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Optimal Quadrature Rules for Numerical Solution of the Nonlinear Fredholm Integral Equations

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Abstract. In this paper, an iterative method of successive approximations to the approximate solution of nonlinear Hammerstein- Fredholm integral equations using an optimal quadrature formula for classes of functions of Lipschitz types is provided. Also, the convergence analysis and numerical stability of the proposed method are proved. Finally, some numerical examples verify the theoretical results and show the accuracy of the method.

1. Introduction

In this investigation, we propose a numerical method for the following nonlinear Hammerstein- Fredholm integral equation of the second kind

$$x(t) = f(t) + \lambda \int_a^b K(t,s)g(s,x(s))ds, \qquad t \in [a,b], \tag{1}$$

where $x(t)$ is an unknown function on $[a,b]$ and also, $f(t)$, $K(t,s)$ are known functions on $[a,b]$, $[a,b] \times [a,b]$, respectively.
The mathematical modeling of physical phenomena, many problems in applied mathematics, engineering, mechanics, mathematical physics and many other fields can be transformed into the second-kind of integral equations [8, 11, 15, 21, 22, 24]. There are many numerical methods for solving these equations. The Galerkin and collocation methods are the two commonly used methods for the numerical solutions of these equations [2, 9]. Numerical solutions of linear and nonlinear integral equations have been presented, including, block-pulse functions (BPFs)[7, 17], degenerate kernel method [1], triangular functions (TFs)[10], Chebyshev polynomials [31], Taylor-series expansion method [16], Least squares approximation method[28], operational matrices [27], Bernoulli polynomials [4], B-spline wavelets [19] and wavelet method [3, 20]. Classical theorems on the existence and uniqueness of the solution of nonlinear integral equations can be found in [11, 29]. Existence results for functional integral equations are obtained using the measure of noncompactness and Darbo conditions in [14] and [18] respectively. The method of

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successive approximations and its iterative methods are applied in [6, 13]. Wu in [30] figured out the optimal quadrature formula for classes of crisp continuous functions of Lipschitz type. In this paper, we will discuss iterative method of successive approximations according to the Optimal quadrature to acquire the numerical solution of nonlinear Hammerstein integral equation (1). To prove the convergence and numerical stability of the method, we just used Lipschitz conditions relevant to the function g and it do not need smoothness conditions, while there are some numerical methods to prove convergence which is used smoothness conditions. This paper is divided into five sections. Second section deals with the basic concepts. In Section 3, a sequence of successive approximations is introduced by using the explained Optimal quadrature formula. Also, the convergence and numerical stability of the method of successive approximations used to approximate the solution of nonlinear Hammerstein integral equation (1), are proved. In Section 4, some numerical problems are carried out. Some conclusions are drawn in Section 5.

2. Preliminaries

2.1. Quadrature formula

Definition 2.1. *A function $f : [a, b] \to \mathbb{R}$ is called Lipschitz, if there exists a constant $L > 0$ such that the inequality*

$$|f(x) - f(t)| \le L|x - t|,$$

holds for all $x, t \in [a, b]$. Also, for $0 < \zeta \le 1$, a function $f : [a, b] \to \mathbb{R}$ is Lipschitz of order ζ if $|f(x) - f(t)| \le L|x - t|^{\zeta}$, for any $x, t \in [a, b]$.

For Lipschitzian function the following result holds:

Theorem 2.2. *Let $f : [a, b] \to R$, be a L-Lipschitz function. Then, for any divisions $a = x_0 < x_1 < ... < x_n = b$ and any points $\xi_i \in [x_{i-1}, x_i], i = 1, 2, ..., n$ we have*

$$\begin{aligned} \left| \int_a^b f(t)dt - \sum_{i=1}^{n} (x_i - x_{i-1}) f(\xi_i) \right| &\le \frac{L}{2} \sum_{i=1}^{n} [(\xi_i - x_{i-1})^2 + (x_i - \xi_i)^2] \\ &\le \frac{L}{2} \sum_{i=1}^{n} (x_i - x_{i-1})^2. \end{aligned}$$

Proof. It is known that the integrals are additive related to interval. This leads us to

$$\begin{aligned} \left| \int_a^b f(t)dt - \sum_{i=1}^{n} (x_i - x_{i-1}) f(\xi_i) \right| &= \left| \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} f(t)dt - \sum_{i=1}^{n} (x_i - x_{i-1}) f(\xi_i) \right| \\ &= \left| \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} f(t)dt - \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} f(\xi_i)dt \right| \\ &\le \sum_{i=1}^{n} \left| \int_{x_{i-1}}^{x_i} f(t)dt - \int_{x_{i-1}}^{x_i} f(\xi_i)dt \right| \\ &\le \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} \left| f(t) - f(\xi_i) \right| dt. \end{aligned}$$

By the definition of a L-Lipschitz function, we have

$$\begin{aligned}\sum_{i=1}^{n}\int_{x_{i-1}}^{x_i}\left|f(t)-f(\xi_i)\right|dt &\le L\sum_{i=1}^{n}\int_{x_{i-1}}^{x_i}|t-\xi_i|dt\\ &\le L\sum_{i=1}^{n}(\int_{x_{i-1}}^{\xi_i}(\xi_i-t)dt+\int_{\xi_i}^{x_i}(t-x_i)dt)\\ &= \frac{L}{2}\sum_{i=1}^{n}[(\xi_i-x_{i-1})^2+(x_i-\xi_i)^2]\\ &\le \frac{L}{2}\sum_{i=1}^{n}(x_i-x_{i-1})^2,\end{aligned}$$

which completes the proof. □

Corollary 2.3. *Assume that $f:[a,b]\to R$, is a L-Lipschitz function. Then*

$$\left|\int_a^b f(t)dt-[(x-a)f(u)+(b-x)f(v)]\right|\le L[\frac{1}{4}(b-a)^2+(x-\frac{a+b}{2})^2],$$

for any $x\in[a,b]$,$u\in[a,x]$, $v\in[x,b]$.

