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Multiplicity of Solutions for Kirchhoff Type Problem Involving Eigenvalue

A. Rezvani[a], M. Alimohammady[a,b], B. Agheli[a]

[a]*Department of Mathematics, Qaemshahr Branch, Islamic Azad University, Qaemshahr, Iran*
[b]*Department of Mathematics, University of Mazandran, Babolsar, Iran*

Abstract. This paper is concerned with the existence and multiplicity of weak solutions for a $p(x)$-Kirchhoff problem by using variational method and genus theory. We prove the simplicity and boundedness of the principal eigenvalue.

1. Introduction

In this paper, the authors study the following nonlocal $p(x)$-Kirchhoff problem:

$$\begin{cases} -\left(a + b\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) = \lambda|u|^{q(x)-2}u - g(x)|u|^{p(x)-2}\,u & in\,\Omega, \\ u = 0 & on\,\partial\Omega, \end{cases} \tag{1}$$

where $a, b > 0$ are constants, $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, λ is a positive real parameter and g is a continuous function and $p(x), q(x)$ are real continuous functions on $\bar{\Omega}$ with $1 < q(x) < p(x) < p^*(x) = \frac{Np(x)}{N-p(x)}$ and $p(x) < N$ for all $x \in \overline{\Omega}$.

Based on variational methods, especially the Krasnoselskii's genus theorem, the existence of infinitely many solutions has been proved. The main conclusion of the paper is correct and improves the related results on this topic.

In recent years, there has been a great deal of work done on Kirchhoff $p(x)$-Laplacian equations, especially concerning the existence, multiplicity, uniqueness and reqularity of solutions. Some important and interesting results can be found, for example, in [1, 3, 6, 9, 16–18] and references therein.

Many results have been obtained on this kind of problems, for example [3, 9, 19]. With the aid of the three-critical-point theorem related to local linking due to Brezis and Nirenberg [1], Zeng-Ou, -Li [19] proved the existence of at least two nontrivial solutions for the following nonlocal Kirchhoff type problem:

$$\begin{cases} -\left(a + b\int_\Omega |\nabla u|^2 dx\right)\Delta u = f(x,u) & in\,\Omega, \\ u = 0 & on\,\partial\Omega, \end{cases} \tag{2}$$

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Corresponding author: M. Alimohammady
Email addresses: Asieh.Rezvani@gmail.com (A. Rezvani), amohsen@umz.ac.ir (M. Alimohammady), b.agheli@qaemiau.ac.ir (B. Agheli)

where Ω is a smooth bounded domain in $\mathbb{R}^N(N \geq 1)$, $a, b > 0$ are real numbers and $f(x, u)$ is a continuous function which 3-sublinear or asymptotically critical growth at infinity.

Recently, Che-Chen [3] extended problem (2) to the following Kirchhoff-type equations:

$$\begin{cases} -\left(a + b\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)\Delta u + V(x)u + \mu\,\phi|u|^{p-2}u = f(x,u) + g(x,u) & \text{in } \mathbb{R}^3, \\ (-\Delta)^{\frac{\alpha}{2}}\phi = \mu|u|^p & \text{in } \mathbb{R}^3, \end{cases} \tag{3}$$

where $a > 0$, $b, \mu \geq 0$ are constants, $\alpha \in (0, 3)$, $p \in [2, 3 + 2\alpha)$ and the potential $V(x)$ may be unbounded from below. Under some mild conditions on $f(x, u)$ and $g(x, u)$, the authors proved that the above system has infinitely many nontrivial solutions.

In recent years, the study of new Kirchhoff type equations (1) was firstly extended by Hamdani et al. [9] to the case involving the $p(x)$-Laplacian of type:

$$\begin{cases} -\left(a - b\int_{\Omega} \frac{1}{p(x)}|\nabla u|^{p(x)} dx\right)\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) = \lambda|u|^{p(x)-2}u + g(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4}$$

where $a \geq b > 0$ are constants, $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $p \in C(\overline{\Omega})$ with $N > p(x) > 1$, λ is a real parameter and g is a continuous function. The authors in [9] proved the existence and multiplicity of solutions by using the Mountain pass theorem and Fountain theorem.

Motivated by the works above, we shall study the existence and multiplicity of the solutions for problem (1) by using variational method and Krasnoselskiis genus theory. Moreover, we study the simplicity and the boundedness of the principal eigenvalue of problem (1) , with $p(x) \equiv p$.

The outline of this paper is the following: In Section 2, we give some preliminary results. Section 3 is devoted to the proof of Theorem 3.5. Finally, we give the proofs of Theorem 4.5, Theorem 4.6 and Theorem 4.8 in Section 4.

2. Preliminaries

For the reader's convenience, we recall some necessary knowledge and propositions concerning the Lebesgue and Sobolev spaces.

Let Ω be a bounded domain of $\mathbb{R}^N$, denote $C_+(\bar{\Omega}) = \{p(x)\,;\, p(x) \in C(\bar{\Omega}), p(x) > 1, \text{ for all } x \in \bar{\Omega}\}$.

For any continuous function $p : \Omega \to (1, \infty)$,

$$p^- := \inf_{x\in\Omega} p(x) \quad \text{and} \quad p^+ := \sup_{x\in\Omega} p(x).$$

Let $p \in C_+(\bar{\Omega})$, the variable exponent Lebesgue space is defined by

$$L_{p(x)}(\Omega) := \left\{u : \Omega \to \mathbb{R} \text{ is a measurable function} : \int_\Omega |u(x)|^{p(x)} dx < +\infty\right\}.$$

The space is equipped with the so-called Luxemburg norm:

$$|u|_{p(x)} := \inf\left\{\mu > 0\colon \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \leq 1\right\}.$$

$L_{p(x)}(\Omega)$ is a separable and reflexive Banach space [9].

