le \varpi_n\rho\|x_n - q^*\| + \varpi_n\|h(q^*) - \psi(q^*) + \zeta_n f(q^*)\| + (1-\varpi_n)\|\psi(x_n) - \psi(q^*)\| \\
&\le \varpi_n\rho/\tau\|\psi(x_n) - \psi(q^*)\| + \varpi_n\|h(q^*) - \psi(q^*) + \zeta_n f(q^*)\| + (1-\varpi_n)\|\psi(x_n) - \psi(q^*)\| \\
&= [1 - (1-\rho/\tau)\varpi_n]\|\psi(x_n) - \psi(q^*)\| + \varpi_n\|h(q^*) - \psi(q^*) + \zeta_n f(q^*)\| \\
&\le [1 - (1-\rho/\tau)\varpi_n]\|\psi(x_n) - \psi(q^*)\| + \varpi_n(\|h(q^*) - \psi(q^*)\| + 2\varpi\|f(q^*)\|).
\end{aligned} \tag{29}$$

According to (27) and (29), we obtain

$$\begin{aligned}
\|z_n - \psi(q^*)\|^2 &\le \|\varpi_n(h(x_n) - \psi(q^*) + \zeta_n f(q^*)) + (1-\varpi_n)((\psi(x_n) - \zeta_n f(x_n)) - (\psi(q^*) - \zeta_n f(q^*)))\|^2 \\
&\le \varpi_n\|h(x_n) - \psi(q^*) + \zeta_n f(q^*)\|^2 + (1-\varpi_n)\|(\psi(x_n) - \zeta_n f(x_n)) - (\psi(q^*) - \zeta_n f(q^*))\|^2 \\
&\le \varpi_n\|h(x_n) - \psi(q^*) + \zeta_n f(q^*)\|^2 + (1-\varpi_n)[\|\psi(x_n) - \psi(q^*)\|^2 + \zeta_n(\zeta_n - 2\varpi)\|f(x_n) - f(q^*)\|^2].
\end{aligned} \tag{30}$$

Letting $z^\dagger = \psi(q^*)$ in (23), we have $\langle v_n, z_n - \psi(q^*)\rangle \ge (1-\varsigma)\|z_n - y_n\|^2$. This together with (18) implies that

$$\begin{aligned}
\|u_n - \psi(q^*)\|^2 &\le \left\| z_n - \psi(q^*) - \alpha(1-\varsigma)\|y_n - z_n\|^2 \frac{v_n}{\|v_n\|^2} \right\|^2 \\
&= \|z_n - \psi(q^*)\|^2 + \frac{\alpha^2(1-\varsigma)^2\|y_n - z_n\|^4}{\|v_n\|^2} - \frac{2\alpha(1-\varsigma)\|y_n - z_n\|^2}{\|v_n\|^2}\langle v_n, z_n - \psi(q^*)\rangle \\
&\le \|z_n - \psi(q^*)\|^2 - (2-\alpha)\alpha(1-\varsigma)^2\frac{\|y_n - z_n\|^4}{\|v_n\|^2} \\
&\le \|z_n - \psi(q^*)\|^2.
\end{aligned} \tag{31}$$

Combining (19), (29) and (31), we obtain

$$\begin{aligned}
\|\psi(x_{n+1}) - \psi(q^*)\| &\le (1-\alpha_n)\|\psi(x_n) - \psi(q^*)\| + \alpha_n\|u_n - \psi(q^*)\| \\
&\le (1-\alpha_n)\|\psi(x_n) - \psi(q^*)\| + \alpha_n\|z_n - \psi(q^*)\| \\
&\le (1-\alpha_n)\|\psi(x_n) - \psi(q^*)\| + \alpha_n[1 - (1-\rho/\tau)\varpi_n]\|\psi(x_n) - \psi(q^*)\| \\
&\quad + \alpha_n\varpi_n(\|h(q^*) - \psi(q^*)\| + 2\varpi\|f(q^*)\|) \\
&= [1 - (1-\rho/\tau)\alpha_n\varpi_n]\|\psi(x_n) - \psi(q^*)\| \\
&\quad + (1-\rho/\tau)\alpha_n\varpi_n\frac{\|h(q^*) - \psi(q^*)\| + 2\varpi\|f(q^*)\|}{1-\rho/\tau}.
\end{aligned} \tag{32}$$

