Filomat 36:11 (2022), 3797–3808
https://doi.org/10.2298/FIL2211797J

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Saddle Point Optimality Criteria and Duality for Convex Continuous-Time Programming Problem

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Abstract. In this paper, convex continuous-time programming problem with inequality type of constraints is considered. We derive new saddle point optimality conditions and classical duality results such as weak and strong duality properties, under additional regularity assumption. A fundamental tool, employed in the derivation of the necessary saddle point optimality criteria and strong duality result for convex continuous-time programming, is a new version of a theorem of the alternative in infinite-dimensional spaces.

1. Introduction

Continuous-time programming problems originated from a class of production-inventory "bottleneck" problems studied by Bellman [3]. His work was expended and built upon by Tyndall [25], who gave mathematical rigorous treatment of duality to the problems in the linear case and since then the theory was intensively developed.

In [11], the authors generalized these duality theorems to the case where the objective functional is concave and derived the complementary slackness principle and Kuhn-Tucker necessary and sufficient conditions. They considered a generalization of the linear constrained nonlinear continuous-time program by replacing the constant matrices B and K by time-dependent matrices $B(t)$ and $K(t,s)$ with piecewise continuous elements. Their proof of the duality theorem was based on extending some results from [14]. However, their proof was invalid which was later indicated and corrected in [26]. Afterwards, Hanson [12] obtained duality results for the linear continuous-time programming problem with differentiability assumptions. He established the duality relationship for this problem by a method originally utilized by Dorn [8] in expanding the duality theorem of linear programming to convex nonlinear programming. Farr and Hanson [9] further generalized the continuous-time programming problem by introducing nonlinear smooth constraints and establishing the complementary slackness theorem and Kuhn-Tucker theorem in

2020 *Mathematics Subject Classification*. 90C30; 90C46; 90C47
Keywords. Continuous-time programming, Optimality conditions, Duality
Received: 12 December 2020; Revised: 17 August 2021; Accepted: 06 March 2022
Communicated by Predrag Stanimirović
This research was supported by the Science Fund of the Republic of Serbia, Grant No. 7744592, Integrability and Extremal Problems in Mechanics, Geometry and Combinatorics-MEGIC and by the Ministry of Education, Science and Technological Development of the Republic of Serbia (Contract No. 451-03-68/2022-14/200135).
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their setup. Thereafter, in [24], the authors considered the continuous-time programming problem

$$\begin{aligned} &F(x(\cdot)) = \int_0^T f(x(t))dt \to \sup; \\ &\text{subject to } h(x(t)) \le c(t) + \int_0^t l(s,t,x(s))ds, \text{ a.e. in } [0,T], \\ &x(t) \ge 0, \text{ a.e. in } [0,T], \end{aligned} \tag{P0}$$

where $f(\cdot)$ is a real-valued function defined on $L_\infty([0,T];\mathbb{R}^n)$, $x(\cdot)$ is an $n \times 1$ vector-valued function defined on $[0,T]$, $h(\cdot)$ is an $k \times 1$ vector-valued function defined on the space $L_\infty([0,T];\mathbb{R}^n)$, $l(\cdot,\cdot,\cdot)$ is an $k \times 1$ vector-valued function defined on $[0,t] \times [0,T] \times L_\infty([0,T];\mathbb{R}^n)$ for each $t \in [0,T]$, $c(\cdot)$ is an $k \times 1$ vector-valued function defined on $[0,T]$, and all the integrals are given in the Lebesgue sense. They established optimality criteria of Kuhn-Tucker and Fritz-John type for (P0) without assuming the differentiability of the functions involved. However, all authors assumed that the functions are convex or concave. It should be noted that the main results in that paper are based on the theorem of the alternative (Theorem 7 [24]). However, the proof of their main theorem of the alternative needed for obtaining necessary optimality criteria is incorrect because in their proof the authors invoke a separation theorem to separate the origin $\{(0,0)\}$ from a subset of $\mathbb{R} \times L_1([0,T];\mathbb{R}^m)$ which may not have an interior point. It is well known that a separation theorem and theorem od the alternative cannot be applied if the cone has a empty interior.

Having in mind all of the above, we conclude that valid results without differentiability do not appear in the literature. Also, it should be highlighted that in the aforementioned papers, the nonegativity of the function $x(\cdot)$ on the interval $[0,T]$ is required. Also the function $f(\cdot)$ depends only on $x(\cdot)$, while this is not required in our paper. We will give more detailed explanations in the following paragraphs. However, our aim in this paper is to establish new saddle point optimality conditions and duality results for nonsmooth convex continuous-time programming problems, using a new tool under more general assumptions compared to existing problems in literature.

In 1980, the articles [22, 23] were published by Reiland generalizing previous results encountered in the literature until then. However, in Reiland's work, to apply a generalized Farkas lemma [7], he had assumed that the kernel of a given operator, between infinite dimensional spaces, has finite dimension and the image of that operator is a closed subspace. In that work, a generalized Slater regularity qualification was also required. We conclude that author had to make very restrictive assumptions of the problem, which are very difficult to verify. After Reiland's papers, in 1985, Zalmai published a series of articles on smooth nonlinear continuous-time programming. See [27–30], for example. Necessary and sufficient conditions for smooth nonlinear continuous-time programming can be found in [28, 30]. Zalmai made assumptions which are less restrictive than Reiland's ones. One of the fundamental tools used by Zalmai was a generalization of the Gordan Transposition Theorem in the continuous-time context [27]. Also, in many later articles, after Zalmai's work, the main tool was the aforementioned theorem. However, in [1], the authors point out that such a result is not valid. In [29], the author established the Kuhn-Tucker saddle point optimality criteria and Lagrangian duality for nonlinear convex continuous-time programming problem, using a perturbation approach in infinite dimensional spaces. (The interested reader is referred to [2] for the use of perturbation approach within the scope of duality theory.) The aforementioned results were obtained in the concept of stable problem. More precisely, the assumption of stability is essential for obtaining the Kuhn-Tucker saddle point theorem. (The hypothesis of stability is not required in our work.) This was achieved by generalizing some of the Geoffrion's results [10] to a certain infinite-dimensional setting. Zalmai highlighted that, using Mclinden's logical equivalence [16] of duality and the theorem of the alternative, his main duality results can be used to derive an important generalization of Gordan's Transposition Theorem [27] which will lead to obtaining further optimality conditions and duality results in continuous time programming in a manner analogous to the finite-dimensional case. However, in [1], it was proved that this is not possible under these assumptions.