Proof. Taking $n=2$, $x_1=x$, $\xi_1=u$, $\xi_2=v$ in Theorem (2.2) we obtain the required inequality. □

Remark 2.4. *If we put $u=a$, $v=b$, $x=\frac{a+b}{2}$, then we acquire trapezoidal formula:*

$$\left|\int_a^b f(t)dt-\frac{(b-a)}{2}[f(a)+f(b)]\right|\le\frac{L}{4}(b-a)^2,$$

which can be extended for uniform partitions,

$$D: a=t_0<t_1<t_2<...<t_{n-1}<t_n=b,$$

with $t_i=a+ih$, $h=\frac{b-a}{n}$, as can be viewed in the following result:

Corollary 2.5. *For uniform partition D of $[a,b]$, the following trapezoidal inequality holds:*

$$\left|\int_a^b f(t)dt-\sum_{i=1}^{n}\frac{(t_i-t_{i-1})}{2}[f(t_{i-1})+f(t_i)]\right|\le\frac{L}{4n}(b-a)^2. \tag{2}$$

Proof. By previous Remark, we have

$$\left|\int_{t_{i-1}}^{t_i} f(t)dt-\frac{(t_i-t_{i-1})}{2}[f(t_{i-1})+f(t_i)]\right|\le\frac{L}{4}(t_i-t_{i-1})^2,$$

where L is the Lipschitz constant of f. We obtain

$$\begin{aligned}\left|\int_a^b f(t)dt-\sum_{i=1}^{n}\frac{(t_i-t_{i-1})}{2}[f(t_{i-1})+f(t_i)]\right| &= \left|\sum_{i=1}^{n}\int_{x_{i-1}}^{x_i} f(t)dt-\sum_{i=1}^{n}\frac{(t_i-t_{i-1})}{2}[f(t_{i-1})+f(t_i)]\right|\\ &\le \sum_{i=1}^{n}\left|\int_{x_{i-1}}^{x_i} f(t)dt-\frac{(t_i-t_{i-1})}{2}[f(t_{i-1})+f(t_i)]\right|\\ &\le \sum_{i=1}^{n}\frac{L}{4}(t_i-t_{i-1})^2\\ &= \sum_{i=1}^{n}\frac{L}{4}\frac{(b-a)^2}{n^2}\\ &= \frac{L}{4n}(b-a)^2.\end{aligned}$$

□

2.2. Optimal quadrature formulas

In this section, we use acquired optimal quadrature formulas with given nodes among all quadrature formulas for classes of functions of Lipschitz type in [30] to obtain numerical method to approximate the solution Eq.(1).

Theorem 2.6. *([30]). Let $f:[a,b]\to\mathbb{R}$ be a Riemann integrable and Lipschitz of order ζ function, then the following quadrature formula*

$$I_n(f)=(\frac{\eta_1+\eta_2}{2}-a)f(\eta_1)+\sum_{i=2}^{n-1}(\frac{\eta_{i+1}-\eta_{i-1}}{2})f(\eta_i)+(b-\frac{\eta_{n-1}+\eta_n}{2})f(\eta_n), \tag{3}$$

to approximate $\int_a^b f(x)dx$ has the minimal error among all quadrature formulas that use given nodes $a\le\eta_1<...<\eta_n\le b$.

Remark 2.7. *If we take $\eta_1=a, \eta_n=b, \eta_{i+1}-\eta_i=h=\frac{b-a}{n}, i=1,2,...,n-1$, in the above Theorem, then we obtain that*

$$I_n(f)=\frac{h}{2}\sum_{i=1}^{n-1}(f(\eta_i)+f(\eta_{i+1})),$$

which is the classical trapezoidal rule.

Theorem 2.8. *([30]). Let $f:[a,b]\to\mathbb{R}$ be a Riemann integrable function on $[a,b]$ of Lipschitz type with constant L and order $0<\zeta\le 1$, then the following variant of classical trapezoidal rule*

$$S_n(f)=\frac{b-a}{2}\sum_{i=1}^{n}f(a+\frac{(2i-1)(b-a)}{2n}), \tag{4}$$

is the optimal quadrature formula for $\int_a^b f(x)dx$ among all formulas (3). Also, we have

$$E_n(\int_a^b f(x)dx, S_n(f))=\sup|\int_a^b f(x)dx-S_n(f)|\le L\frac{(b-a)^{\zeta+1}}{(\zeta+1)2^{\zeta}n^{\zeta}}. \tag{5}$$

In this paper, it is assumed that $\zeta=1$.

3. Main results

Here, we provide a sequence of successive approximations for approximate the solution of (1) then we illustrate the existence and uniqueness of the solution for this equation. Also, by using optimal quadrature we present an efficient numerical method for approximating the solution of (1).

3.1. The sequence of successive approximations

Here, we consider the Fredholm integral equations (1). Assume that

1°. $f \in C([a,b],\mathbb{R})$, $g \in C([a,b]\times\mathbb{R},\mathbb{R})$, $K \in C([a,b]\times[a,b],\mathbb{R})$,

2°. there exists $\beta \geq 0$, such that $|f(t) - f(t')| \leq \beta|t - t'|$, for all $t, t' \in [a,b]$,

3°. there exist $\gamma, \alpha \geq 0$, such that $|g(s,u) - g(s',v)| \leq \gamma|s - s'| + \alpha|u - v|$, for all $s, s' \in [a,b], u, v \in \mathbb{R}$,

4°. $\alpha\lambda M_K(b-a) < 1$, where $M_k \geq 0$ is such that $|K(t,s)| \leq M_K, \forall t, s \in [a,b]$, according to the continuity of K,

5°. there exist $\mu, \delta \geq 0$, such that $|K(t,s) - K(t',s')| \leq \mu|t - t'| + \delta|s - s'|$, for all $t, t', s, s' \in [a,b]$.

Here, we consider $\mathbf{X} = \{f : [a,b] \to \mathbb{R}; f \;\; is \;\; continuous\}$ be the space of continuous functions with the metric

$$d(f,g) = \left\|f - g\right\| = \sup\{\left|f(s) - g(s)\right|; s \in [a,b]\}.$$

Now, we shall prove the existence and uniqueness of the solution of Eq. (1) by the method of successive approximations. We define the operators $T : \mathbf{X} \to \mathbf{X}$ by

$$T(x)(t) = f(t) + \lambda \int_a^b K(t,s)g(s,x(s))ds, \quad t \in [a,b], \forall x \in \mathbf{X}.$$

Theorem 3.1. *Under the above assumptions, equation* (1) *has a unique solution $x^* \in X$. Moreover, for any $x_0 \in \mathbf{X}$, the sequence of successive approximations $(x_k)_{m\in N} \subset C([a,b],\mathbb{R})$, defined by*

$$x_k = T(x_{k-1}), \tag{6}$$

with initial value $x_0 := f(t)$ converges to $x^* \in \mathbf{X}$. Furthermore, the following error estimates hold

$$d(x^*, x_k) \leq \frac{(M_k\alpha\lambda(b-a))^k}{1 - M_k\alpha\lambda(b-a)} d(x_0, x_1), \tag{7}$$

$$d(x^*, x_k) \leq \frac{M_k\alpha\lambda(b-a)}{1 - M_k\alpha\lambda(b-a)} d(x_{k-1}, x_k), \tag{8}$$

and choosing $x_0 = f \in \mathbf{X}$, the inequality 7 becomes

$$d(x^*, x_k) \leq \frac{(M_k\alpha\lambda(b-a))^{k+1}}{\alpha(1 - M_k\alpha\lambda(b-a))} M_0, \tag{9}$$