An induction yields

$$\|\psi(x_n) - \psi(q^*)\| \le \max\left\{\|\psi(x_0) - \psi(q^*)\|, \frac{\|h(q^*) - \psi(q^*)\| + 2\varpi\|f(q^*)\|}{1-\rho/\tau}\right\}.$$

It follows that

$$\|x_n - q^*\| \le \frac{1}{\tau}\|\psi(x_n) - \psi(q^*)\| \le \frac{1}{\tau}\max\left\{\|\psi(x_0) - \psi(q^*)\|, \frac{\|h(q^*) - \psi(q^*)\| + 2\varpi\|f(q^*)\|}{1-\rho/\tau}\right\}.$$

Thus, $\{\psi(x_n)\}$, $\{x_n\}$, $\{z_n\}$ and $\{u_n\}$ are bounded.

By (19), we derive

$$\psi(x_{n+1}) - \psi(x_n) = \alpha_n(u_n - \psi(x_n)),\ n \ge 0. \tag{33}$$

Then,

$$\langle \psi(x_{n+1}) - \psi(x_n), \psi(x_n) - \psi(q^*) \rangle = \alpha_n \langle u_n - \psi(x_n), \psi(x_n) - \psi(q^*) \rangle. \tag{34}$$

It follows that

$$\begin{aligned} &\|\psi(x_{n+1}) - \psi(q^*)\|^2 - \|\psi(x_n) - \psi(q^*)\|^2 - \|\psi(x_{n+1}) - \psi(x_n)\|^2 \\ &\qquad = \alpha_n[\|u_n - \psi(q^*)\|^2 - \|\psi(x_n) - \psi(q^*)\|^2 - \|u_n - \psi(x_n)\|^2]. \end{aligned} \tag{35}$$

By (31), (33) and (35), we obtain

$$\begin{aligned} &\|\psi(x_{n+1}) - \psi(q^*)\|^2 - \|\psi(x_n) - \psi(q^*)\|^2 \\ &\quad = \alpha_n[\|u_n - \psi(q^*)\|^2 - \|\psi(x_n) - \psi(q^*)\|^2 - \|u_n - \psi(x_n)\|^2] + \alpha_n^2\|u_n - \psi(x_n)\|^2 \\ &\quad = \alpha_n[\|u_n - \psi(q^*)\|^2 - \|\psi(x_n) - \psi(q^*)\|^2] - \alpha_n(1-\alpha_n)\|u_n - \psi(x_n)\|^2 \\ &\quad \le \alpha_n[\|z_n - \psi(q^*)\|^2 - \|\psi(x_n) - \psi(q^*)\|^2] - \alpha_n(1-\alpha_n)\|u_n - \psi(x_n)\|^2. \end{aligned} \tag{36}$$

In terms of (29), we get

$$\|z_n - \psi(q^*)\|^2 \le [1 - (1-\rho/\tau)\varpi_n]\|\psi(x_n) - \psi(q^*)\|^2 + (1-\rho/\tau)\varpi_n\Big(\frac{\|h(q^*) - \psi(q^*)\| + 2\varpi\|f(q^*)\|}{1-\rho/\tau}\Big)^2. \tag{37}$$

There are two cases. Case 1. For some large enough $N_0 > 0$, $\{\|\psi(x_n) - \psi(q^*)\|\}$ is decreasing when $n \ge N_0$. Hence, $\lim_{n\to\infty}\|\psi(x_n) - \psi(q^*)\|$ exists. Based on (36), (37) and ($C1$), we have

$$\begin{aligned} \alpha_n(1-\alpha_n)\|u_n - \psi(x_n)\|^2 &\le \|\psi(x_n) - \psi(q^*)\|^2 - \|\psi(x_{n+1}) - \psi(q^*)\|^2 + \alpha_n[\|z_n - \psi(q^*)\|^2 - \|\psi(x_n) - \psi(q^*)\|^2] \\ &\le \|\psi(x_n) - \psi(q^*)\|^2 - \|\psi(x_{n+1}) - \psi(q^*)\|^2 + (1-\rho/\tau)\varpi_n\Big(\frac{\|h(q^*) - \psi(q^*)\| + 2\varpi\|f(q^*)\|}{1-\rho/\tau}\Big)^2 \\ &\to 0. \end{aligned}$$