In [5, 6], the authors have established optimality conditions for nonsmooth continuous-time programming. One of the fundamental tools used in [5, 28, 30] was a generalization of the Gordan Transposition

Theorem in infinite-dimensional spaces [27]. It can be concluded, that many of the results, that have been using the Gordan Transposition Theorem and its consequences, have now come into question. Therefore, the work by Reiland became one of the main references in the fields of smooth continuous-time programming and linear continuous-time programming, but assumptions and qualification given in Reiland's work is, in general, difficult to verify. Therefore, the alternative theorem stated in [1] has now become one of the main tools for solving continuous-time programming problems.

In [17] the authors provided new first-order Karush-Kuhn-Tucker necessary optimality conditions for smooth continuous-time problems, with both equality and inequality constraints, under Mangasarian-Fromovitz constraint qualification, using new Theorem of the alternative presented in [1] and uniform implicit theorem [21]. In [18, 19], the authors derived new first-and second order necessary optimality conditions for smooth continuous-time problems, with both equality and inequality constraints, under a full and constant rank type constraint qualification, respectively, using the aforementioned uniform implicit theorem. However, it should be noted that these are the first improvements of Zalmai's results under differentiability hypotheses. Oliveira and Monte [17–19] have resolved the smooth nonlinear continuous-time problem successfully and with excellence.

In [22], the author established duality theorems for linear continuous-time programming problems under some constraint qualifications. He also presented an example showing that constraint qualifications are essential on obtaining such results. In 2020, Oliveira [20] obtained classical duality results (weak and strong) for linear continuous-time problems with inequality constraints. He also established the complementary slackness theorem under a new regularity qualification which is simpler to be verified in comparison with the one used in [22].

Recently, in [13], the authors derived new optimality conditions for convex multiobjective continuous-time problem, using results presented in [1].

It is known, sadlle point optimality criteria and duality are well studied in finite dimensional space. For example, see [4, 15]. Based on the aforementioned, we conclude that valid results for convex and nonsmooth continuous-time programming problems without stability hypothesis under more general assumptions, do not appear in the literature. Having in mind all of the above, the alternative theorem for convex inequality systems [1] has now become one of the fundamental tools for solving convex continuous-time programming problems. Our aim in this paper is to provide saddle point optimality criteria and duality results for this type of problem under certain regularity qualification.

The paper is organized in the following way. Some preliminaries about the problem are given in Section 2. In Section 3, we formulate and discuss Lagrangian type function for a convex continuous-time problem and establish necessary and sufficient saddle point optimality conditions. We propose a new regularity qualification which is simple to be verified and essential for obtaining the main results. We also derive new saddle point optimality conditions for a special case of convex continuous-time programming, when the objective function is linear in the second argument. Also, illustrative examples are provided to demonstrate the usefulness of this optimality criteria. In Section 4, we prove the weak and strong duality theorems.

2. Preliminaries

In this work, we consider the following continuous-time problem:

$$\begin{aligned} &J_0(x(\cdot)) = \int_0^T f_0(t, x(t))dt \to \inf; \\ &\text{subject to } f_i(t, x(t)) \leq 0,\ \ i \in I = \{1, \ldots, m\},\ \text{a.e. in } [0, T], \\ &\qquad\qquad x(\cdot) \in L_\infty([0, T]; \mathbb{R}^n), \end{aligned} \tag{CTP}$$

where $f_i : [0, T] \times \mathbb{R}^n \to \mathbb{R}$, $i = 0, 1, \ldots, m$, are given functions. Here for each $t \in [0, T]$, $x_i(t)$ is the ith component of $x(t) \in \mathbb{R}^n$ and all integrals are in the Lebesgue sense. Let Ω_P be the set of all feasible solutions of the problem (CTP) i.e.,

$$\Omega_P = \{x(\cdot) \in L_\infty([0, T]; \mathbb{R}^n) : f_i(t, x(t)) \leq 0,\ \ i \in I,\ \text{a.e. in } [0, T]\}.$$

For each $i = 0, \ldots, m$ function $f_i(t, \cdot)$ is convex and continuous on $\mathbb{R}^n$, for a.e. $t \in [0, T]$. For each $i = 0, \ldots, m$ function $f_i(\cdot, x)$ is Lebesgue measurable for all $x \in \mathbb{R}^n$ and for each $K \geq 0$ there exist $M = M(K) \geq 0$ such that

$$\|x\| \leq K \Rightarrow |f_i(t, x)| \leq M \text{ a.a. } t \in [0, T], \quad \forall x \in \mathbb{R}^n, \quad \forall\, i = 0, \ldots, m.$$

Also, all vectors in our paper are column vectors. Inequality signs between vectors should be read componentwise. The minimization in the initial problem is in the sense of the global minimum.