The modular of $L_{p(x)}(\Omega)$ is defined by

$$\rho_{p(x)}(u) := \int_\Omega |u(x)|^{p(x)} dx.$$

Proposition 2.1. *[18]. The space* $\left(L_{p(x)}(\Omega), |u|_{p(x)}\right)$ *is separable, uniformly convex, reflexive and its conjugate space is* $\left(L_{q(x)}(\Omega), |u|_{q(x)}\right)$*, where* $q(x)$ *is the conjugate function of* $p(x)$*, i.e.,*

$$\frac{1}{p(x)} + \frac{1}{q(x)} = 1, \quad \forall x \in \Omega.$$

For all $u \in L_{p(x)}(\Omega)$, $v \in L_{q(x)}(\Omega)$*, the Hölder's type inequality*

$$\left|\int_\Omega u\, v\, dx\right| \le \left(\frac{1}{p^-} + \frac{1}{q^-}\right)|u|_{p(x)}\,|v|_{q(x)} \le 2|u|_{p(x)}\,|v|_{q(x)} \tag{5}$$

holds.

Proposition 2.2. *[5]. Suppose that* $u_n, u \in L_{p(x)}(\Omega)$*, then the following properties hold:*

$$|u|_{p(x)} > 1 \Rightarrow |u|^{p^-}_{p(x)} \le \rho_{p(x)}(u) \le |u|^{p^+}_{p(x)};$$
$$|u|_{p(x)} < 1 \Rightarrow |u|^{p^+}_{p(x)} \le \rho_{p(x)}(u) \le |u|^{p^-}_{p(x)};$$
$$|u|_{p(x)} < 1\,(\text{respectively}, = 1; > 1) \Leftrightarrow \rho_{p(x)}(u) < 1\ (\text{respectively},\ = 1; > 1)\,;$$
$$|u_n|_{p(x)} \longrightarrow 0(\text{respectively}, \longrightarrow +\infty) \Leftrightarrow \rho_{p(x)}(u_n) \longrightarrow 0(\text{respectively}, \longrightarrow +\infty);$$
$$\lim_{n\to\infty} |u_n - u|_{p(x)} = 0 \Longleftrightarrow \lim_{n\to\infty} \rho_{p(x)}(u_n - u) = 0.$$

The Sobolev space $W^{1,p(x)}(\Omega)$ is defined as

$$W^{1,p(x)}(\Omega) := \left\{u \in L_{p(x)}(\Omega) \colon |\nabla u| \in L_{p(x)}(\Omega)\right\},$$

is a separable and reflexive Banach spaces. For more details, we refer to [7, 15, 17].

$W^{1,p(x)}(\Omega)$ is equipped with the norm

$$\|u\|_{1,p(x)} = \|u\|_{p(x)} + \|\nabla u\|_{p(x)}.$$

On $W^{1,p(x)}(\Omega)$ we may consider the following equivalent norm

$$\|u\|_{p(x)} = |\nabla u|_{p(x)}.$$

$W_0^{1,p(x)}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\| = \inf\left\{\mu > 0\colon \int_\Omega \left|\frac{\nabla u(x)}{\mu}\right|^{p(x)} dx \le 1\right\}.$$

It is well known that

$$W_0^{1,p(x)}(\Omega) = \left\{u\,;\, u\Big|_{\partial\Omega} = 0,\ u \in L_{p(x)}(\Omega),\ |\nabla u| \in L_{p(x)}(\Omega)\right\}.$$

For more details, we refer to [8, 9].

Proposition 2.3. *(Sobolev Embedding[16]) For* $p, q \in C_+(\bar{\Omega})$ *such that* $1 < q(x) < p^*(x)$ *for all* $x \in \bar{\Omega}$*, there is a Continuous embedding*

$$W_0^{1,p(x)}(\Omega) \hookrightarrow L_{q(x)}(\Omega)$$

is continuous and compact. Therefore, there is a constant $C_0 > 0$*, such that* $\|u_n\|_{q(x)} \le C_0\|u_n\|$.

Proposition 2.4. *(Poincare Inequality[7]) There is a constant $C > 0$, such that*

$$|u|_{p(x)} \leq C\|\nabla u\|_{p(x)} \tag{6}$$

for all $u \in W_0^{1,p(x)}(\Omega)$.

Definition 2.5. *Let E be a real Banach space.*
Set $R := \{A \subset E - \{0\}\,;\ A$ is compact and symmetric$\}$. Let $A \in R$ and we define the genus of A as follows:

$$\gamma(A) := \inf\{m \geq 1\,;\ \exists f \in C(A, \mathbb{R}^m \setminus \{0\})\,;\ f \text{ is odd}\}$$

and $\gamma(A) = \infty$, if does not exist such a map f. $\gamma(\emptyset) = 0$ by definition.
For more details, we refer to [6].

Theorem 2.6. *[6]. Let $\Omega \subset \mathbb{R}^N$ be bounded symmetric with boundary $\partial\Omega$. Assume that $0 \in \Omega$, then $\gamma(\partial\Omega) = N$.*

Corollary 2.7. *[6]. The genus of unit sphere S^{N-1} of the space $\mathbb{R}^N$ is N.*

3. Main results

Definition 3.1. *$u \in W_0^{1,p(x)}(\Omega)$ is called a weak solution of (1) if*

$$\left(a + b\int \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx\right)\int_\Omega |\nabla u|^{p(x)-2}\nabla u\,\nabla\varphi\,dx = \lambda\int_\Omega |u|^{q(x)-2}u\,\varphi\,dx - \int_\Omega g(x)|u|^{p(x)-2}u\varphi\,dx$$

for all $\varphi \in W_0^{1,p(x)}(\Omega)$.

The energy functional associated with problem (1)

$$J(u) = a\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx + \frac{b}{2}\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx\right)^2 - \lambda\int_\Omega \frac{1}{q(x)}|u|^{q(x)}\,dx + \int_\Omega \frac{g(x)}{p(x)}|u|^{p(x)}\,dx$$

for all $u \in W_0^{1,p(x)}(\Omega)$ is well defined, C^1 functional and for all $u, \varphi \in W_0^{1,p(x)}(\Omega)$

$$\begin{aligned}\langle J'(u), \varphi\rangle = &\left(a + b\int \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx\right)\int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla\varphi\,dx\\ &- \lambda\int_\Omega |u|^{q(x)-2}u\varphi\,dx + \int_\Omega g(x)|u|^{p(x)-2}u\varphi\,dx,\end{aligned}$$

we can observe that the critical points of this energy functional are weak solutions of the problem (1).