where

$$M_0 = \max\{|g(s,f)|; s \in [a,b], f = x_0 \in R\}.$$

Proof. Firstly, we prove that $T(\mathbf{X}) \subset \mathbf{X}$. To this aim, we see that for all $\varepsilon > 0$ there are $\varepsilon_1, \varepsilon_2 > 0$ such that $\varepsilon_1 + M_0\lambda(b-a)\varepsilon_2 < \varepsilon$. Since f is continuous on compact set of $[a,b]$, we infer that it is uniformly continuous, therefore for $\varepsilon_1 > 0$ exists $\delta' > 0$ such that

$$|f(t_1) - f(t_2)| < \varepsilon_1 \;\; \forall t_1, t_2 \in [a,b],$$

with $|t_1 - t_2| < \delta'$.
As mentioned above, K also is uniformly continuous thus, for $\varepsilon_2 > 0$ exists $\delta'' > 0$ such that

$$\Big|K(t_1, s) - K(t_2, s)\Big| < \varepsilon_2 \quad \forall t_1, t_2 \in [a, b],$$

with $|t_1 - t_2| < \delta''$.
Let $\delta = \min\{\delta', \delta''\}$ and $t_1, t_2 \in [a, b]$, with $|t_1 - t_2| < \delta$. We obtain

$$\begin{aligned} | T(x)(t_1) - T(x)(t_2) | &\leq | f(t_1) - f(t_2) | + \lambda \int_a^b |K(t_1, s) - K(t_2, s)||g(s, x(s))| ds \\ &< \varepsilon_1 + \lambda M_0 (b - a)\varepsilon_2 < \varepsilon, \end{aligned}$$

we derive

$$| T(x)(t_1) - T(x)(t_2) | < \varepsilon.$$

This shows that $T(x)$ is uniformly continuous for any $x \in \mathbf{X}$, so continuous on $[a, b]$, and hence T maps $\mathbf{X}$ into $\mathbf{X}$, (i.e. $T(\mathbf{X}) \subset \mathbf{X}$).
Now, we show that the operator T is a contraction map. So, for $x, y \in \mathbf{X}$ and $t \in [a, b]$, we have

$$\begin{aligned} |T(x)(t) - T(y)(t)| &\leq \lambda \int_a^b |K(t, s)g(s, x(s)) - K(t, s)g(s, y(s))| ds \\ &\leq \lambda M_k \alpha (b - a)\|x - y\|. \end{aligned}$$

Consequently,

$$\|T(x) - T(y)\| \leq \lambda M_k \alpha (b - a)\|x - y\|.$$

Since $\lambda M_k \alpha(b - a) < 1$, the operator T is a contraction on Banach space $(\mathbf{X}, \|.\|)$. Using the Banach's fixed point principle implies that Eq. (1) has a unique solution x^* in $\mathbf{X}$.
The same Banach's fixed point principle leads to the estimates (7) and (8).
Choosing $x_0 = f$, we have

$$\begin{aligned} |x_0(t) - x_1(t)| &\leq \lambda \int_a^b | K(t, s)g(s, x_0(t)) | ds \\ &\leq \lambda \int_a^b \max_{a \leq s \leq b} | g(s, f) | ds \\ &\leq \lambda (b - a) M_k M_0. \end{aligned}$$

Taking supremum from the above inequality we get

$$\|x_0 - x_1\| \leq \lambda (b - a) M_k M_0.$$

In this way we obtain the inequality (9), which completes the proof. □

Now, we consider a uniform partition $D : a = t_0 < t_1 < ... < t_n = b$ of $[a, b]$ with $t_i = a + ih$, where $h = \frac{b-a}{n}$, $i = \overline{0, n}$. Applying the quadrature rule (4) and (5) to approximate of the integral in (6) we obtain,

$$\begin{aligned} &\bar{x}_0(t) = f(t), \\ &\bar{x}_k(t) = f(t) + h \sum_{i=1}^{n} K(t, a + \frac{(2i - 1)(b - a)}{2n}) g\Big(a + \frac{(2i - 1)(b - a)}{2n}, \bar{x}_{k-1}(a + \frac{(2i - 1)(b - a)}{2n})\Big). \end{aligned} \tag{10}$$

3.2. Convergence Analysis

In this section, we investigate the convergence of the iterative proposed method to the solution of equation (1).

Proposition 3.2. *Under the conditions (i)-(iii) of Theorem (3.1), the sequence of successive approximations (6) are uniformly bounded. Moreover, let $G_k(s) = g(s, x_k(s)), k \in N, s \in [a, b]$ then the functions $G_k(s), k \in N$ is uniformly Lipschitz with constant $L' = \alpha + \beta(\theta + \lambda(b-a)(d-c)M\zeta)$, where M is given in (12).*

Proof. Let $G_0 : [a, b] \to \mathbb{R}, G_0(s) = g(s, f(s))$. Since G, f are continuous, we infer that G_0 is continuous on the compact set $[a, b]$ and therefore $M_0 \geq 0$ exist, such that

$$|G_0(s)| \leq M_0 \qquad \forall s \in [a, b]. \tag{11}$$

So,

$$\begin{aligned}\left|x_1(t) - x_0(t)\right| &\leq \lambda \int_a^b \left|K(t,s)g(s, x_0(s)\right|ds \\ &\leq \lambda \int_a^b M_k M_0 ds = \lambda M_k M_0 (b-a).\end{aligned}$$

For arbitrary $t \in [a, b]$, it follows that

$$\begin{aligned}\left|x_k(t) - x_{k-1}(t)\right| \leq& \lambda\left|K(t,s)\right| \int_a^b \left|g(s, x_{k-1}(s)) - g(s, x_{k-2}(s))\right|dxdy \\ &\leq \lambda M_k \int_a^b \left|g(s, x_{k-1}(s)) - g(s, x_{k-2}(s))\right|dxdy \\ &\leq \alpha\lambda M_k(b-a) \max_{a \leq x \leq b} |x_{k-1}(t) - x_{k-2}(t)|,\end{aligned}$$

and by induction,

$$\left|x_k(t) - x_{k-1}(t)\right| \leq (\alpha\lambda M_K(b-a))^{k-1}\left\|x_1 - x_0\right\|.$$

So,

$$\begin{aligned}\left|x_k(t) - x_0(t)\right| \leq& \left|x_k(t) - x_{k-1}(t)\right| + ... + \left|x_1(t) - x_0(t)\right| \\ &\leq \left((\alpha\lambda M_K(b-a))^{k-1} + (\alpha\lambda M_K(b-a))^{k-2} + ... + \alpha\lambda M_K(b-a) + 1\right)\left\|x_1 - x_0\right\| \\ &= \frac{1 - (\alpha\lambda M_K(b-a))^k}{1 - \alpha\lambda M_K(b-a)} \lambda M_k M_0 (b-a) \\ &\leq \frac{\lambda M_k(b-a)M_0}{1 - \lambda\alpha M_k(b-a)} \qquad \forall t \in [a, b].\end{aligned}$$