Definition 2.1. *A point $\hat{x}(\cdot) \in \Omega_P$ is said to be an optimal solution for* (CTP) *if $J_0(\hat{x}(\cdot)) \leq J_0(x(\cdot))$, $\forall x(\cdot) \in \Omega_P$.*

3. Saddle point optimality criteria

For mathematical programming the relationships between the solutions of a constrained programming problem and the points which fulfill certain conditions known as the saddle point optimality criteria, are well known. In this section we extend these results to continuous-time programming problems under convexity assumptions. To do this we begin by giving new definitions of saddle points in continuous-time context.

We define the Lagrange-type function $\mathcal{L} : L_\infty([0, T]; \mathbb{R}^n \times \mathbb{R}^m) \to \mathbb{R}$ with respect to Problem (CTP) as

$$\mathcal{L}(x(\cdot), \lambda(\cdot)) = \int_0^T \left(f_0(t, x(t)) + \sum_{i \in I} \lambda_i(t) f_i(t, x(t)) \right) dt.$$

Definition 3.1. *A point $(\hat{x}(\cdot), \hat{\lambda}(\cdot)) \in L_\infty([0, T]; \mathbb{R}^n \times \mathbb{R}^m)$ is said to be a Karush-Kuhn-Tucker saddle point for* (CTP) *if $\hat{\lambda}(t) \geq 0$ a.e. in $[0, T]$ and*

$$\mathcal{L}(\hat{x}(\cdot), \lambda(\cdot)) \leq \mathcal{L}(\hat{x}(\cdot), \hat{\lambda}(\cdot)) \leq \mathcal{L}(x(\cdot), \hat{\lambda}(\cdot)), \tag{1}$$

for all $x(\cdot) \in L_\infty([0, T]; \mathbb{R}^n)$ and all $\lambda(\cdot) \in L_\infty([0, T]; \mathbb{R}^m)$, $\lambda(t) \geq 0$ a.e. in $[0, T]$.

Theorems of the alternative, also referred to as transposition theorems, are fundamental tools for establishing optimality conditions and duality results in a wide class of optimization problems. A transposition theorem is an assertion about the solvability of two alternative systems, say *I* and *II*, of inequalities and/or equalities, and may be stated as follows:

Either system *I* has a solution, or system *II* has a solution, but never both.

Aryutunov et al. in [1] presented an alternative theorem for convex inequality systems which are related to the existence of multipliers. The aforementioned theorem will be needed in this section. To the best of our knowledge, all alternative theorems in infinite-dimensional spaces require some regularity condition.

We say that Slater's constraint qualification (SQ) is satisfied, if there exists $y(\cdot) \in L_\infty([0, T]; \mathbb{R}^n)$ such that $f_i(t, y(t)) < 0$, $i \in I$ a.e. in $[0, T]$. Let $\tilde{x}(\cdot) \in \Omega_P$ be an optimal solution for (CTP). The system below will be referred to in the next theorem:

$$\begin{aligned} &\chi_0(t, x) := \int_0^T \left(f_0(t, x) - f_0(t, \tilde{x}(t))\right) dt < 0, \\ &\chi_i(t, x) := f_i(t, x) \leq 0, \quad i \in I, \\ &x \in \mathbb{R}^n. \end{aligned} \tag{2}$$

Let $I_0 = \{0\} \cup I$ and

$$\mathcal{I}(t, x) = \left\{ j \in I_0 : \chi_j(t, x) = \max\{\chi_0(t, x), \chi_1(t, x), \ldots, \chi_m(t, x)\} \right\}, \quad t \in [0, T], \quad x \in \mathbb{R}^n.$$

Definition 3.2. *We say that the regularity condition* (RC) *holds, if there exist a function $\bar{x}(\cdot) \in L_\infty([0, T]; \mathbb{R}^n)$, real numbers $R \geq 0$ and $\alpha > 0$ such that for a.e. $t \in [0, 1]$ and for all $x \in \mathbb{R}^n$ with $\|x - \bar{x}(t)\| \geq R$, there exists $e = e(t, x) \in \mathbb{R}^n$ with $\|e\| = 1$, satisfying*

$$\langle \partial_x \chi_j(t, x), e \rangle \geq \alpha \quad \forall j \in \mathcal{I}(t, x),$$

where $\partial_x \chi_j(t, x)$ denotes the partial subdifferential of χ_j at (t, x) in the sense of convex analysis.

Below, we state Karush-Kuhn-Tucker saddle-point necessary optimality theorem.

Theorem 3.3. *Let $\hat{x}(\cdot)$ be an optimal solution of the problem* (CTP). *Assume that the problem* (CTP) *satisfies* (RC) *and Slater's constraint qualification* (SQ). *Then there exist the multipliers $\hat{\lambda}_i(\cdot) \in L_\infty([0,T];\mathbb{R})$, $i \in I$, such that $\hat{\lambda}_i(t) f_i(t,\hat{x}(t)) = 0$, $i \in I$, a.e. in* $[0,T]$ *and $(\hat{x}(\cdot),\hat{\lambda}(\cdot))$ is a Karush-Kuhn-Tucker saddle point for* (CTP).