We consider $\Omega \subset \mathbb{R}^N$ $(N > 3)$ is a bounded domain with smooth boundary and $p, q \in C_+(\Omega)$ such that:

$$1 < q^- \leq q(x) \leq q^+ < p^- \leq p(x) \leq p^+ < 2p^- < p^*(x) = \frac{Np(x)}{N - p(x)} \tag{7}$$

and $p(x) < N$ for all $x \in \bar{\Omega}$.

Furthermore, we assume that the function $g(x)$ satisfies the hypothesis:

(H) $g : \bar{\Omega} \to [0, \infty)$, $g \in L^\infty(\bar{\Omega})$.

Proposition 3.2. *[1]. Define the functional $\Lambda = \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx$. Then $\Lambda : W_0^{1,p(x)}(\Omega) \to \mathbb{R}$ is convex. The mapping $\Lambda' : W_0^{1,p(x)}(\Omega) \to \left(W_0^{1,p(x)}(\Omega)\right)^*$ is a strictly monotone, bounded homeomorphism, and of (S_+) type, namely $u_n \rightharpoonup u$ (weakly) and $\overline{\lim}_{n\to\infty}\,(\Lambda'(u_n), u_n - u) \leq 0$ implies $u_n \to u$ (strongly).*

Definition 3.3. *The functional J satisfies the Palais-Smale condition at the level $C \in \mathbb{R}$ (" $(PS)_C$ condition" for short) if for every sequence $\{u_n\} \subset W_0^{1,p(x)}(\Omega)$ satisfying*

$$J(u_n) \to C \ \text{ and } \ J'(u_n) \to 0 \ \text{ as } \ n \to \infty,$$

has a convergence subsequence of $\{u_n\}$.

Theorem 3.4. *[6]. Let $J \in C^1(W_0^{1,p(x)}, \mathbb{R})$ and satisfies the $(PS)_C$ Condition. We assume the following conditions*

i) J is bounded from below and even;

ii) There is a compact set $T \in R$ such that $\gamma(T) = k$ and

$$\sup_{x \in T} J(x) < J(0).$$

Then problem, (1) has at least k pairs of distinct critical points, and their corresponding critical values are less than $J(0)$.

Theorem 3.5. *If (7) holds. Then there are at least k pairs of different critical point for (1).*

Lemma 3.6. *The functional J satisfies the $(PS)_C$ condition.*

Proof. We prove that $\{u_n\}$ is bounded in $W_0^{1,p(x)}(\Omega)$.
Let $\{u_n\} \subset W_0^{1,p(x)}(\Omega)$ be a $(PS)_C$ sequence. Arguing by contradiction we assume that, passing eventually to a subsequence, still denote by $\{u_n\}$, we have $\|u_n\| \to +\infty$ as $n \to +\infty$. we choose $0 < \theta < \left\{\frac{1}{q^+}, \frac{1}{p^+}, \frac{p^-}{2p^{+2}}\right\}$.

By Definition 3.3, for n large enough, we have

$$\begin{aligned} C + \|u_n\| &\geq J(u_n) - \theta\langle J'(u_n), u_n\rangle \\ &\geq a\left(\frac{1}{p^+} - \theta\right)\int_\Omega |\nabla u_n|^{p(x)}\,dx + b\left(\frac{1}{2p^{+2}} - \frac{\theta}{p^-}\right)\left(\int_\Omega |\nabla u_n|^{p(x)}\,dx\right)^2 \\ &\quad - \lambda\left(\frac{1}{q^-} - \theta\right)\int_\Omega |u_n|^{q(x)}\,dx + \left(\frac{1}{p^+} - \theta\right)\int_\Omega g(x)|u_n|^{p(x)}\,dx. \end{aligned}$$

By Proposition 2.3, there is a constant $C_0 > 0$, such that

$$-\lambda\left(\frac{1}{q^-} - \theta\right)\|u_n\|_q \geq -\lambda C_0\left(\frac{1}{q^-} - \theta\right)\|u_n\|.$$

So

$$C + \|u_n\| \geq a\left(\frac{1}{p^+} - \theta\right)\|u_n\|^{p^-} + b\left(\frac{1}{2p^{+2}} - \frac{\theta}{p^-}\right)\|u_n\|^{2p^-} - \lambda C_0\left(\frac{1}{q^-} - \theta\right)\|u_n\|.$$

Dividing the above inequality by $\|u_n\|$ and passing to the limit as $n \to \infty$, we obtain a contradiction. It follows from (7) that $\{u_n\}$ is bounded in $W_0^{1,p(x)}(\Omega)$.

It follows from Proposition 2.3, where $1 \leq s(x) < p^*(x)$ and $W_0^{1,p(x)}(\Omega)$ is reflexive Banach space, we may assume that

$$\begin{aligned} &u_n \rightharpoonup u \quad \text{in} \quad W_0^{1,p(x)}(\Omega), \quad u_n \to u \quad \text{in} \quad L_{s(x)}(\Omega), \\ &u_n(x) \to u(x), \ \text{a.e. in } \Omega. \end{aligned} \tag{8}$$

Hölder's inequality and (8), imply that

$$\left|\int_\Omega |u_n|^{p(x)-2}u_n(u_n-u)\,dx\right| \le \int_\Omega |u_n|^{p(x)-1}|u_n-u|_{p(x)}$$
$$\le \left||u_n|^{p(x)-1}\right|_{\frac{p(x)}{p(x)-1}} |u_n-u|_{p(x)} \to 0 \quad \text{as} \quad n\to\infty.$$

Thus

$$\lim_{n\to\infty}\int |u_n|^{p(x)-2}u_n(u_n-u)\,dx = 0, \tag{9}$$

similarly

$$\lim_{n\to\infty}\int |u_n|^{q(x)-2}\,u_n(u_n-u)\,dx \to 0.$$

Then $\langle J'(u_n), u_n-u\rangle \to 0$,

$$\langle J'(u_n), u_n-u\rangle = \left(a+b\int \frac{1}{p(x)}|\nabla u_n|^{p(x)}\,dx\right)\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n\,(\nabla u_n-\nabla u)\;dx$$
$$-\lambda\int_\Omega |u_n|^{q(x)-2}u_n(u_n-u)\,dx + \int_\Omega g(x)|u_n|^{p(x)-2}u_n(u_n-u)\,dx \to 0.$$