Let $M_f \geq 0$ such that $|f(t)| \leq M_f$ for all $t \in [a, b]$. Then

$$\left|x_k(t)\right| \leq \left|x_k(t) - x_0(t)\right| + \left|x_0(t)\right| \leq \frac{\lambda M_K(b-a)M_0}{1 - \alpha\lambda M_K(b-a)} + M_f = l,$$

for all $t \in [a, b]$. Moreover, considering

$$M = max\Big(M_0, max\{|g(t,u)| : t \in [a, b], u \in [-l, l]\}\Big), \tag{12}$$

we get

$$|G_k(t)| = |g(t, x_k(t))| \leq M,$$

for all $t \in [a, b]$ and $k \in \mathbf{N}$. Let $t, t' \in [a, b]$. We obtain

$$\left|x_0(t) - x_0(t')\right| \le \beta|t - t'|,$$

$$\begin{aligned}\left|X_m(t) - X_m(t')\right| &\le \left|f(t) - f(t')\right| \\ &+ \lambda \int_a^b | K(t,s) - K(t^{'}, s)) || g\Big(s, x_{k-1}(s)\Big) | dxdy \\ &\le \beta\left|t - t'\right| + \lambda(b-a)M\mu\left|t - t'\right| \\ &= L_0\left|t - t'\right|,\end{aligned}$$

with $L_0 = \beta + \lambda(b - a)M\mu$ and

$$\begin{aligned}\left|G_0(t) - G_0(t')\right| &\le \gamma\left|t - t'\right| + \alpha\left|x_0(t) - x_0(t')\right| \\ &\le (\gamma + \alpha\beta)\left|t - t'\right|,\end{aligned}$$

$$\begin{aligned}\left|G_k(t) - G_k(t')\right| &\le \gamma\left|t - t'\right| + \alpha\left|x_k(t) - x_k(t')\right| \\ &\le \gamma\left|t - t'\right| + \alpha L_0\left|t - t'\right| = (\gamma + \alpha L_0)\left|t - t'\right| = L'\left|t - t'\right|,\end{aligned}$$

for all $t \in [a, b]$ and $k \in \mathbf{N}$. So, the sequence of functions $(G_k)_{k\in\mathbf{N}}$ are uniformly Lipschitz with the constant $L' = \gamma + \alpha(\beta + \lambda(b - a)M\mu)$. □

Corollary 3.3. *Under the conditions (i)-(vi), the functions $K(t,s)g(s, x_k(s))$, for arbitrary fixed $t \in [a, b]$ and for all $k \in \mathbf{N}$, are uniformly Lipschitz with constant*

$$L = M_k\Big(\gamma + \alpha(\beta + \lambda(b - a)M\mu)\Big) + M\delta.$$

Proof. *Let arbitrary $s, s' \in [a, b]$, we have*

$$\begin{aligned}\left|K(t,s)g(s, x_k(s)) - K(t,s')g(s', x_k(s'))\right| &\le \left|K(t,s)g(s, x_k(s)) - K(t,s)g(s', x_k(s'))\right| \\ &+ \left|K(t,s)g(s', x_k(s')) - K(t,s')g(s', x_k(s'))\right| \\ &\le |K(t,s)|.|g(s, x_k(s)) - g(s', x_k(s'))| \\ &+ |g(s', x_k(s'))|.|K(t,s) - K(t,s')| \\ &\le M_kL'|s - s'| + M\delta|s - s'|, \end{aligned} \tag{13}$$

for $k \in \mathbf{N}$. Then, according to (12), (13), and denoting $L = M_kL' + M\delta = M_k\Big(\gamma + \alpha(\beta + \lambda(b - a)M\mu)\Big) + M\delta$, it follows that

$$\left|K(t,s)g(s, x_k(s)) - K(t,s')g(s', x_k(s'))\right| \le L|s - s'|,$$

for any fixed $t \in [a, b]$ and $k \in \mathbf{N}$. Thus, the functions $K(t,s)g(s, x_k(s))$ for all k are Lipschitzian. □

Theorem 3.4. *Consider the Eq. (1) with the hypotheses of Theorem 3.1. Then the iterative procedure (10) converges to the unique solution of Eq. (1), x^*, and its error estimate is as follows*

$$d(x^*, \bar{x}_k) \le \frac{(M_k\alpha\lambda(b - a))^{k+1}}{\alpha(1 - M_k\alpha\lambda(b - a))}M_0 + \frac{L(b - a)^2}{4n(1 - \lambda M_k\alpha(b - a))}. \tag{14}$$

Proof. Using (9) we have

$$d(x^*,\bar{x}_k) \leq d(x^*,x_k) + d(x_k,\bar{x}_k) \leq \frac{(M_k\alpha\lambda(b-a))^{k+1}}{\alpha(1-M_k\alpha\lambda(b-a))}M_0 + \|x_k(t)-\bar{x}_k(t)\|. \tag{15}$$

Therefore, we shall to obtain the estimates for $\|x_k(t)-\bar{x}_k(t)\|$. We apply the quadrature formula (5) in (6) obtaining

$$\begin{aligned} x_0(t) &= f(t),\\ x_k(t) &= f(t) + h\sum_{i=1}^{n} K(t, a+\frac{(2i-1)(b-a)}{2n})g\Big(a+\frac{(2i-1)(b-a)}{2n}, x_{k-1}(a+\frac{(2i-1)(b-a)}{2n})\Big)\\ &\quad + E_k(t), \end{aligned} \tag{16}$$

with

$$\mid E_k(t) \mid \leq \frac{L(b-a)^2}{4n}. \tag{17}$$

Form (16), (10) and (17), for $k=1$, we obtain

$$\mid x_1(t) - \bar{x}_1(t) \mid \leq \mid E_1(s) \mid \leq \frac{L(b-a)^2}{4n}. \tag{18}$$