Proof. Since $\hat{x}(\cdot)$ solves (CTP), we have that there is no $x(\cdot) \in L_\infty([0,T];\mathbb{R}^n)$ such that the following system is consistent

$$\begin{aligned} &\int_0^T (f_0(t,x(t)) - f_0(t,\hat{x}(t)))\, dt < 0, \\ &f_i(t,x(t)) \le 0, \quad i \in I, \ \text{a.e. in } [0,T]. \end{aligned} \tag{3}$$

It is clear that all assumptions of the alternative theorem (Theorem 1 [1]) are satisfied. It follows that there exists a nonzero function $(\hat{v}_0(\cdot),\hat{v}_1(\cdot),\dots,\hat{v}_m(\cdot)) \in L_\infty([0,T];\mathbb{R}^{m+1})$, with $\hat{v}_0(t) \ge 0$, $\hat{v}_i(t) \ge 0$, $i \in I$, $t \in [0,T]$, and $\hat{v}_0(t) \not\equiv 0$, such that

$$\hat{v}_0(t)\int_0^T f_0(t,x(t))dt + \sum_{i=1}^m \hat{v}_i(t) f_i(t,x(t)) \ge \hat{v}_0(t)\int_0^T f_0(t,\hat{x}(t))\,dt, \ \forall x(\cdot) \in L_\infty([0,T];\mathbb{R}^n) \ \text{a.e. in } [0,T]. \tag{4}$$

By letting $x(\cdot) = \hat{x}(\cdot)$ in the above, we have

$$\sum_{i=1}^m \hat{v}_i(t) f_i(t,\hat{x}(t)) \ge 0 \quad \text{a.e. in } [0,T].$$

But since $\hat{v}_i(t) \ge 0$, $i \in I$, a.e. in $[0,T]$ and $\hat{x}(\cdot) \in \Omega_P$, so that we get the opposite inequality above. Then

$$\hat{v}_i(t) f_i(t,\hat{x}(t)) = 0, \quad i \in I, \ \text{a.e. in } [0,T]. \tag{5}$$

Integrating (4) from 0 to T, we have

$$w\int_0^T f_0(t,x(t))dt + \int_0^T \sum_{i=1}^m \hat{v}_i(t) f_i(t,x(t))\,dt \ge w\int_0^T f_0(t,\hat{x}(t))\,dt, \ \forall x(\cdot) \in L_\infty([0,T];\mathbb{R}^n), \tag{6}$$

where

$$w = \int_0^T \hat{v}_0(t)\,dt > 0.$$

Setting

$$\hat{\lambda}_i(t) = \frac{\hat{v}_i(t)}{w} \ge 0, \quad i \in I, \ \text{a.e. in } [0,T],$$

from (5) and (6) we have

$$\int_0^T \left(f_0(t,x(t)) + \sum_{i=1}^m \hat{\lambda}_i(t) f_i(t,x(t)) \right) dt \ge \int_0^T \left(f_0(t,\hat{x}(t)) + \sum_{i=1}^m \hat{\lambda}_i(t) f_i(t,\hat{x}(t)) \right) dt = \int_0^T f_0(t,\hat{x}(t))\,dt \tag{7}$$

$$\ge \int_0^T \left(f_0(t,\hat{x}(t)) + \sum_{i=1}^m \lambda_i(t) f_i(t,\hat{x}(t)) \right) dt, \ \forall x(\cdot) \in (L_\infty[0,T];\mathbb{R}^n), \ \forall \lambda(\cdot) \in (L_\infty[0,T];\mathbb{R}^m), \lambda(t) \ge 0 \ \text{a.e. in } [0,T].$$

Therefore, $(\hat{x}(\cdot),\hat{\lambda}(\cdot)) \in L_\infty([0,T];\mathbb{R}^n \times \mathbb{R}^m)$ is a Karush-Kuhn-Tucker saddle point of (CTP) with $\hat{\lambda}_i(t) f_i(t,\hat{x}(t)) = 0$, $i \in I$ a.e. in $[0,T]$. □

Now, we give sufficient Karush-Kuhn-Tucker optimality criteria for (CTP). The assertions of Theorem 3.3 are also sufficient for the optimality of the point $\hat{x}(\cdot)$.

Theorem 3.4. *If* $(\hat{x}(\cdot), \hat{\lambda}(\cdot))$ *is a Karush-Kuhn-Tucker saddle point, then* $\hat{x}(\cdot)$ *is an optimal solution for* (CTP).

Proof. If $\lambda_i \equiv 0$, $i \in I$, (1) can be written as

$$\int_0^T f_0(t, \hat{x}(t))\, dt \leq \int_0^T \left(f_0(t, x(t)) + \sum_{i=1}^m \hat{\lambda}_i(t) f_i(t, x(t)) \right) dt, \;\; \forall x(\cdot) \in L_\infty([0,T]; \mathbb{R}^n).$$

For the all admissible point $x(\cdot)$ holds

$$\begin{aligned} \int_0^T f_0(t, x(t)) dt &\geq \int_0^T \left(f_0(t, x(t)) + \sum_{i=1}^m \hat{\lambda}_i(t) f_i(t, x(t)) \right) dt \\ &\geq \int_0^T f_0(t, \hat{x}(t))\, dt, \end{aligned}$$

so that

$$\int_0^T f_0(t, \hat{x}(t))\, dt \leq \int_0^T f_0(t, x(t))\, dt, \;\; \forall x(\cdot) \in \Omega_P.$$

Therefore, $\hat{x}(\cdot)$ is an optimal solution of (CTP). □

As an illustration, we will consider the following example:

Example 3.5.

$$\begin{aligned} \int_0^1 \Big(|x(t) - t| + x^2(t) - 2tx(t) + t^2 + 1 \Big)\, dt &\to \inf; \\ -x(t) \leq 0 \text{ a.e. in } [0,1], \\ e^{x(t)-t} - 1 \leq 0 \text{ a.e. in } [0,1], \\ x(\cdot) \in L_\infty([0,1]; \mathbb{R}), \end{aligned} \tag{P}$$

where $f_0(t, x(t)) := |x(t) - t| + x^2(t) - 2tx(t) + t^2 + 1$, $f_1(t, x(t)) := -x(t)$, $f_2(t, x(t)) := e^{x(t)-t} - 1$. It can be easily verified that $\hat{x}(t) = t$ a.e. t in $[0,1]$, is an optimal solution of preceding problem. We have that Slater's condition (SQ) is satisfied for $y(t) = \frac{t}{2}$.
Given $x \in \mathbb{R}$, for almost everywhere in $[0,1]$, $\chi_1(t, x) = -x$, $\chi_2(t, x) = e^{x-t} - 1$ and

$$\chi_0(t, x) = \begin{cases} x^2 - 2x + \frac{5}{6}, & x \leq 0, \\ 2x^2 - 2x + \frac{5}{6}, & x \in (0,1), \\ x^2 - \frac{1}{6}, & x \geq 1. \end{cases}$$