This together with ($C2$) implies that

$$\lim_{n\to\infty}\|u_n - \psi(x_n)\| = 0. \tag{38}$$

Moreover, by (33), we have

$$\lim_{n\to\infty}\|\psi(x_{n+1}) - \psi(x_n)\| = 0. \tag{39}$$

Thanks to (19), (30) and (31), we deduce

$$\begin{aligned} \|\psi(x_{n+1}) - \psi(q^*)\|^2 &= \|(1-\alpha_n)(\psi(x_n) - \psi(q^*)) + \alpha_n(u_n - \psi(q^*))\|^2 \\ &\le (1-\alpha_n)\|\psi(x_n) - \psi(q^*)\|^2 + \alpha_n\|u_n - \psi(q^*)\|^2 \\ &\le (1-\alpha_n)\|\psi(x_n) - \psi(q^*)\|^2 + \alpha_n\|z_n - \psi(q^*)\|^2 \\ &\le (1-\alpha_n)\|\psi(x_n) - \psi(q^*)\|^2 + \alpha_n\varpi_n\|h(x_n) - \psi(q^*) + \zeta_n f(q^*)\|^2 \\ &\quad + \alpha_n(1-\varpi_n)\zeta_n(\zeta_n - 2\varpi)\|f(x_n) - f(q^*)\|^2 + \alpha_n(1-\varpi_n)\|\psi(x_n) - \psi(q^*)\|^2 \\ &\le \|\psi(x_n) - \psi(q^*)\|^2 + \alpha_n\varpi_n\|h(x_n) - \psi(q^*) + \zeta_n f(q^*)\|^2 \\ &\quad + \alpha_n(1-\varpi_n)\zeta_n(\zeta_n - 2\varpi)\|f(x_n) - f(q^*)\|^2. \end{aligned} \tag{40}$$

It results in that

$$\begin{aligned} \alpha_n(1-\varpi_n)\zeta_n(2\varpi - \zeta_n)\|f(x_n) - f(q^*)\|^2 &\le \|\psi(x_n) - \psi(q^*)\|^2 - \|\psi(x_{n+1}) - \psi(q^*)\|^2 + \alpha_n\varpi_n\|h(x_n) - \psi(q^*) + \zeta_n f(q^*)\|^2 \\ &\le (\|\psi(x_n) - \psi(q^*)\| + \|\psi(x_{n+1}) - \psi(q^*)\|)\|\psi(x_{n+1}) - \psi(x_n)\| \\ &\quad + \alpha_n\varpi_n\|h(x_n) - \psi(q^*) + \zeta_n f(q^*)\|^2 \\ &\to 0. \end{aligned}$$

Accordingly,

$$\lim_{n\to\infty}\|f(x_n)-f(q^*)\|=0. \tag{41}$$

Set $w_n=\psi(x_n)-\zeta_n f(x_n)-(\psi(q^*)-\zeta_n f(q^*))$ for all $n\geq 0$. Applying inequality (14) to (15), we have