From (16), (10) we obtain

$$\begin{aligned} \mid x_k(t)-\bar{x}_k(t)\mid &\leq \frac{L(b-a)^2}{4n} + \lambda\frac{b-a}{n}\sum_{i=1}^{n}\Big(|K(t,s_i)|\Big|g\big(s_i,x_{k-1}(s_i)\big) - g\big(s_i,\bar{x}_{k-1}(s_i)\big)\Big|\Big)\\ &\leq \frac{L(b-a)^2}{4n} + \lambda M_k\alpha\frac{b-a}{n}\sum_{i=1}^{n}\Big|x_{k-1}(s_i)-\bar{x}_{k-1}(s_i)\Big|, \end{aligned}$$

where

$$s_i = a + \frac{(2i-1)(b-a)}{2n}. \tag{19}$$

Now, from (16), (10) for $k=2$ it follow that

$$\mid x_2(t)-\bar{x}_2(t)\mid \leq \frac{L(b-a)^2}{4n} + \lambda M_k\alpha\frac{b-a}{n}\sum_{i=1}^{n}\frac{L(b-a)^2}{4n} \tag{20}$$

$$\leq (1+\lambda M_k\alpha(b-a))\frac{L(b-a)^2}{4n}. \tag{21}$$

By induction, for $k \in N$, $k \geq 3$, we obtain

$$\begin{aligned} \mid x_k(t)-\bar{x}_k(t)\mid &\leq [1+\lambda M_k\alpha(b-a)... + (\lambda M_k\alpha(b-a))^{k-1}]\frac{L(b-a)^2}{4n}\\ &\leq \frac{1-(\lambda M_k\alpha(b-a))^k}{1-\lambda M_k\alpha(b-a)}\frac{L(b-a)^2}{4n}\\ &\leq \frac{L(b-a)^2}{4(1-\lambda M_k\alpha(b-a))n}. \end{aligned} \tag{22}$$

Hence, from (15), (18 and (22) we conclude that

$$d(x_k(t)-\bar{x}_k(t)) \leq \frac{L(b-a)^2}{4(1-\lambda M_k\alpha(b-a))n}.$$

Remark 3.5. *From the error estimation (14), since $\alpha\lambda M_k(b-a) < 1$, we see that for $k\to\infty$, $n\to\infty$, it follows $d(x^*,\bar{x}_k)\to 0$, which is the convergence of the proposed method.*

3.3. The numerical stability analysis

An important property for an algorithm to have is that small changes in the initial data produce correspondingly small changes in the final results. An algorithm that satisfies this property is called numerically stable. So, with the purpose of studying the numerical stability of the iterative method (10), considering the small changes in the first iteration, an another first iteration term $y_0(t) \in C([a,b],R)$ is considered in such a way that there exists $\varepsilon > 0$ for which $\left|y_0(t) - x_0(t)\right| < \varepsilon, \forall s \in [a,b]$. Suppose that there exist $M_y, \beta' \geq 0$ with $| y_o(t) - y_0(t') | \leq \beta' | t - t' |$, $\forall t, t' \in [a,b]$ and $|y_0(t)| \leq M_y$, for all $t \in [a,b]$. The obtained new sequence of successive approximations is:

$$y_k(t) = f(t) + \lambda \int_a^b K(t,s)g(s, y_{k-1}(s))ds, \quad k \geq 1. \tag{23}$$

Using the same iterative method (10) to solve (1) we have

$$\begin{aligned} &\bar{y}_0(t) = y_0(t), \\ &\bar{y}_k(t) = f(t) + \frac{b-a}{2}\sum_{i=1}^{n} K(t, a + \frac{(2i-1)(b-a)}{2n})g\Big(a + \frac{(2i-1)(b-a)}{2n}, \bar{y}_{k-1}(a + \frac{(2i-1)(b-a)}{2n})\Big). \end{aligned} \tag{24}$$

Definition 3.6. *Let $x_0, y_0 \in \mathbf{X}$ be two initial value such that $\|x_0 - y_0\| < \varepsilon$, for arbitrary small $\varepsilon > 0$. We say that the algorithm of successive approximation applied to the integral equation* (1) *is numerically stable with respect to the choice of the first iteration iff there exist the constants the constants $\xi_1, \xi_2 > 0$ which are independent by h, such that:*

$$\|x_k - y_k\| < \xi_1 \varepsilon + \xi_2 h, \qquad k \in \mathbb{N} \cup \{0\}.$$

Theorem 3.7. *Assume the conditions of Theorem 3.4 are fulfilled. Then the iterative approach (10) is numerically stable with respect to the selection of the first iteration.*

Proof. In order to obtain the numerical stability we reproduce the proof of Theorem (3.4) and deliver the corresponding constants $\tilde{M}_0, \tilde{M}, \tilde{L_0}, \tilde{L'}, \tilde{L}$ given by $|g(s, y(s)| \leq \tilde{M}_0, \tilde{L_0} = \beta + \lambda(b-a)\tilde{M}\mu$, $\tilde{L'} = \gamma + \alpha(\beta + \lambda(b-a)\tilde{M}\mu), \tilde{L} = M_k\big(\gamma + \alpha(\beta + \lambda(b-a)\tilde{M}\mu)\big) + \tilde{M}\delta$, Similarly as above it follows that

$$d(y_k(t) - \bar{y}_k(t)) \leq \frac{\tilde{L}(b-a)^2}{4(1 - \lambda M_k \alpha(b-a))n},$$

and we have

$$\begin{aligned} | \bar{x}_k(t) - \overline{y}_k(t) | &\leqslant | \bar{x}_k(t) - x_k(t) | + | x_k(t) - y_k(t) | + |y_k(t) - \overline{y}_k(t) | \\ &\leq \frac{L(b-a)^2}{4(1 - \lambda M_k \alpha(b-a))n} + | x_k(t) - y_k(t) | + \frac{\tilde{L}(b-a)^2}{4(1 - \lambda M_k \alpha(b-a))n}. \end{aligned}$$

We have

$$| x_0(t) - y_0(t) | < \varepsilon, \qquad \forall t \in [a,b],$$

and

$$\begin{aligned} \left|x_1(t) - y_1(t)\right| &\leq |f(t) - f(t)| + \lambda \int_a^b |K(t,s)g(s, x_0(s)) - K(t,s)g(s, y_o(s))|ds \\ &\leq \alpha\lambda M_k \int_a^b | x_0(s) - y_0(s) | \, ds \\ &< \alpha\lambda M_k (b-a)\varepsilon. \end{aligned}$$

For $k \geq 2$, by induction, we have

$$\left|x_2(t) - y_2(t)\right| \leq \alpha\lambda M_k \int_a^b \mid x_1(s) - y_1(s) \mid ds < (\alpha\lambda M_k(b-a))^2\varepsilon,$$

for all $t \in [a, b]$, $k \in \mathbb{N} \cup \{0\}$ and $\alpha\lambda M_k(b-a) < 1$, Then,

$$d(x_k(t) - y_k(t)) \leq \alpha\lambda M_k \int_a^b \mid x_{k-1}(s) - y_{k-1}(s) \mid ds \leq (\alpha\lambda M_k(b-a))^k\varepsilon < \varepsilon.$$

Now, we get

$$\begin{aligned} \mid \bar{x}_k(t) - \overline{y}_k(t) \mid &\leq \varepsilon + \frac{L(b-a)^2}{4(1-\lambda M_k\alpha(b-a))n} + \frac{\tilde{L}(b-a)^2}{4(1-\lambda M_k\alpha(b-a))n} \\ &< \varepsilon + \frac{(L+\tilde{L})(b-a)}{4(1-\lambda M_k\alpha(b-a))}\frac{b-a}{n} = \xi_1\varepsilon + \xi_2 h, \end{aligned}$$

where

$$\xi_1 = 1, \qquad \xi_2 = \frac{(L+\tilde{L})(b-a)}{4(1-\lambda M_k\alpha(b-a))}.$$

□

Remark 3.8. *Since $\alpha\lambda M_k(b-a) < 1$, it is easy to see that*

$$\lim_{h,\varepsilon\to 0} d(\overline{x}_k, \overline{y}_k) = 0.$$

This shows the stability of the method.