Take $\bar{x}(\cdot) \equiv 2$, $R = 3$ and $\alpha = \frac{1}{2}$. It is clear,

$$\mathcal{I}(t, x) = \begin{cases} \{0\}, & \text{for } x \leq -1,\; t \in [0,1], \\ \{2\}, & \text{for } x \geq 5, \quad t \in [0,1]. \end{cases}$$

The regularity condition (RC) is verified with $e = e(t, x) = 1$, for almost every $t \in [0,1]$, for $x \geq 5$, i.e.,

$$\langle \partial_x \chi_2(t, x), e \rangle \geq \alpha.$$

Similarly, the regularity condition (RC) is verified with $e = (t, x) = -1$, for almost every $t \in [0,1]$, for $x \leq -1$, i.e.,

$$\langle \partial_x \chi_0(t, x), e \rangle = 1 \geq \alpha.$$

Indeed, for $x \in \mathbb{R} \setminus (-1, 5)$, we have

- $\langle \partial_x \chi_0, e\rangle = \langle 2x-2, -1\rangle \geq \alpha$, for $x \leq -1$, $e = -1$ and $t \in [0,1]$,
- $\langle \partial_x \chi_2, e\rangle = \langle e^{x-t}, 1\rangle \geq \alpha$, for $x \geq 5$, $e = 1$ and $t \in [0,1]$.

In the sequel, from inequalities

$$\begin{aligned}
\mathcal{L}(x(\cdot),\hat{\lambda}(\cdot)) &= \int_0^1 \left(|x(t)-t| + x^2(t) - 2tx(t) + t^2 + e^{x(t)-t}\right) dt \\
&\geq \int_0^1 \left(|x(t)-t| + (x(t)-t)^2 + 1 + x(t) - t\right) dt \\
&\geq \mathcal{L}(\hat{x}(\cdot),\hat{\lambda}(\cdot)) \\
&= 1 \\
&\geq \mathcal{L}(\hat{x}(\cdot),\lambda(\cdot)),\ \ \forall x(\cdot) \in L_\infty([0,1];\mathbb{R}),\ \lambda(\cdot) = (\lambda_1(\cdot),\lambda_2(\cdot)) \in L_\infty([0,1];\mathbb{R}^2),\ \lambda(t) \geq 0 \text{ a.e in } [0,1],
\end{aligned}$$

it can be easily verified that $(\hat{x}(t), \hat{\lambda}_1(t), \hat{\lambda}_2(t)) = (t, 0, 1)$ a.e. in $[0,1]$ is a Karush-Kuhn-Tucker saddle point of the problem (P) and complementary slackness condition holds. Indeed, $\hat{\lambda}_i(t) f_i(t, \hat{x}(t)) = 0$, $i = 1, 2$ a.e. in $[0,1]$.

Definition 3.6. *We say a function $f : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ is linear if for any vectors $x, y \in \mathbb{R}^n$ and $t \in [0,T]$,*

$$f(t, x+y) = f(t,x) + f(t,y) \text{ and } f(t,\alpha x) = \alpha f(t,x),\ \forall \alpha \in \mathbb{R}.$$

Let L be the subset of indices from the set I for which the function f_i, $i \in I$ is linear in second argument and let N be the subset of indices from the set I for which the function f_i, $i \in I$ is nonlinear in second argument. It is clear that $I = L \sqcup N$. For all $i \in I$, note that by $\lambda_i(t)$ we denote the multiplier that corresponds to the constraint function f_i.

Using the same approach, with an additional regularity assumption we can obtain new saddle point optimality criteria for the initial problem, where is at least one of the multipliers to the correspond to the nonlinear constraint functions is nonzero.

Theorem 3.7. *Let the function $f_0(t,\cdot)$ be linear and let $\hat{x}(\cdot)$ be an optimal solution of the problem* (CTP). *Assume that the problem* (CTP) *satisfies* (RC) *and Slater's constraint qualification*(SQ). *Further, assume that there exists $z(\cdot) \in L_\infty([0,T];\mathbb{R}^n)$ such that $f_0(t,z(t)) < 0$ and $f_i(t,z(t)) \leq 0$ for $i \in L$ a.e. in $[0,T]$. Then there exist the multipliers $\hat{\lambda}_i(\cdot) \in L_\infty([0,T];\mathbb{R})$, $i \in I$ such that $\hat{\lambda}_i(t) f_i(t,\hat{x}(t)) = 0$, $i \in I$ a.e. in $[0,T]$ and $(\hat{x}(\cdot), \hat{\lambda}(\cdot))$ is a Karush-Kuhn-Tucker saddle point for* (CTP), *where $\hat{\lambda}_i(t) \not\equiv 0$ for some $i \in N$, $t \in [0,T]$,*

Proof. As in the proof Theorem 3.3, $(\hat{x}(\cdot), \hat{\lambda}(\cdot))$ is a Karush-Kuhn-Tucker saddle point for (CTP) and $\hat{\lambda}_i(t) f_i(t,\hat{x}(t)) = 0$, $i \in I$ a.e. in $[0,T]$. Now, we shall prove that there exists $i \in N$ such that $\hat{\lambda}_i(t) \not\equiv 0$ a.e. in $[0,T]$. We will suppose that is not true. Let $\hat{\lambda}_i \equiv 0$, $\forall i \in N$. Define the function $\Phi : L_\infty([0,T];\mathbb{R}^n) \to \mathbb{R}$ by

$$\Phi(x(\cdot)) = \int_0^T \left(f_0(t,x(t)) + \sum_{i \in L} \hat{\lambda}_i(t) f_i(t,x(t)) - f_0(t,\hat{x}(t)) \right) dt.$$