So, we can deduce from (9) that

$$\left(a+b\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\,dx\right)\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n(\nabla u_n-\nabla u)\,dx \to 0. \tag{10}$$

So, we have

$$\langle \Lambda'(u_n), u_n-u\rangle = \int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n\,(\nabla u_n-\nabla u)\;dx \longrightarrow 0,$$

from Proposition 3.2, $u_n \to u$ in $W_0^{1,p(x)}(\Omega)$. □

Lemma 3.7. *The functional J is coercive and bounded from below.*

Proof. Indeed, for any $u \in W_0^{1,p(x)}(\Omega)$, we have

$$J(u) \ge \frac{a}{p^+}\int |\nabla u|^{p(x)} + \frac{b}{2p^{+2}}\left(\int_\Omega |\nabla u|^{p(x)}\,dx\right)^2 - \frac{\lambda}{q^-}\int_\Omega |u|^{q(x)}\,dx + \frac{1}{p^+}\int g(x)|u|^{p(x)}\,dx.$$

Let $\rho_p(u) = \int_\Omega |u|^{p(x)}$, by (6), we have some cases:
i) If $\rho_p(u) > 1$,

$$J(u) \ge \frac{a}{p^+}\|u\|^{p^-} + \frac{b}{2p^{+2}}\|u\|^{2p^-} - \frac{\lambda C_1}{q^-}\|u\|^{q^+}.$$

Since (7), so J is coercive and bounded from below.
ii) If $\rho_p(u) < 1$,

$$J(u) \ge \frac{a}{p^+}\|u\|^{p^+} + \frac{b}{2p^{+2}}\|u\|^{2p^+} - \frac{\lambda C_1}{q^-}\|u\|^{q^-},$$

Since $2p^+ > p^+$ and $2p^+ > p^-$, then J is coercive and bounded from below. □

proof of theorem 3.5. We notice that $W_0^{1,p^+}(\Omega) \subset W_0^{1,p(x)}(\Omega)$.
Let us consider $(e_n)_{n=1}^{\infty}$ a schauder basis for $W_0^{1,p^+}(\Omega)$ [14] and $X_k = \mathrm{span}\{e_1, e_2, \ldots, e_k\}$, the subspace of $W_0^{1,p^+}(\Omega)$ generated by $e_1, e_2, \ldots, e_k$. Clearly X_k is subspace of $W_0^{1,p(x)}(\Omega)$.
So, we notice that $X_k \subset L^{q(x)}(\Omega)$ because $X_k \subset W_0^{1,p^+}(\Omega) \subset L_{q(x)}$. Thus, the norms $\|\cdot\|$ and $|\cdot|_{q(x)}$ are equivalent on X_k, because X_k is a finite dimension space. Consequently, there exists a positive constant C_k such that

$$|u|_{q(x)} \geq C_k\|u\|, \text{ for all } u \in X_k.$$

Let $u \in X_k$; $\|u\| < 1$, by (H) we have

$$\begin{aligned}
J(u) &= a\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx + \frac{b}{2}\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx\right)^2 \\
&- \lambda \int \frac{1}{q(x)}|u|^{q(x)} + \int \frac{g(x)}{p(x)}|u|^{p(x)}\,dx \\
&\leq \frac{a}{p^-}\int_\Omega |\nabla u|^{p(x)} + \frac{b}{2p^{-^2}}\left(\int |\nabla u|^{p(x)}\,dx\right)^2 - \frac{\lambda}{q^+}\int_\Omega |u|_{q(x)}^{q(x)} + \frac{1}{p^-}\int g(x)|u|^{p(x)}\,dx \\
&\leq \alpha_1\left(\|u\|^{p^-} + \|u\|^{2p^-}\right) - \alpha_2\|u\|_{q(x)}^{q^+} + \frac{\|g\|_\infty}{p^-}\|u\|^{p^-} \\
&\leq \alpha_3\left(\|u\|^{p^-} + \|u\|^{2p^-}\right) - \alpha_2 C\|u\|^{q^+} \\
&= \|u\|^{q^+}\left[\alpha_3\left(\|u\|^{p^- - q^+} + \|u\|^{2p^- - q^+}\right) - C\alpha_2\right].
\end{aligned}$$

There exists $r_1 \in (0,1)$ be enough small in which $r_1^{q^+} < 1$ and $\alpha_3 r_1^{p^- - q^+} + \alpha_2 r_1^{2p^- - q^+} \leq \frac{C\alpha_2}{2}$.
Considering $T = S_r^k = \left\{u \in X_k \,\middle|\, \|u\| = r_1\right\}$, $J(u) \leq r_1^{q^+}\left(\alpha_3 r_1^{p^- - q^+} + \alpha_3 r_1^{2p^- - q^+} - C\alpha_2\right)$, $\forall\, u \in T$. Then

$$\sup_T J(u) \leq 1 \cdot \left(\frac{C\alpha_2}{2} - C\alpha_2\right) = -\frac{C\alpha_2}{2} < 0 = J(0).$$

Since X_k and $\mathbb{R}^k$ are isomorphic so S_r^k and S^{k-1} are homeomorphic so $\gamma(S_r^k) = k$.
J is even, so by Theorem 3.4, J has least k pairs of different critical points.

Corollary 3.8. *If* (7) *holds. Then there are infinitely many solution for* (1).

Proof. Since k is arbitrary, so there are infinitely many critical points of J. □

4. Regularity results on eigenfunctions

In this section we shall prove boundedness and the simplicity of the problem (1) typical conditions. We consider the following problem

$$\begin{cases} -\left(a + b\int_\Omega \frac{|\nabla u|^p}{p}\right)\Delta_p u = \lambda|u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{11}$$

The pair $(u, \lambda) \in W_0^{1,p}(\Omega) \times \mathbb{R}^+$ is a weak solution of (11) if

$$a\int |\nabla u|^{p-2}\nabla u \cdot \nabla v\,dx + \left(\frac{b}{p}\int |\nabla u|^{p-2}\nabla u \cdot \nabla v\right) \cdot \int |\nabla u|^p\,dx = \lambda\int |u|^{p-2}uv\,dx. \tag{12}$$

Lemma 4.1. *[4].*

(i) Let $p \geq 2$ *then for all* $x, y \in \mathbb{R}^N$

$$|y|^p \geq |x|^p + p|x|^{p-2}x \cdot (y-x) + C(p)|x-y|^p. \tag{13}$$

(ii) Let $1 < p < 2$, *then for all* $x, y \in \mathbb{R}^n$,

$$|y|^p \geq |x|^p + p|x|^{p-2}x \cdot (y-x) + C(p)\frac{|x-y|^2}{(|x|+|y|)^{2-p}}. \tag{14}$$

(iii) For any $x \neq y$, $p > 1$,

$$|y|^p > |x|^p + p|x|^{p-2}\, x \cdot (y-x).$$

In the above $C(p)$ *is a constant depending only on* p.