Remark 3.9. *The "a posteriori" error estimate is useful to get the stopping criterion. Such estimate can be obtained as follows:*
For given $\varepsilon^{'} > 0$ (previously chosen) consider the first natural number k such that

$$|x_k(t) - x_{k-1}(t)| < \varepsilon^{'},$$

and we stop to this k retaining the approximations $x_k(t)$ of solution. We observe

$$\begin{aligned} \mid x_k^*(t) - \bar{x}_k(t) \mid &\leq \mid x^*(t) - x_k(t) \mid + \mid x_k(t) - \bar{x}_k(t) \mid \\ &\leq \frac{\alpha\lambda M_k(b-a)}{1-\alpha\lambda M_k(b-a)} \mid x_k(t) - x_{k-1}(t) \mid + \frac{L(b-a)^2}{4(1-\lambda M_k\alpha(b-a))n}, \end{aligned}$$

and

$$\begin{aligned} \mid x_k(t) - x_{k-1}(t) \mid &\leq \mid x_k(t) - \bar{x}_k(t) \mid + \mid \bar{x}_k(t) - \bar{x}_{k-1}(t) \mid + \mid \bar{x}_{k-1}(t) - x_{k-1}(t) \mid \\ &\leq \frac{L(b-a)^2}{2(1-\lambda M_k\alpha(b-a))n} + \mid \bar{x}_k(t) - \bar{x}_{k-1}(t) \mid. \end{aligned}$$

So,

$$|x_k^*(t) - \bar{x}_k(t)| \leq \frac{1+M_k(b-a)}{(1-\alpha\lambda M_k(b-a))^2}\frac{L(b-a)^2}{4n} + \frac{\alpha\lambda M_k(b-a)}{1-\alpha\lambda M_k(b-a)} \mid \bar{x}_k(t) - \bar{x}_{k-1}(t) \mid,$$

and therefore, in order to obtain $| x_k^*(t) - \bar{x}_k(t) |< \varepsilon$ *we require*

$$\frac{1 + M_k(b-a)}{(1-\alpha\lambda M_k(b-a))^2}\frac{L(b-a)^2}{4n} < \frac{\varepsilon}{2},$$

and

$$\frac{\alpha\lambda M_k(b-a)}{1-\alpha\lambda M_k(b-a)} \mid \bar{x}_k(t) - \bar{x}_{k-1}(t) \mid < \frac{\varepsilon}{2}.$$

Then we choose the smallest natural number $n \in \mathbb{N}$ *that is,*

$$n > \frac{1 + M_k(b-a)}{(1-\alpha\lambda M_k(b-a))^2}\frac{L(b-a)^2}{2\varepsilon}.$$

Now, we get the last number $k \in \mathbb{N}$ *that is,*

$$|\bar{x}_k(t) - \bar{x}_{k-1}(t)| < \frac{\varepsilon}{2}.\frac{1-\alpha\lambda M_k(b-a)}{\alpha\lambda M_k(b-a)} = \varepsilon^{'}.$$

3.4. Algorithm of the approach

The iterative procedure 10 gives the following algorithm of computation for the solution of Eq. (1):

Step 0: Input $a, b, h, \lambda, \varepsilon^{'}, n$ and the functions K, f.

Step 1 (the first iterative step): For $j = \overline{0, n}$ compute $\bar{x}_1(t_j)$ by (10).

Step 2 (the generic iterative step): For $j = \overline{0, n}$ compute $\bar{x}_k(t_j)$ by (10).

Step 3 (a condition of "do- while" type): If $| \bar{x}_k(t_j) - \bar{x}_{k-1}(t_j) |< \varepsilon^{'}$, print k and print $\bar{x}_k(t_j)$, $j = \overline{0, n}$, STOP.

4. Numerical experiments

We have applied our method on some numerical examples, to observe the accuracy and efficiency of the present method for solving Eq. (1). Also, we compare the numerical solutions obtained by using the proposed method with the exact solutions. In order to analyze the error of the method we introduce notations

$$e_n = |x^*(t) - \bar{x}(t)|, \tag{25}$$

and

$$||e_n||_\infty := \max\{|e_n(t_j)|, j = 0, 1, 2, ..., n\}, \tag{26}$$

where $\bar{x}(t)$ and $x^*(t)$ are the approximate solution and the exact solution of the nonlinear equation (1), respectively, which is computed by the algorithm described in Section 3. The results, show that the errors were obtained from our method is much smaller than the errors of the classical quadrature and whatever the magnitude of n is much larger, the convergence will be faster. The computations associated with the examples were executed using MAPLE 17.

Example 4.1. *Consider the following nonlinear Fredholm integral equation*

$$x(t) = \frac{13}{16}t^2 - 1 - \frac{11}{21}t + \int_0^1 \frac{1}{2}(3s^2t + st^2)(2s^2 + x^3(s))ds, \quad t \in [0, 1], \tag{27}$$

t_j	$x^*(t_j)$	$e_j, n=10$	d_j	$e_j, n=20$	$e_j, n=40$
0.1	-0.99	1.679719×10^{-6}	.1000	4.199074×10^{-7}	1.049757×10^{-8}
0.2	-0.96	3.476824×10^{-6}	.1000	8.691449×10^{-7}	2.172829×10^{-8}
0.3	-0.91	5.391310×10^{-6}	.1000	1.347712×10^{-6}	3.369217×10^{-7}
0.4	-0.84	7.423192×10^{-6}	.1000	1.855610×10^{-6}	4.638919×10^{-7}
0.5	-0.75	9.572456×10^{-6}	.1000	2.392837×10^{-6}	5.981936×10^{-7}
0.6	-0.64	1.183912×10^{-5}	.1001	2.959394×10^{-6}	7.398267×10^{-7}
0.7	-0.51	1.422314×10^{-5}	.1001	3.555281×10^{-6}	8.887913×10^{-7}
0.8	-0.36	1.672456×10^{-5}	.1001	4.180499×10^{-6}	1.045087×10^{-6}
0.9	-0.19	1.934337×10^{-5}	.1003	4.835046×10^{-6}	1.208715×10^{-6}
NI		18		19	19

Table 1: Numerical results for $n = 10, n = 20$ and $n = 40$ in Example 4.1.