Since $(\hat{x}(\cdot), \hat{\lambda}(\cdot))$ is a Karush-Kuhn-Tucker saddle point, we obtain

$$\Phi(x(\cdot)) \geq 0,\ \ \forall x(\cdot) \in L_\infty([0,T];\mathbb{R}^n).$$

For the admissible point $x(\cdot) = \hat{x}(\cdot)$, and from slackness condition $\hat{\lambda}_i(t) f_i(t,\hat{x}(t)) = 0$, $i \in I$ a.e. in $[0,T]$, we have that $\Phi(\hat{x}(\cdot)) = 0$ holds. Since the function $\Phi(x(\cdot))$ is linear in x, nonnegative and vanishes at the point $\hat{x}(\cdot)$, we can conclude that the preceding function vanishes on whole space $L_\infty([0,T];\mathbb{R}^n)$. Also, we have that

$$\Phi(\hat{x}(t) + z(t)) = \int_0^T \left(f_0(t,z(t)) + \sum_{i \in L} \hat{\lambda}_i(t) f_i(t,z(t)) \right) dt = 0,$$

holds, which is a contradiction with assumptions $f_i(t,z(t)) \leq 0$ for $i \in L$ and $f_0(t,z(t)) < 0$ a.e. in $[0,T]$. Indeed, from that and from the fact that $\hat{\lambda}(t) \geq 0$ a.e. in $[0,T]$, we obtain $\Phi(\hat{x}(t) + z(t)) < 0$. □

As an illustration, we will consider the following simple example:

Example 3.8.

$$\begin{aligned} &\int_0^1 (x_1(t) - x_2(t))\, dt \to \inf; \\ &-x_1(t) - 1 \le 0 \text{ a.e. in } [0,1], \\ &3x_1(t) + 2x_2(t) \le 0 \text{ a.e. in } [0,1], \\ &x_i(\cdot) \in L_\infty([0,1];\mathbb{R}),\ i = 1,2. \end{aligned}$$

It is obvious that $\hat{x}(t) = (\hat{x}_1(t), \hat{x}_2(t)) = (-1, \frac{3}{2})$ *a.e. in* $[0,1]$ *is an optimal solution of preceding problem, where* $f_0(t, x(t)) = x_1(t) - x_2(t)$, $f_1(t, x(t)) = -x_1(t) - 1$ *and* $f_2(t, x(t)) = 3x_1(t) + 2x_2(t)$. *Further, we have that* (SQ) *is satisfied for* $y(t) = (\frac{t}{3}, -\frac{t+1}{2})$ *a.e. in* $[0,1]$ *and there exists point* $z(t) = (-t-1, t)$ *a.e. in* $[0,1]$, *such that* $f_0(t, z(t)) < 0$ *and* $f_2(t, z(t)) \le 0$ *a.e. in* $[0,1]$.

Given $x = (x_1, x_2) \in \mathbb{R}^2$, *for almost every* t *in* $[0,1]$, $\chi_0(t,x) = x_1 - x_2 + \frac{5}{2}$, $\chi_1(t,x) = -x_1 - 1$, $\chi_2(t,x) = 3x_1 + 2x_2$.

Define

$$\begin{aligned} A &= \{(x_1,x_2) \in \mathbb{R}^2 : \frac{2}{3}x_1 + x_2 - \frac{5}{6} \ge 0,\ 2x_1 + x_2 + \frac{1}{2} \ge 0\}, \\ B &= \{(x_1,x_2) \in \mathbb{R}^2 : 2x_1 + x_2 + \frac{1}{2} \le 0,\ -2x_1 + x_2 - \frac{7}{2} \ge 0\}, \\ C &= \{(x_1,x_2) \in \mathbb{R}^2 : -2x_1 + x_2 - \frac{7}{2} \le 0,\ \frac{2}{3}x_1 + x_2 - \frac{5}{6} \le 0\}. \end{aligned}$$

It is obvious that $\partial_x \chi_0(t,x) = (1,-1)$, $\partial_x \chi_1(t,x) = (-1,0)$ *and* $\partial_x \chi_2(t,x) = (3,2)$. *It can be easily verified that* $A \cap B \cap C = \{(-1, \frac{3}{2})\}$, $A \cup B \cup C = \mathbb{R}^2$ *and*

$$\begin{aligned} &\mathcal{I}(t,x) = \{2\} \text{ in } \operatorname{int}(A), \\ &\mathcal{I}(t,x) = \{1\} \text{ in } \operatorname{int}(B), \\ &\mathcal{I}(t,x) = \{0\} \text{ in } \operatorname{int}(C), \\ &\mathcal{I}(t,x) = \{1,2\} \text{ in } (A \cap B) \setminus (A \cap B \cap C), \\ &\mathcal{I}(t,x) = \{0,2\} \text{ in } (A \cap C) \setminus (A \cap B \cap C), \\ &\mathcal{I}(t,x) = \{0,1\} \text{ in } (B \cap C) \setminus (A \cap B \cap C), \\ &\mathcal{I}(t,x) = \{0,1,2\} \text{ in } A \cap B \cap C. \end{aligned}$$