$$\begin{aligned}\|z_n-\psi(q^*)\|^2 &= \|proj_C[\varpi_n h(x_n)+(1-\varpi_n)(\psi(x_n)-\zeta_n f(x_n))]-proj_C[\psi(q^*)-\zeta_n f(q^*)]\|^2\\ &\leq \langle \varpi_n h(x_n)+(1-\varpi_n)(\psi(x_n)-\zeta_n f(x_n))-\psi(q^*)-\zeta_n f(q^*), z_n-\psi(q^*)\rangle\\ &= \varpi_n\langle h(x_n)-\psi(q^*)+\zeta_n f(q^*), z_n-\psi(q^*)\rangle+(1-\varpi_n)\langle w_n, z_n-\psi(q^*)\rangle\\ &\leq \varpi_n\langle h(x_n)-\psi(q^*)+\zeta_n f(q^*), z_n-\psi(q^*)\rangle\\ &\quad+\frac{1}{2}\Big\{\|w_n\|^2+\|z_n-\psi(q^*)\|^2-\|\psi(x_n)-z_n-\zeta_n(f(x_n)-f(q^*))\|^2\Big\}\\ &\leq \varpi_n\|h(x_n)-\psi(q^*)+\zeta_n f(q^*)\|\|z_n-\psi(q^*)\|\\ &\quad+\frac{1}{2}\Big\{\|\psi(x_n)-\psi(q^*)\|^2+\|z_n-\psi(q^*)\|^2-\|\psi(x_n)-z_n\|^2-\zeta_n^2\|f(x_n)-f(q^*)\|\\ &\quad+2\zeta_n\langle\psi(x_n)-z_n, f(x_n)-f(q^*)\rangle\Big\}.\end{aligned}$$

It yields

$$\begin{aligned}\|z_n-\psi(q^*)\|^2 \leq{}& \|\psi(x_n)-\psi(q^*)\|^2-\|\psi(x_n)-z_n\|^2+2\zeta_n\|\psi(x_n)-z_n\|\|f(x_n)-f(q^*)\|\\ &+2\varpi_n\|h(x_n)-\psi(q^*)+\zeta_n f(q^*)\|\|z_n-\psi(q^*)\|.\end{aligned} \tag{42}$$

According to (31), (40) and (42), we obtain

$$\begin{aligned}\|\psi(x_{n+1})-\psi(q^*)\|^2 &\leq (1-\alpha_n)\|\psi(x_n)-\psi(q^*)\|^2+\alpha_n\|z_n-\psi(q^*)\|^2\\ &\leq \|\psi(x_n)-\psi(q^*)\|^2-\alpha_n\|\psi(x_n)-z_n\|^2+2\zeta_n\|\psi(x_n)-z_n\|\|f(x_n)-f(q^*)\|\\ &\quad+2\varpi_n\|h(x_n)-\psi(q^*)+\zeta_n f(q^*)\|\|z_n-\psi(q^*)\|,\end{aligned}$$

which implies that

$$\begin{aligned}\alpha_n\|\psi(x_n)-z_n\|^2 \leq{}& (\|\psi(x_n)-\psi(q^*)\|+\|\psi(x_{n+1})-\psi(q^*)\|)\|\psi(x_{n+1})-\psi(x_n)\|\\ &+2\zeta_n\|\psi(x_n)-z_n\|\|f(x_n)-f(q^*)\|\\ &+2\varpi_n\|h(x_n)-\psi(q^*)+\zeta_n f(q^*)\|\|z_n-\psi(q^*)\|.\end{aligned} \tag{43}$$

On the basis of $(C1)$, $(C2)$, (39), (41) and (43), we deduce

$$\lim_{n\to\infty}\|\psi(x_n)-z_n\|=0. \tag{44}$$

As a result of (31) and (40), we get

$$\begin{aligned}\|\psi(x_{n+1})-\psi(q^*)\|^2 &\leq (1-\alpha_n)\|\psi(x_n)-\psi(q^*)\|^2+\alpha_n\|u_n-\psi(q^*)\|^2\\ &\leq (1-\alpha_n)\|\psi(x_n)-\psi(q^*)\|^2+\alpha_n\|z_n-\psi(q^*)\|^2-\alpha_n(2-\alpha)\alpha(1-\varsigma)^2\frac{\|y_n-z_n\|^4}{\|v_n\|^2},\end{aligned}$$

which together with (44) implies that

$$\begin{aligned}\alpha_n(2-\alpha)\alpha(1-\varsigma)^2\frac{\|y_n-z_n\|^4}{\|v_n\|^2} &\leq (1-\alpha_n)(\|\psi(x_n)-\psi(q^*)\|^2-\|\psi(x_{n+1})-\psi(q^*)\|^2)\\ &\quad+\alpha_n(\|z_n-\psi(q^*)\|+\|\psi(x_{n+1})-\psi(q^*)\|)\|\psi(x_{n+1})-z_n\|\\ &\to 0.\end{aligned}$$