Theorem 4.2. *[14]. Let* $f \in C(\mathbb{R})$ *be a piecewise smooth function with* $f' \in L^\infty(\mathbb{R})$. *Then*

(a) If $u \in W^{1,p}(\Omega)$, *then* $f \circ u \in W^{1,p}(\Omega)$.

(b) If $u \in W_0^{1,p}(\Omega)$ *and* $f(0) = 0$, *then* $f \circ u \in W_0^{1,p}(\Omega)$.

In all cases, we have

$$\nabla(f \circ u) = \begin{cases} f'(u)\nabla u & \text{if} \quad u \notin L, \\ 0 & \text{if} \quad u \in L, \end{cases}$$

where L *denotes the set of corner points of* f.

Lemma 4.3. *[14]. Let* u *be in* $W^{1,p}(\Omega)$, $u \geq 0$. *Let* $\{u_n\}$ *be a sequence such that* $u_n \to u$ *in* $W^{1,p}(\Omega)$. *Then the sequence* $\{u_n^+\}$ *also converges to* u *in* $W^{1,p}(\Omega)$.

Lemma 4.4. *Let* u *be in* $W_0^{1,p}(\Omega)$, $u \geq 0$. *Then there exist a sequence of nonnegative functions in* $C_0^\infty(\Omega)$ *converging to* u *in* $W_0^{1,p}(\Omega)$.

Theorem 4.5. *Let* X *be* $W_0^{1,p}(\Omega)$ *or* $W^{1,p}(\Omega)$ *and let* $(u, \lambda) \in X \times \mathbb{R}^+$ *be an eigen pair of the weak form* (11), *then* $u \in L^\infty(\Omega)$.

Proof. By Sobolev's embedding theorem it suffices to consider the case $p \leq N$. Assume first that $u \geq 0$. For $M > 0$ define $v_M(x) = \min\{u(x), M\}$.
Letting $f(x) = x$, if $x \leq M$ and $f(x) = M$, if $x > M$, it follows from Theorem 4.2 that $v_M \in X \cap L^\infty(\Omega)$.
For $k > 0$ define $\varphi = v_M^{kp+1}$, then $\nabla\varphi = (kp+1)v_M^{kp} \cdot \nabla v_M$. It follows that $\varphi \in X \cap L^\infty(\Omega)$.
Using φ as a test function in (11), one obtains

$$a(kr+1)\int_\Omega |\nabla u|^{p-2}\,\nabla u\,\nabla v_M \cdot v_M^{kp}\,dx + (kp+1)\frac{b}{p}\left(\int |\nabla u|^{p-2}\,\nabla u \cdot \nabla v_M^{kp}\right).$$
$$\int_\Omega |\nabla u|\,dx = \lambda \int_\Omega |u|^{p-2}u\,v_M^{kp+1}\,dx \leq \lambda \int_\Omega |u|^{(k+1)p}\,dx.$$

Then

$$a(kp+1)\int_\Omega |\nabla v_M \cdot v_M^k|^p\,dx + (kp+1)\frac{b}{p}\left(\int_\Omega \nabla v_M^{kp+p-1}\right)\int_\Omega |\nabla v_M|^p\,dx$$
$$\leq \lambda \int_\Omega |u|^{(k+1)p}\,dx.$$

Since

$$a\frac{(kp+1)}{(k+1)^p}\int_\Omega |\nabla v_{_M}^{k+1}|^p \leq \lambda |u|^{(k+1)p}\,dx$$

So

$$\begin{aligned} & a\frac{kp+1}{(k+1)^p}\int_\Omega |\nabla v_{_M}^{k+1}|^p + \frac{kp+1}{(k+1)^p}\int_\Omega |v_{_M}^{k+1}|^p\,dx \\ & \leq \left(\lambda + \frac{kp+1}{(k+1)^p}\right)\int_\Omega u^{(k+1)p}\,dx. \end{aligned}$$

Then

$$a\int_\Omega |\nabla v_{_M}^{k+1}|^p\,dx + \int_\Omega |v_{_M}^{k+1}|^p\,dx \leq \left(\lambda\frac{(k+1)^p}{(kp+1)} + 1\right)\int_\Omega u^{(k+1)p}\,dx.$$

By (6), we have

$$\|v_{_M}^{k+1}\|^p \leq \left(\lambda\frac{(k+1)^p}{(kp+1)} + 1\right)\|u\|_{(k+1)p}^{(k+1)p}.$$

By Sobolev's embedding theorem, there is a constant $C_1 > 0$ such that

$$\|v_{_M}^{k+1}\|_{p^*} \leq C_1\|v_{_M}^{k+1}\|,$$

we take $p^* = \frac{Np}{N-p}$, if $p < N$ and $p^* = 2p$, if $p = N$. Thus

$$\begin{aligned} \|v_{_M}\|_{(k+1)p^*} & \leq \|v_{_M}^{k+1}\|_{p^*}^{\frac{1}{k+1}} \\ & \leq C_1^{\frac{1}{k+1}}\left(\lambda\frac{(k+1)^p}{(kp+1)} + 1\right)^{\frac{1}{p(k+1)}}\|u\|_{(k+1)p}. \end{aligned}$$

We can find a constant $C_2 > 0$ such that

$$\left(\lambda\frac{(k+1)^p}{(kp+1)} + 1\right)^{\frac{1}{p\sqrt{k+1}}} \leq C_2$$

for any $k > 0$.
Thus

$$\|v_{_M}\|_{(k+1)p^*} \leq C_1^{\frac{1}{k+1}}\,C_2^{\frac{1}{\sqrt{k+1}}}\|u\|_{(k+1)p}.$$