The regularity of the system

$$\begin{aligned} &\chi_0(t,x) = x_1 - x_2 + \frac{5}{2} < 0, \\ &\chi_1(t,x) = -x_1 - 1 \le 0, \\ &\chi_2(t,x) = 3x_1 + 2x_2 \le 0, \\ &x = (x_1, x_2) \in \mathbb{R}^2, \end{aligned}$$

is verified with $\bar{x}(\cdot) \equiv (-1, \frac{3}{2})$, $R = 2$, $\alpha = \frac{1}{10}$ *and for almost every* $t \in [0,1]$, $e(t,x) = (1,0)$ *for* $x \in \operatorname{int}(A)$ *or* $x \in \operatorname{int}(C)$. *Also, for almost every* $t \in [0,1]$, $e(t,x) = (1,0)$ *for* $x \in (A \cap C) \setminus (A \cap B \cap C)$, $e(t,x) = (-\frac{3}{5}, -\frac{4}{5})$ *for* $x \in \operatorname{int}(B)$ *or* $x \in (B \cap C) \setminus (A \cap B \cap C)$ *and* $e(t,x) = (-\frac{1}{3}, \frac{2\sqrt{2}}{3})$ *for* $x \in (A \cap B) \setminus (A \cap B \cap C)$. *Consequently, regularity condition (RC) is satisfied.*

Therefore, it can be easily verified that $(\hat{x}_1(t), \hat{x}_2(t), \hat{\lambda}_1(t), \hat{\lambda}_2(t)) = (-1, \frac{3}{2}, \frac{5}{2}, \frac{1}{2})$ *a.e. in* $[0,1]$ *is a Karush-Kuhn-Tucker saddle point and the complementary slackness condition holds. Indeed, we have* $\hat{\lambda}_i(t) f_i(t, \hat{x}(t)) = 0$, $i = 1,2$ *a.e. in* $[0,1]$. *Also, from Theorem 3.7 immediately follows that the multiplier* $\hat{\lambda}_1(\cdot) \in L_\infty([0,T];\mathbb{R})$ *must be nonzero in* $[0,1]$.

4. Duality theorems

A duality theorem in continuous-time programming is, generally speaking, the statement of a relationship of a certain kind between two continuous-time programming problems. In this section we will focus our attention on developing some duality relationships between problems (CTP) and (DCTP). We define the dual problem for (CTP) as

$$\begin{aligned} &F(\lambda(\cdot)) \to \sup;\\ \text{subject to } &\lambda(t) \geq 0, \text{ a.e. in } [0,T],\\ &\lambda(\cdot) \in L_\infty([0,T];\mathbb{R}^m), \end{aligned} \tag{DCTP}$$

where

$$F(\lambda(\cdot)) = \inf_{x(\cdot)\in L_\infty([0,T];\mathbb{R}^n)} \mathcal{L}(x(\cdot),\lambda(\cdot)).$$

Here, Ω_D denotes the set of all feasible solutions of the problem (DCTP) i.e.,

$$\Omega_D = \{\lambda(\cdot) \in L_\infty([0,T];\mathbb{R}^m) : \lambda(t) \geq 0, \text{ a.e. in } [0,T]\}.$$

In the sequel, the duality results are stated. The first result, called the weak duality theorem, is a simple consequence of the definition of (DCTP). However, it has some important corollaries.

Theorem 4.1. *(Weak duality theorem) Let $x(\cdot) \in L_\infty([0,T];\mathbb{R}^n)$ and $\lambda(\cdot) \in L_\infty([0,T];\mathbb{R}^m)$ be feasible solutions of* (CTP) *and* (DCTP)*, respectively. Then*

$$F(\lambda(\cdot)) \leq J_0(x(\cdot)).$$

Proof. From definition of F, we obtain

$$\begin{aligned} F(\lambda(\cdot)) &= \inf_{\tilde{x}(\cdot)\in L_\infty([0,T];\mathbb{R}^n)} \int_0^T \left(f_0(t,\tilde{x}(t)) + \sum_{i\in I} \lambda(t) f_i(t,\tilde{x}(t)) \right) dt\\ &\leq \int_0^T f_0(t,x(t))dt + \int_0^T \sum_{i\in I} \lambda_i(t) f_i(t,x(t))dt. \end{aligned}$$

Since $\lambda(\cdot) \in \Omega_D$ and $x(\cdot) \in \Omega_P$, i.e., $\lambda(t) \geq 0$ and $f_i(t,x(t)) \leq 0$, $i \in I$ a.e. in $[0,T]$, we have

$$\int_0^T \sum_{i\in I} \lambda_i(t) f_i(t,x(t))dt \leq 0.$$

Hence

$$F(\lambda(\cdot)) \leq \int_0^T f_0(t,x(t))dt + \int_0^T \sum_{i\in I} \lambda_i(t) f_i(t,x(t))dt \leq \int_0^T f_0(t,x(t))dt = J_0(x(\cdot)),$$

and the result follows. □

We then have, as a corollaries of the previous theorem, the following results.

Corollary 4.2.

$$\sup_{\lambda(\cdot)\in\Omega_D} F(\lambda(\cdot)) \leq \inf_{x(\cdot)\in\Omega_P} J_0(x(\cdot)).$$

Note from the previous theorem that the optimal objective value of the primal problem is greater than or equal to the optimal objective value of the dual problem.

Corollary 4.3. *If*

$$F(\lambda(\cdot)) \geq J_0(x(\cdot))$$

for any feasible solution $x(\cdot)$ *for* (CTP) *and any feasible solution* $\lambda(\cdot)$ *for* (DCTP)*, then* $x(\cdot)$ *and* $\lambda(\cdot)$ *are optimal solutions of* (CTP) *and* (DCTP)*, respectively.*

Proof. From Theorem 4.1, we have that for all (CTP) feasible points $\tilde{x}(\cdot)$, $F(\lambda(\cdot)) \leq J_0(\tilde{x}(\cdot))$ holds. If in addition $F(\lambda(\cdot)) \geq J_0(x(\cdot))$ then $J_0(\tilde{x}(\cdot)) \geq J_0(x(\cdot))\ \forall \tilde{x}(\cdot) \in \Omega_P$. Therefore, this guarantees that $x(\cdot)$ is an optimal solution of (CTP).