Therefore,

$$\lim_{n\to\infty}\frac{\|y_n-z_n\|^2}{\|v_n\|}=0. \tag{45}$$

Since $\{v_n\}$ is bounded, it follows from (45) that

$$\lim_{n\to\infty} \|y_n - z_n\| = 0. \tag{46}$$

Note that $\{x_n\}$ and $\{z_n\}$ are bounded. Choose a subsequence $\{n_i\}$ of $\{n\}$ verifying $x_{n_i} \rightharpoonup p^\dagger$ and

$$\limsup_{n\to\infty} \langle h(q^*) - \psi(q^*), z_n - \psi(q^*)\rangle = \lim_{i\to\infty} \langle h(q^*) - \psi(q^*), z_{n_i} - \psi(q^*)\rangle. \tag{47}$$

Thus, $\psi(x_{n_i}) \rightharpoonup \psi(p^\dagger)$, $y_{n_i} \rightharpoonup \psi(p^\dagger)$ and $z_{n_i} \rightharpoonup \psi(p^\dagger)$.

Next, we prove $z \in VI(C, f, \psi)$. Set

$$T(u^\dagger) = \begin{cases} f(u^\dagger) + N_C(u^\dagger), & u^\dagger \in C, \\ \emptyset, & u^\dagger \notin C. \end{cases}$$

Then, T is maximal ψ-monotone. Pick up $(u^\dagger, u) \in G(T)$. Hence, $u - f(u^\dagger) \in N_C(u^\dagger)$ and $\langle \psi(u^\dagger) - \psi(x_n), u - f(u^\dagger)\rangle \geq 0$. Observe that

$$\langle \psi(u^\dagger) - z_n, z_n - [\varpi_n h(x_n) + (1 - \varpi_n)(\psi(x_n) - \zeta_n f(x_n))]\rangle \geq 0.$$

It follows that

$$\langle \psi(u^\dagger) - z_n, \frac{z_n - \psi(x_n)}{\zeta_n} + f(x_n)\rangle + \frac{\varpi_n}{\zeta_n}\langle \psi(u^\dagger) - z_n, \psi(x_n) - \zeta_n f(x_n) - h(x_n)\rangle \geq 0.$$

Thus,

$$\begin{aligned} \langle \psi(u^\dagger) - \psi(x_{n_i}), u\rangle &\geq \langle \psi(u^\dagger) - \psi(x_{n_i}), f(u^\dagger)\rangle \\ &\geq \langle \psi(u^\dagger) - \psi(x_{n_i}), f(u^\dagger)\rangle - \langle \psi(u^\dagger) - z_{n_i}, \frac{z_{n_i} - \psi(x_{n_i})}{\zeta_{n_i}} + f(x_{n_i})\rangle \\ &\quad - \frac{\varpi_{n_i}}{\zeta_{n_i}}\langle \psi(u^\dagger) - z_{n_i}, \psi(x_{n_i}) - \zeta_{n_i} f(x_{n_i}) - h(x_{n_i})\rangle \\ &= \langle \psi(u^\dagger) - \psi(x_{n_i}), f(u^\dagger) - f(x_{n_i})\rangle + \langle z_{n_i} - \psi(x_{n_i}), f(x_{n_i})\rangle \\ &\quad - \frac{\varpi_{n_i}}{\zeta_{n_i}}\langle \psi(u^\dagger) - z_{n_i}, \psi(x_{n_i}) - \zeta_{n_i} f(x_{n_i}) - h(x_{n_i})\rangle \\ &\quad - \langle \psi(u^\dagger) - z_{n_i}, \frac{z_n - \psi(x_{n_i})}{\zeta_{n_i}}\rangle \\ &\geq \langle z_{n_i} - \psi(x_{n_i}), f(x_{n_i})\rangle - \langle \psi(u^\dagger) - z_{n_i}, \frac{z_n - \psi(x_{n_i})}{\zeta_{n_i}}\rangle \\ &\quad - \frac{\varpi_{n_i}}{\zeta_{n_i}}\langle \psi(u^\dagger) - z_{n_i}, \psi(x_{n_i}) - \zeta_{n_i} f(x_{n_i}) - h(x_{n_i})\rangle. \end{aligned} \tag{48}$$