Letting $M \to \infty$, Fatou's lemma implies that

$$\|u\|_{(k+1)p^*} \leq C_1^{\frac{1}{k+1}}\,C_2^{\frac{1}{\sqrt{k+1}}}\|u\|_{(k+1)p}. \tag{15}$$

Choosing k_1 such that $(k_1+1)p = p^*$, then (15) becomes

$$\|u\|_{(k_1+1)p^*} \leq C_1^{\frac{1}{k_1+1}}\,C_2^{\frac{1}{\sqrt{k_1+1}}}\|u\|_{p^*}.$$

Next, we choose k_2 such that $(k_2+1)p = (k_1+1)p^*$, then taking $k_2 = k$ in (15), we have

$$\begin{aligned} \|u\|_{(k_2+1)p^*} & \leq C_1^{\frac{1}{k_2+1}}\,C_2^{\frac{1}{\sqrt{k_2+1}}}\,\|u\|_{(k_2+1)p} \\ & = C_1^{\frac{1}{k_2+1}}\,C_2^{\frac{1}{\sqrt{k_2+1}}}\,\|u\|_{(k_1+1)p^*}. \end{aligned}$$

Then we obtain

$$\|u\|_{(k_n+1)p^*} \le C_1^{\frac{1}{k_n+1}}\, C_2^{\frac{1}{\sqrt{k_n+1}}}\, \|u\|_{(k_{n-1}+1)p^*},$$

where the sequence $\{k_n\}$ is chosen such that $(k_n+1)p = (k_{n-1}+1)p^*$, $k_0 = 0$. We see that $k_n + 1 = (\frac{p^*}{p})^n$. Hence

$$\|u\|_{(k_n+1)p^*} \le C_1^{\sum_{i=1}^{n}\frac{1}{k_i+1}}\, C_2^{\sum_{i=1}^{n}\frac{1}{\sqrt{k_i+1}}}\, \|u\|_{p^*}.$$

As $\frac{p}{p^*} < 1$, there is $C > 0$ such that for any $n = 1, 2, \ldots$

$$\|u\|_{(k_n+1)p^*} \le C\|u\|_{p^*},$$

with $r_n = (k_n+1)p^* \to \infty$ as $n \to \infty$.

We show that $u \in L^\infty(\Omega)$.
Suppose that $u \notin L^\infty(\Omega)$, then there exist $\varepsilon > 0$ and a set A of positive measure in Ω such that

$$|u(x)| > C\|u\|_{p^*} + \varepsilon = K, \ \text{ for all } x \in A.$$

Then

$$\liminf_{n\to\infty} \|u\|_{r_n} \ge \liminf_{n\to\infty} \left(\int_A K^{r_n}\right)^{\frac{1}{r_n}} = \liminf_{n\to\infty} K|A|^{\frac{1}{r_n}} = K > C\|u\|_{p^*}.$$

Which is contradiction.

If u (as an eigen function of (11)) changes sign, we consider u^+. By Lemma 4.3, $u^+ \in X$.
Define for each $M > 0$, $v_M(x) = \min\{u^+(x), M\}$. Taking again $\varphi = v_M^{kp+1}$ as a test function in X, we obtain

$$\begin{aligned} & a(kp+1)\int_\Omega |\nabla u|^{p-2}\,\nabla u\cdot\nabla v_M\cdot v_M^{kp}\,dx + \frac{b}{p}(kp+1)\left(\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla v_M^{kp}\right)\cdot\int_\Omega |\nabla u|^p\,dx \\ & = \lambda\int_\Omega |u|^{p-2}\,u\,v_M^{kp+1}\,dx, \end{aligned}$$

which implies

$$\begin{aligned} & a(kp+1)\int_\Omega |\nabla u^+|^{p-2}\,\nabla u^+\cdot\nabla v_M\, v_M^{kp}\,dx + (kp+1)\frac{b}{p}\left(\int |\nabla u^+|^{p-2}\,\nabla u^+\cdot\nabla v_M^{kp}\right)\int_\Omega |\nabla u^+|^p\,dx \\ & = \lambda\int_\Omega |u^+|^{p-2}u^+\, v_M^{kp+1}\,dx. \end{aligned}$$

Proceding the same way as above we conclude that $u^+ \in L^\infty(\Omega)$. Similary we have $u^- \in L^\infty(\Omega)$. Therefore $u = u^+ + u^-$ is in $L^\infty(\Omega)$. □

We define the following quantities for problem (1)

$$\lambda^* = \inf_{u\in W_0^{1,p(x)}(\Omega)\setminus\{0\}} \frac{a\int \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx + \frac{b}{2}\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx\right)^2 + \int \frac{g(x)}{p(x)}|u|^{p(x)}\,dx}{\int_\Omega \frac{1}{q(x)}|u|^{q(x)}\,dx}, \tag{16}$$

and

$$\lambda_1 = \inf_{u\in W_0^{1,p(x)}(\Omega)\setminus\{0\}} \frac{\left(a + b\int \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx\right)\int_\Omega |\nabla u|^{p(x)}|\,dx + \int g(x)|u|^{p(x)}\,dx}{\int_\Omega |u|^{q(x)}\,dx}, \tag{17}$$

λ_1 is smalest eigenvalue.

Theorem 4.6. *The problem* (1) *has no-solution for every* $\lambda < \lambda_1$.