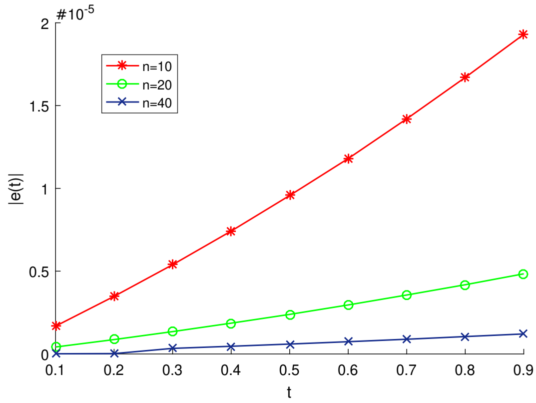

Figure 1: The absolute errors for $n = 10, n = 20$ and $n = 40$ in Example 4.1.

with the exact solution

$$x(t) = t^2 - 1.$$

Applying the algorithm for $n = 10$, $\varepsilon' = 10^{-15}$, we obtain the number of iterations NI $= k = 18$ iterations. For more details, please see Table 1. In order to test the numerical stability regarding the choice of the first iteration, we take $\varepsilon = 0.1$ $(f(t) := f(t) + 0.1)$, and the differences between the effective computed values $d_j = |\overline{x}_{18}(t_j) - \overline{y}_{18}(t_j)|$, $t_j = \frac{j}{10}$, $j = \overline{1,9}$, are in Table 1 that confirm the numerical stability of the algorithm.
In order to more detailed testing of convergence, we consider $n = 20$ and for $\varepsilon' = 10^{-25}$ the number of iterations is $k = 19$. It is seen that e_j, $j = \overline{0,n}$ tend to zero as h decrease. For $n = 40$, $\varepsilon' = 10^{-25}$, we have $k = 19$ iterations. The results $\|e_n\|_\infty$ for $\varepsilon' = 10^{-15}$ and $n \in \{10, 20, 40\}$, respectively, are 1.934×10^{-5}, 4.835×10^{-6} and 1.209×10^{-6}. The comparisons of the absolute errors for $n = 10$, $n = 20$ and $n = 40$ have been graphically shown in Figure 1.

Example 4.2. *The following nonlinear Fredholm integral equation has been considered by other authors as a numerical test [5, 12, 23, 25, 26],*

$$x(t) = f(t) + \int_0^1 K(t,s)g(s,x(s))ds, \qquad t \in [0,1], \tag{28}$$

where

$$f(t) = \sin(\pi t),$$
$$K(t,s) = \frac{1}{5}\cos(\pi t)\sin(\pi s),$$
$$g(s,x(s)) = (x(s))^3,$$

t_j	$x^*(t_j)$	$e_j, n = 10$	$e_j, n = 20$	$e_j, n = 40$
0.1	0.380752038	$2.41822181708501 \times 10^{-5}$	$4.5782956593141 \times 10^{-6}$	8.285×10^{-7}
0.2	0.648806725	$2.05706234347790 \times 10^{-5}$	$3.8945309034702 \times 10^{-6}$	7.009×10^{-7}
0.3	0.853351689	$1.49454327529507 \times 10^{-5}$	$2.8295423280022 \times 10^{-6}$	5.076×10^{-7}
0.4	0.974364644	$7.98727898231050 \times 10^{-6}$	$1.4875784348888 \times 10^{-6}$	2.784×10^{-7}
0.5	1	0	0	0
0.6	0.9277483875	$7.98727898231050 \times 10^{-6}$	$1.4875784348887 \times 10^{-6}$	2.784×10^{-7}
0.7	0.7646822990	$1.49454327529507 \times 10^{-5}$	$2.8295423280022 \times 10^{-6}$	5.076×10^{-7}
0.8	0.5267637791	$2.05706234347790 \times 10^{-5}$	$3.8945309034702 \times 10^{-6}$	7.009×10^{-7}
0.9	0.2372819503	$2.41822181708501 \times 10^{-5}$	$4.5782956593141 \times 10^{-6}$	8.284×10^{-7}
NI		8	9	9

Table 2: Numerical results for $n = 10$, $n = 20$ and $n = 40$ in Example 4.2.

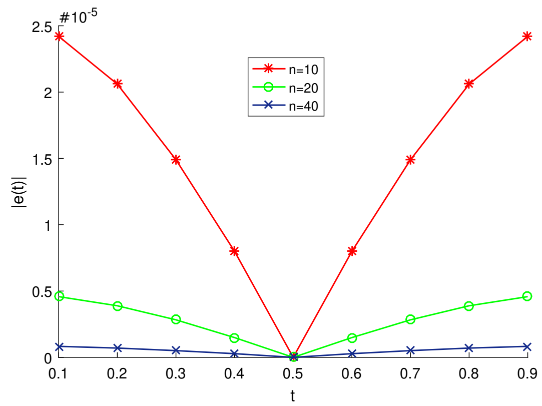

Figure 2: The absolute errors for $n = 10$, $n = 20$ and $n = 40$ in Example 4.2.

with the exact solution

$$x(t) = \sin(\pi t) + \frac{20 - \sqrt{391}}{3} \cos(\pi t).$$

Ezquerro et al. studied existence of the solutions of the above equation 28 in [12]. Moreover, Rashidinia et al. in [25] analytically found solutions for the mentioned equation including $x_1(t) = \sin(\pi t) + \frac{20-\sqrt{391}}{3}\cos(\pi t)$ *and* $x_2(t) = \sin(\pi t) + \frac{20+\sqrt{391}}{3}\cos(\pi t)$*. But in [5] just* $x(t) = \sin(\pi t) + \frac{20-\sqrt{391}}{3}\cos(\pi t)$ *has been considered. In [5] the minimum absolute errors of approximation is* 3.6765×10^{-7} *and the error just in the point* $t = 0.5$ *is zero. Also, in [26] the minimum absolute errors of approximation is* 7.796×10^{-3}*. By using the proposed method, we can present the approximate solution for this example. Table 2 shows that the numerical results for this example. In Figure 2, we have graphically shown the comparisons between the absolute errors for* $n = 10$*,* $n = 20$ *and* $n = 40$*.*

Remark 4.3. *The method can be extended even for Fredholm functional integral equations of the form*

$$x(t) = f(t) + \int_a^b g(t, s, x(s))ds, \qquad t \in [a, b],$$

and an example to illustrate this extension is:

Example 4.4.

$$x(t) = \frac{1}{2t+2}\Big(2t(t+1)^2 \ln(\frac{t+1}{t+2}) + 2t^2 + t + 1\Big) + \int_0^1 \frac{s}{1 + t|x(s)|}ds, \qquad t \in [0, 1], \tag{29}$$

t_j	$x^*(t_j)$	$\overline{x}_{14}(t_j)$	$e_j, n = 10$	d_j
(0.1)	0.90909090909090909091	0.90908942291695715390	148617395193701 $\times 10^{-6}$	0.10000
(0.2)	0.83333333333333333333	0.83333070000821070300	263332512263033 $\times 10^{-6}$	0.10000
(0.3)	0.76923076923076923077	0.76922724200733572660	352722343350417 $\times 10^{-6}$	0.10000
(0.4)	0.71428571428571428571	0.71428148585719846040	422842851582531 $\times 10^{-6}$	0.10000
(0.5)	0.66666666666666666667	0.66666188591315761720	478075350904947 $\times 10^{-6}$	0.10001
(0.6)	0.62500000000000000000	0.62499478344626185110	521655373814890 $\times 10^{-6}$	0.10001
(0.7)	0.58823529411764705882	0.58822973397162923020	556014601782862 $\times 10^{-6}$	0.10002
(0.8)	0.55555555555555555556	0.55554972547086757670	583008468797886 $\times 10^{-6}$	0.10003
(0.9)	0.52631578947368421053	0.52630974875845654670	604071522766383 $\times 10^{-6}$	0.10005
(1.0)	0.50000000000000000000	0.49999379674175039680	6.2032582496032 $\times 10^{-6}$	0.10009

Table 3: The results for Example 4.4 for $n = 10$.

t_j	$x^*(t_j)$	$\overline{x}_{14}(t_j)$	$e_j, n = 20$	d_j
(0.1)	0.90909090909090909091	0.90909053751471516023	3.7157619393068 $\times 10^{-7}$	0.10000
(0.2)	0.83333333333333333333	0.83333267495385743373	6.5837947589960 $\times 10^{-7}$	0.10000
(0.3)	0.76923076923076923077	0.76922988737077813346	8.8185999109731 $\times 10^{-7}$	0.10000
(0.4)	0.71428571428571428571	0.71428465712384497075	1.0571618693149 $\times 10^{-6}$	0.10000
(0.5)	0.66666666666666666667	0.66666547142587757111	1.1952407890955 $\times 10^{-6}$	0.10001
(0.6)	0.62500000000000000000	0.62499869581303321033	1.3041869667896 $\times 10^{-6}$	0.10001
(0.7)	0.58823529411764705882	0.58823390403722967708	1.3900804173817 $\times 10^{-6}$	0.10001
(0.8)	0.55555555555555555556	0.55555409799535300818	1.4575602025473 $\times 10^{-6}$	0.10002
(0.9)	0.52631578947368421053	0.52631427926071434792	1.5102129698626 $\times 10^{-6}$	0.10003
(1.0)	0.50000000000000000000	0.49999844915596956520	1.5508440304348 $\times 10^{-6}$	0.10004

Table 4: The results for Example 4.4 for $n = 20$.

with the exact solution

$$x(t) = \frac{1}{1+t}.$$

The results for $\varepsilon' = 10^{-15}$ and $n = 10$ (with the values d_j, generated by initial perturbation $f(t) := f(t) + 0.1$), $n = 20$ and $n = 40$ are in Tables 3, 4 and 5, respectively. Comparing the results in optimal and classical quadrature formulas confirms the correctness of the theoretical results. We present these results in Table 6.

5. Conclusions

In this paper, an iterative method has been presented for approximating the solution of nonlinear Hammerstein integral equation (1) based on optimal quadrature formula for classes of Lipschitz functions. One of the advantages of the proposed method is easy to implement without complicated computations of the integral terms. In Theorem 3.1 sufficient conditions for existence and uniqueness solution of nonlinear Hammerstein integral equation (1) are given. In Proposition (3.2), we proved that the sequence of successive approximations (6) are uniformly bounded and Lipschitz. Proof of the convergence and the error estimation of the proposed method in terms of Lipschitz condition are given in Theorem 3.4.

Acknowledgment

The authors are grateful to the editor for handling the paper and the reviewers for the valuable comments and suggestions.

t_j	$x^*(t_j)$	$\overline{x}_{14}(t_j)$	$e_j, n = 40$	d_j
(0.1)	0.90909090909090909091	0.90909081619401617083	9.2896892920081 $\times 10^{-8}$	0.10000
(0.2)	0.83333333333333333333	0.83333316873401629663	1.6459931703670 $\times 10^{-7}$	0.10000
(0.3)	0.76923076923076923077	0.76923054876044470726	2.2047032452351 $\times 10^{-7}$	0.10000
(0.4)	0.71428571428571428571	0.71428544998947520455	2.6429623908116 $\times 10^{-7}$	0.10000
(0.5)	0.66666666666666666667	0.66666636785051602191	2.9881615064476 $\times 10^{-7}$	0.10000
(0.6)	0.62500000000000000000	0.62499967394728426653	3.2605271573347 $\times 10^{-7}$	0.10001
(0.7)	0.58823529411764705882	0.58823494659164595108	3.4752600110774 $\times 10^{-7}$	0.10001
(0.8)	0.55555555555555555556	0.55555519115974392238	3.6439581163318 $\times 10^{-7}$	0.10001
(0.9)	0.52631578947368421053	0.52631541191485000112	3.7755883420941 $\times 10^{-7}$	0.10002
(1.0)	0.50000000000000000000	0.49999961228358741264	3.8771641258740 $\times 10^{-7}$	0.10002

Table 5: The results for Example 4.4 for $n = 40$.

	$\|\|e_n\|\|_\infty$(Opti)			$\|\|e_n\|\|_\infty$(Clas)		
	n=10	n=20	n=40	n=10	n=20	n=40
Exa. 4.1	1.934×10^{-5}	4.835×10^{-6}	1.209×10^{-6}	1.749×10^{-4}	4.440×10^{-5}	1.103×10^{-5}
Exa. 4.2	2.418×10^{-5}	4.578×10^{-6}	8.284×10^{-7}	3.096×10^{-4}	1.386×10^{-5}	6.1309×10^{-6}
Exa. 4.4	6.203×10^{-6}	1.551×10^{-6}	3.877×10^{-7}	4.961×10^{-5}	1.240×10^{-5}	3.101×10^{-6}

Table 6: Comparing of $||e_n||_\infty$ (opti) and $||e_n||_\infty$ (clas) in Examples 4.1, 4.2 and 4.4.

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