Note that $\|z_{n_i} - \psi(x_{n_i})\| \to 0$, $\varpi_{n_i} \to 0$ and $\psi(x_{n_i}) \rightharpoonup \psi(p^\dagger)$. Letting $i \to \infty$ in (48), we conclude that $\langle \psi(u^\dagger) - \psi(p^\dagger), u\rangle \geq 0$. Thus, $p^\dagger \in T^{-1}(0)$. So, $p^\dagger \in VI(C, f, \psi)$.

Next, we show $\psi(p^\dagger) \in VI(C, g)$. From (22), we have

$$\langle z_{n_i} - \beta\varrho_n g(z_{n_i}) - y_{n_i}, y_{n_i} - x^\dagger\rangle \geq 0, \ \forall x^\dagger \in C.$$

It yields

$$\langle g(z_{n_i}), x^\dagger - z_{n_i}\rangle \geq \langle g(z_{n_i}), y_{n_i} - z_{n_i}\rangle + \frac{1}{\beta\varrho_n}\langle y_{n_i} - x^\dagger, y_{n_i} - z_{n_i}\rangle, \ \forall x^\dagger \in C. \tag{49}$$

Owing to (46) and (49), we obtain

$$\liminf_{i\to\infty} \langle g(z_{n_i}), x^\dagger - z_{n_i}\rangle \geq 0, \ \forall x^\dagger \in C. \tag{50}$$

In view of (50), there exists a positive real numbers sequence $\{\sigma_j\}$ such that $\lim_{j\to\infty}\sigma_j = 0$. For each σ_j, there exists the smallest positive integer k_i such that

$$\langle g(z_{n_{i_j}}), x^\dagger - z_{n_{i_j}}\rangle + \sigma_j \geq 0,\ \forall j \geq k_i. \tag{51}$$

Moreover, for each $j > 0$, $g(z_{n_{i_j}}) \neq 0$. Setting $\varphi(z_{n_{i_j}}) = \frac{g(z_{n_{i_j}})}{\|g(z_{n_{i_j}})\|^2}$, we have $\langle g(z_{n_{i_j}}), \varphi(z_{n_{i_j}})\rangle = 1$. According to (51), we obtain

$$\langle g(z_{n_{i_j}}), x^\dagger + \sigma_j\varphi(z_{n_{i_j}}) - z_{n_{i_j}}\rangle \geq 0.$$

By the pseudomonotonicity of f, we get

$$\langle g(x^\dagger + \sigma_j\varphi(z_{n_{i_j}})), x^\dagger + \sigma_j\varphi(z_{n_{i_j}}) - z_{n_{i_j}}\rangle \geq 0,$$

which implies that

$$\begin{aligned}\langle g(x^\dagger), x^\dagger - z_{n_{i_j}}\rangle \geq\ & \langle g(x^\dagger) - g(x^\dagger + \sigma_j\varphi(z_{n_{i_j}})), x^\dagger + \sigma_j\varphi(z_{n_{i_j}}) - z_{n_{i_j}}\rangle \\ & + \langle g(x^\dagger), -\sigma_j\varphi(z_{n_{i_j}})\rangle.\end{aligned} \tag{52}$$

Because of $g(z_{n_{i_j}}) \rightharpoonup g(\psi(p^\dagger))$, we have

$$\liminf_{j\to\infty}\|g(z_{n_{i_j}})\| \geq \|g(\psi(p^\dagger))\| > 0.$$

Then,

$$\lim_{j\to\infty}\|\sigma_j\varphi(z_{n_{i_j}})\| = \lim_{j\to\infty}\frac{\sigma_j}{\|g(z_{n_{i_j}})\|} = 0.$$

This together with (52), we deduce

$$\langle g(x^\dagger), x^\dagger - \psi(p^\dagger)\rangle \geq 0. \tag{53}$$

It follows from Lemma 2.1 that $\psi(p^\dagger) \in VI(C, g)$. Therefore, $p^\dagger \in VI(C, f, \psi) \cap \psi^{-1}(VI(C, g)) = \Gamma$.