Proof. Let

$$P(x,u) = \left(a + b\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx\right)\int_\Omega |\nabla u|^{p(x)}\,dx + \int_\Omega g(x)|u|^{p(x)}\,dx$$

and

$$Q(x,u) = \int_\Omega |u|^{q(x)}\,dx.$$

We have $\lambda_1 := \inf_{u\in W_0^{1,p(x)}(\Omega)\setminus\{0\}} \frac{P(x,u)}{Q(x,u)}$.
Fix $\lambda < \lambda_1$ we argue by contradiction and assume that λ is an eigenvalue of the problem (1).
Therefore we may find $u \in W_0^{1,p(x)}(\Omega)\setminus\{0\}$ such that

$$\begin{aligned}&\left(a + b\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx\right)\int_\Omega |\nabla u|^{p(x)-2}\,\nabla u\,\nabla\varphi\,dx + \int_\Omega g(x)|u|^{p(x)-2}\,u\,\varphi\,dx\\ &= \lambda\int_\Omega |u|^{q(x)-2}\,u\,\varphi\,dx, \quad \text{for all } \varphi \in W_0^{1,p(x)}(\Omega).\end{aligned}$$

Thus we have

$$P'_u(x,u) = \lambda Q'_u(x,u).$$

For now on, taking $\varphi = u$, we get that $P(x,u) = \lambda Q(x,u)$.
Hence,

$$\lambda = \frac{P(x,u)}{Q(x,u)} \geq \inf_{u\in W_0^{1,p(x)}(\Omega)\setminus\{0\}} \frac{P(x,u)}{Q(x,u)} = \lambda_1$$

which contradict by the choice of λ. □

Let us denote the quantity of the problem (11)

$$\lambda^* := \inf_{u\in W_0^{1,p(x)}(\Omega)\setminus\{0\}} \frac{\frac{a}{p}\int_\Omega |\nabla u|^p\,dx + \frac{b}{2p^2}\left(\int_\Omega |\nabla u|^p\,dx\right)^2}{\frac{1}{p}\int_\Omega |u|^p\,dx}. \tag{18}$$

The smallest eigenvalue of problem (11) is

$$\lambda_1 := \inf_{u\in W_0^{1,p(x)}(\Omega)\setminus\{0\}} \frac{a\int |\nabla u|^p\,dx + \frac{b}{p}\left(\int_\Omega |\nabla u|^p\,dx\right)^2}{\int_\Omega |u|^p\,dx}. \tag{19}$$

Lemma 4.7. *[14]. Let u be an eigenfunction associated with* λ_1*, then either* $u > 0$ *or* $u < 0$ *in* Ω.

Theorem 4.8. *The principal eigenvalue* λ_1 *is simple .i.e., if u and v are two eigenfunctions associated with* λ_1*, then there exists C such that* $u = Cv$.

Proof. By Lemma 4.7, we can assume that u and v are positive in Ω. Let $\eta = \frac{(u+\varepsilon)^p-(v+\varepsilon)^p}{(u+\varepsilon)^{p-1}}$ and $\theta = \frac{(v+\varepsilon)^p-(u+\varepsilon)^p}{(v+\varepsilon)^{p-1}}$.
Where ε is a positive parameter. Then

$$\nabla\eta = \left\{1 + (p-1)(\frac{v+\varepsilon}{u+\varepsilon})^p\right\}\nabla u - p\left(\frac{v+\varepsilon}{u+\varepsilon}\right)^{p-1}\nabla v.$$

From Theorem 4.5, since u and v are bounded, $\nabla\eta$ is in $L^p(\Omega)$ and thus η is in $W^{1,p}(\Omega)$.
By symmetry argument the gradient of the test function θ in the corresponding equation for v has a similar expression when u and v interchanged.
Set $u_\varepsilon = u + \varepsilon$ and $v_\varepsilon = v + \varepsilon$. Inserting these test functions in to their equations obtained from (12) and adding these equations, we obtain

$$
\begin{aligned}
&\lambda_1 \int_\Omega u^{p-1} \frac{u_\varepsilon^p - v_\varepsilon^p}{u_\varepsilon^{p-1}} - \lambda_1 \int_\Omega v^{p-1} \frac{v_\varepsilon^p - u_\varepsilon^p}{v_\varepsilon^{p-1}} \\
&= \lambda_1 \int_\Omega \left[\frac{u^{p-1}}{u_\varepsilon^{p-1}} - \frac{v^{p-1}}{v_\varepsilon^{p-1}} \right] (u_\varepsilon^p - v_\varepsilon^p)\, dx \\
&= a \Bigg[\int_\Omega \left(\left\{ 1 + (p-1)(\frac{v_\varepsilon}{u_\varepsilon})^p \right\} |\nabla u_\varepsilon|^p + \left\{ 1 + (p-1)(\frac{u_\varepsilon}{v_\varepsilon})^p \right\} |\nabla v_\varepsilon|^p \right) dx \\
&- \int_\Omega \left(p(\frac{v_\varepsilon}{u_\varepsilon})^{p-1} |\nabla u_\varepsilon|^{p-2}\, \nabla u_\varepsilon \cdot \nabla v_\varepsilon + p(\frac{u_\varepsilon}{v_\varepsilon})^{p-1}\, |\nabla v_\varepsilon|^{p-2}\, \nabla v_\varepsilon \cdot \nabla u_\varepsilon \right) dx \Bigg] \\
&+ \frac{b}{p} \Bigg[\int_\Omega \left(\left\{ 1 + (p-1)(\frac{v_\varepsilon}{u_\varepsilon})^p \right\} |\nabla u_\varepsilon|^p + \left\{ 1 + (p-1)(\frac{u_\varepsilon}{v_\varepsilon})^p \right\} |\nabla v_\varepsilon|^p \right) dx \\
&- \int_\Omega \left(p(\frac{v_\varepsilon}{u_\varepsilon})^{p-1} |\nabla u_\varepsilon|^{p-2}\, \nabla u_\varepsilon \cdot \nabla v_\varepsilon + p(\frac{u_\varepsilon}{v_\varepsilon})^{p-1}\, |\nabla v_\varepsilon|^{p-2}\, \nabla v_\varepsilon \cdot \nabla u_\varepsilon \right) dx \Bigg] \\
&\cdot \Bigg[\int_\Omega \left(\left\{ 1 + (p-1)(\frac{v_\varepsilon}{u_\varepsilon})^p \right\} |\nabla u_\varepsilon|^p + \left\{ 1 + (p-1)(\frac{u_\varepsilon}{v_\varepsilon})^p \right\} |\nabla v_\varepsilon|^p \right) dx \\
&- \int \left(p(\frac{v_\varepsilon}{u_\varepsilon})^{p-1} |\nabla u_\varepsilon|^p + p(\frac{u_\varepsilon}{v_\varepsilon})^{p-1} |\nabla v_\varepsilon|^p \right) \Bigg] \\
&= a \Bigg[\int_\Omega (u_\varepsilon^p - v_\varepsilon^p)\, (|\nabla \ln u_\varepsilon|^p - |\nabla \ln v_\varepsilon|^p)\, dx \\
&- p \int_\Omega v_\varepsilon^p |\nabla \ln u_\varepsilon|^{p-2} \nabla \ln u_\varepsilon (\nabla \ln v_\varepsilon - \nabla \ln u_\varepsilon)\, dx \\
&-p \int_\Omega u_\varepsilon^p |\nabla \ln v_\varepsilon|^{p-2} \nabla \ln v_\varepsilon\, (\nabla \ln u_\varepsilon - \nabla \ln v_\varepsilon)\, dx \Bigg] \\
&+ \frac{b}{p} \Bigg[\int_\Omega (u_\varepsilon^p - v_\varepsilon^p)\, (|\nabla \ln u_\varepsilon|^p - |\nabla \ln v_\varepsilon|^p)\, dx \\
&- p \int_\Omega v_\varepsilon^p \left| \nabla \ln u_\varepsilon^{p-2} \right| \nabla \ln u_\varepsilon\, (\nabla \ln v_\varepsilon - \nabla \ln u_\varepsilon)\, dx \\
&-p \int_\Omega u_\varepsilon^p \left| \nabla \ln v_\varepsilon^{p-2} \right| \nabla \ln v_\varepsilon\, (\nabla \ln u_\varepsilon - \nabla \ln v_\varepsilon)\, dx \Bigg] \\
&\cdot \Bigg[\int_\Omega (u_\varepsilon^p - v_\varepsilon^p)\, (|\nabla \ln u_\varepsilon|^p - |\nabla \ln v_\varepsilon|^p)\, dx \\
&- p \int_\Omega v_\varepsilon^p |\nabla \ln u_\varepsilon|^{p-2} \nabla \ln u_\varepsilon \left(\frac{\nabla u_\varepsilon}{v_\varepsilon} - \nabla \ln u_\varepsilon \right) dx \\
&-p \int_\Omega u_\varepsilon^p |\nabla \ln v_\varepsilon|^{p-2} \nabla \ln v_\varepsilon \left(\frac{\nabla v_\varepsilon}{u_\varepsilon} - \nabla \ln v_\varepsilon \right) dx \Bigg].
\end{aligned}
$$