From (47), we obtain

$$\begin{aligned}\limsup_{n\to\infty}\langle h(q^*) - \psi(q^*), z_n - \psi(q^*)\rangle &= \lim_{i\to\infty}\langle h(q^*) - \psi(q^*), z_{n_i} - \psi(q^*)\rangle \\ &= \langle h(q^*) - \psi(q^*), \psi(p^\dagger) - \psi(q^*)\rangle \leq 0.\end{aligned} \tag{54}$$

By (15), we have

$$\begin{aligned}\|z_n - \psi(q^*)\|^2 =\ & \|proj_C[\varpi_n h(x_n) + (1-\varpi_n)(\psi(x_n) - \zeta_n f(x_n))] \\ & - proj_C[\psi(q^*) - (1-\varpi_n)\zeta_n f(q^*)]\|^2 \\ \leq\ & \langle \varpi_n(h(x_n) - \psi(q^*)) + (1-\varpi_n)w_n, z_n - \psi(q^*)\rangle \\ =\ & \varpi_n\langle h(x_n) - h(q^*), z_n - \psi(q^*)\rangle + \varpi_n\langle h(q^*) - \psi(q^*), z_n - \psi(q^*)\rangle \\ & + (1-\varpi_n)\langle w_n, z_n - \psi(q^*)\rangle \\ \leq\ & [1 - (1-\rho/\tau)\varpi_n]\|\psi(x_n) - \psi(q^*)\|\|z_n - \psi(q^*)\| \\ & + \varpi_n\langle h(q^*) - \psi(q^*), z_n - \psi(q^*)\rangle \\ \leq\ & \frac{1 - (1-\rho/\tau)\varpi_n}{2}\|\psi(x_n) - \psi(q^*)\|^2 + \frac{1}{2}\|z_n - \psi(q^*)\|^2 \\ & + \varpi_n\langle h(q^*) - \psi(q^*), z_n - \psi(q^*)\rangle.\end{aligned}$$

It follows that

$$\|z_n - \psi(q^*)\|^2 \leq [1 - (1-\rho/\tau)\varpi_n]\|\psi(x_n) - \psi(q^*)\|^2 + 2\varpi_n\langle h(q^*) - \psi(q^*), z_n - \psi(q^*)\rangle.$$

Therefore,

$$\begin{aligned}\|\psi(x_{n+1}) - \psi(q^*)\|^2 &\leq (1-\alpha_n)\|\psi(x_n) - \psi(q^*)\|^2 + \alpha_n\|z_n - \psi(q^*)\|^2 \\ &\leq [1-(1-\rho/\tau)\alpha_n\varpi_n]\|\psi(x_n) - \psi(q^*)\|^2 \\ &\quad + 2\alpha_n\varpi_n\langle h(q^*) - \psi(q^*), z_n - \psi(q^*)\rangle.\end{aligned} \tag{55}$$

By Lemma 2.2 and (55), we conclude that $\psi(x_n) \to \psi(q^*)$ and $x_n \to q^*$.

Case 2. There exists an integer $n_0 > N$ such that $\|\psi(x_{n_0}) - \psi(q^*)\| \leq \|\psi(x_{n_0+1}) - \psi(q^*)\|$. Let $\psi_n = \{\|\psi(x_n) - \psi(q^*)\|^2\}$. Then, we have $\psi_{n_0} \leq \psi_{n_0+1}$. Let $\{\beta_n\}$ be an integer sequence defined by, for all $n \geq n_0$,

$$\beta(n) = \max\{l \in \mathbb{N} | n_0 \leq l \leq n, \psi_l \leq \psi_{l+1}\}.$$

Note that $\beta(n)$ is non-decreasing and satisfies $\lim_{n\to\infty}\beta(n) = \infty$ and $\psi_{\beta(n)} \leq \psi_{\beta(n)+1}, \forall n \geq n_0$.