Set $x_1 = u_\varepsilon \nabla \ln v_\varepsilon$, $y_1 = u_\varepsilon \nabla \ln u_\varepsilon$, $x_2 = v_\varepsilon \nabla \ln u_\varepsilon$, $y_2 = v_\varepsilon \nabla \ln v_\varepsilon$ and viceverse, inequality (14) in Lemma 4.1

implies that

$$\begin{aligned}L_\varepsilon = &\int_\Omega (u_\varepsilon^p - v_\varepsilon^p)\left(|\nabla \ln u_\varepsilon|^p - |\nabla \ln v_\varepsilon|^p\right)\, dx\\ &- p\int_\Omega v_\varepsilon^p |\nabla \ln u_\varepsilon|^{p-2}\nabla \ln u_\varepsilon \left(\nabla \ln v_\varepsilon - \nabla \ln u_\varepsilon\right)\, dx\\ &- p\int_\Omega u_\varepsilon^p |\nabla \ln v_\varepsilon|^{p-2}\nabla \ln v_\varepsilon \left(\nabla \ln u_\varepsilon - \nabla \ln v_\varepsilon\right)\, dx \geq 0\end{aligned}$$

and

$$\begin{aligned}S_\varepsilon = &\int_\Omega (u_\varepsilon^p - v_\varepsilon^p)\left(|\nabla \ln u_\varepsilon|^p - |\nabla \ln v_\varepsilon|^p\right)\, dx\\ &- p\int_\Omega v_\varepsilon^p |\nabla \ln u_\varepsilon|^{p-2}\nabla \ln u_\varepsilon \left(\frac{\nabla u_\varepsilon}{v_\varepsilon} - \nabla \ln u_\varepsilon\right) dx\\ &- p\int_\Omega u_\varepsilon^p |\nabla \ln v_\varepsilon|^{p-2}\nabla \ln v_\varepsilon \left(\frac{\nabla v_\varepsilon}{u_\varepsilon} - \nabla \ln v_\varepsilon\right) dx \geq 0.\end{aligned}$$

Dominated convergence theorem implies that

$$\lim_{\varepsilon\to 0^+} \lambda_1 \int_\Omega \left[\frac{u^{p-1}}{u_\varepsilon^{p-1}} - \frac{v^{p-1}}{v_\varepsilon^{p-1}}\right](u_\varepsilon^p - v_\varepsilon^p)\, dx = 0. \tag{20}$$

Theorem 4.6 of [14] implies that u and v are in $C^{1,\alpha}(\bar{\Omega})$.
For the case $p \geq 2$, from to inequality (13) in Lemma 4.1 we have

$$\begin{aligned}0 &\leq C(p)\int_\Omega \left(\frac{1}{v_\varepsilon^p} + \frac{1}{u_\varepsilon^p}\right)|v_\varepsilon \nabla u_\varepsilon - u_\varepsilon \nabla v_\varepsilon|^p\, dx\\ &\leq L_\varepsilon \leq L_\varepsilon + L_\varepsilon S_\varepsilon \leq \lambda_1 \int_\Omega \left[\frac{u^{p-1}}{u_\varepsilon^{p-1}} - \frac{v^{p-1}}{v_\varepsilon^{p-1}}\right](u_\varepsilon^p - v_\varepsilon^p)\, dx\end{aligned}$$

for every $\varepsilon > 0$. Recalling (20), for $\varepsilon \to 0^+$, from Fatou's Lemma we obtain
$\lim v_\varepsilon \nabla u_\varepsilon - u_\varepsilon \nabla v_\varepsilon = 0$ a.e, in Ω and thus

$$v\nabla u = u\nabla v \quad \text{a.e, in } \Omega.$$

We obtain immediatly that $\nabla(\frac{u}{v}) = 0$, i.e., there is a constant k such that $u = kv$ a.e, in Ω. By continuity, $u = kv$ at every point in Ω. □

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