Similarly, we can deduce

$$\limsup_{n\to\infty}\langle h(q^*) - \psi(q^*), z_{\beta(n)} - \psi(q^*)\rangle \leq 0 \tag{56}$$

and

$$\begin{aligned}\psi_{\beta(n)+1} \leq{}& [1 - \frac{2(1-\rho/\tau)\varpi_\beta(n)\alpha_\beta(n)}{1-\varpi_\beta(n)\rho/\tau}]\psi_{\beta(n)} + \frac{2(1-\rho/\tau)\varpi_\beta(n)\alpha_\beta(n)}{1-\varpi_\beta(n)\rho/\tau} \\ &\times\left\{\frac{\varpi_\beta(n)}{2(1-\rho/\tau)}M + \frac{1}{1-\rho/\tau}\langle h(q^*) - \psi(q^*), z_\beta(n) - \psi(q^*)\rangle\right\}.\end{aligned} \tag{57}$$

Note that $\psi_{\beta(n)} \leq \psi_{\beta(n)+1}$. By (57), we have

$$\psi_{\beta(n)} \leq \frac{\varpi_\beta(n)}{2(1-\rho/\tau)}M + \frac{1}{1-\rho/\tau}\langle h(q^*) - \psi(q^*), z_\beta(n) - \psi(q^*)\rangle. \tag{58}$$

Based on (56) and (58), we derive

$$\limsup_{n\to\infty}\psi_{\beta(n)} \leq 0,$$

and thus

$$\lim_{n\to\infty}\psi_{\beta(n)} = 0. \tag{59}$$

From (57), we can deduce

$$\limsup_{n\to\infty}\psi_{\beta(n)+1} \leq \limsup_{n\to\infty}\psi_{\beta(n)}.$$

This together with (59) implies that

$$\lim_{n\to\infty}\psi_{\beta(n)+1} = 0.$$

By Lemma 2.3, we obtain

$$0 \leq \psi_n \leq \max\{\psi_{\beta(n)}, \psi_{\beta(n)+1}\}.$$

Therefore, $\psi_n \to 0$. That is, $\psi(x_n) \to \psi(q^*)$ and thus $x_n \to q^*$. This completes the proof. □

In Algorithm 3.1, choose $\psi = I$, identity operator and $f : C \to H$ is a ϖ-inverse strongly monotone operator. Then, we have the following algorithm and corollary.

Algorithm 3.5. *Let $x_0 \in C$ be a guess. Set $n = 0$.*

Step 1. For given x_n, compute

$$z_n = proj_C[\varpi_n h(x_n) + (1-\varpi_n)(x_n - \zeta_n f(x_n))].$$

Step 2. Compute

$$y_n = proj_C[z_n - \beta\varrho^{n^\dagger} g(z_n)],$$

where $n^\dagger$ is chosen the smallest nonnegative integer number such that

$$\beta\varrho^{n^\dagger}\|g(y_n) - g(z_n)\| \le \varsigma\|y_n - z_n\|.$$

Write $\varrho^{n^\dagger} = \varrho_n$. If $y_n = z_n$, then set $u_n = z_n$ and go to Step 3. Otherwise, compute

$$u_n = proj_C\Big[z_n - \alpha(1-\varsigma)\|y_n - z_n\|^2 \frac{v_n}{\|v_n\|^2}\Big],$$

where $v_n = z_n - y_n + \beta\varrho_n g(y_n)$.

Step 3. Compute

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n u_n.$$

Step 4. Set $n := n+1$ and return to step 1.

Corollary 3.6. *Suppose that $\Gamma_1 := VI(C, f) \cap VI(C, g) \neq \emptyset$. Assume that conditions* (C1)-(C3) *are satisfied. Then the sequence $\{x_n\}$ generated by Algorithm 3.5 converges strongly to $q^* = proj_{\Gamma_1} h(q^*)$.*

Acknowledgments

LijunZhu was supported by the National Natural Science Foundation of China [grant number 11861003], the Major ResearchProjects of Ningxia [grant numbers 2021BEG03049], Major Scientific and Technological Innovation Projects of Yinchuan [grant number NZ17015, 2022RKX03 andNXYLXK2017B09].

Zhichuan Zhu was supported by the National Natural Science Foundation of China [grant number 12126333], the Social Science Foundation of Liaoning province [grant No. L22BTJ001].

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