On the other hand, $F(\lambda(\cdot)) \geq J_0(x(\cdot)) \geq F(\tilde{\lambda}(\cdot))\ \forall \tilde{\lambda}(\cdot) \in \Omega_D$. Then $\lambda(\cdot)$ is an optimal solution of (DCTP). □

Corollary 4.4. *If solution of* (DCTP) *is* ∞ *then solution of* (CTP) *is* ∞*.*

Proof. For all $x(\cdot) \in \Omega_P$, $\lambda(\cdot) \in \Omega_D$, it is verified that $J_0(x(\cdot)) \geq F(\lambda(\cdot))$ and then

$$J_0(x(\cdot)) \geq \sup_{\lambda(\cdot)\in\Omega_D} F(\lambda(\cdot)) = \infty.$$

This implies that $J_0(x(\cdot)) = \infty$, $\forall x(\cdot) \in \Omega_P$. Therefore (CTP) is infeasible. □

Corollary 4.5. *If solution of* (CTP) *is* $-\infty$ *then solution of* (DCTP) *is* $-\infty$*.*

The following result, known as the strong duality theorem, shows that, under convexity assumptions, suitable regularity condition and Slater's constraint qualification, there is no duality gap between primal (CTP) and dual (DCTP).

Theorem 4.6. *(Strong duality theorem) Let* $\hat{x}(\cdot)$ *be an optimal solution of the problem* (CTP)*. Assume that the problem* (CTP) *satisfies* (RC) *and Slater's constraint qualification* (SQ)*. Then there exists* $\hat{\lambda}(\cdot) \in L_\infty([0,T];\mathbb{R}^m)$*,* $\hat{\lambda}(t) \geq 0$ *a.e. in* $[0,T]$*, such that* $\hat{\lambda}(\cdot)$ *is an optimal solution of* (DCTP) *and we have strong duality, i.e.,*

$$F(\hat{\lambda}(\cdot)) = \sup_{\lambda(\cdot)\in\Omega_D} F(\lambda(\cdot)) = \inf_{x(\cdot)\in\Omega_P} J_0(x(\cdot)) = J_0(\hat{x}(\cdot)). \tag{8}$$

Proof. Let

$$\hat{J}_0 = \inf\Big\{ \int_0^T f_0(t,x(t))dt : \ x(\cdot) \in L_\infty([0,T];R^n),\ \ f_i(t,x(t)) \leq 0,\ \ i \in I,\ \ \text{a.e. in } [0,T]\Big\}.$$

If $\hat{J}_0 = -\infty$ we then conclude from the Corollary 4.5 of the Weak Duality Theorem that

$$\sup\Big\{F(\lambda(\cdot)) : \lambda(t) \geq 0, \text{a.e. in } [0,T]\Big\} = -\infty,$$

and, hence (8) is satisfied. Thus, suppose that $\hat{J}_0$ is finite. Since $\hat{x}(\cdot)$ solves (CTP), from Theorem 3.3 we have that there exist the multipliers $\hat{\lambda}_i \in L_\infty([0,T];\mathbb{R})$, $i \in I$, such that $\hat{\lambda}_i(t) f_i(t,\hat{x}(t)) = 0$, $i \in I$, a.e. in $[0,T]$ and $(\hat{x}(\cdot),\hat{\lambda}(\cdot))$ is a Karush-Kuhn-Tucker saddle point for (CTP), i.e.,

$$\mathcal{L}(\hat{x}(\cdot),\lambda(\cdot)) \leq \mathcal{L}(\hat{x}(\cdot),\hat{\lambda}(\cdot)) \leq \mathcal{L}(x(\cdot),\hat{\lambda}(\cdot)), \tag{9}$$

for all $x(\cdot) \in L_\infty([0,T];\mathbb{R}^n)$ and all $\lambda(\cdot) \in L_\infty([0,T];\mathbb{R}^m)$, $\lambda(t) \geq 0$ a.e. in $[0,T]$. Let $\lambda(\cdot)$ be a feasible solution for (DCTP). We have that

$$\begin{aligned} F(\lambda(\cdot)) &= \inf_{x(\cdot)\in L_\infty([0,T];\mathbb{R}^n)} \int_0^T \left(f_0(t,x(t)) + \sum_{i\in I} \lambda_i(t) f_i(t,x(t)) \right) dt \\ &\leq \int_0^T \left(f_0(t,\hat{x}(t)) + \sum_{i\in I} \lambda_i(t) f_i(t,\hat{x}(t)) \right) dt \\ &\leq \int_0^T \left(f_0(t,\hat{x}(t)) + \sum_{i\in I} \hat{\lambda}_i(t) f_i(t,\hat{x}(t)) \right) dt \\ &= \mathcal{L}(\hat{x}(\cdot),\hat{\lambda}(\cdot)). \end{aligned} \tag{10}$$

From (9) we have that $\mathcal{L}(\hat{x}(\cdot), \hat{\lambda}(\cdot)) \leq \mathcal{L}(x(\cdot), \hat{\lambda}(\cdot))$, $\forall x(\cdot) \in L_\infty([0,T];\mathbb{R}^n)$. Therefore, since $\hat{x}(\cdot) \in L_\infty([0,T];\mathbb{R}^n)$, $F(\hat{\lambda}(\cdot)) = \mathcal{L}(\hat{x}(\cdot), \hat{\lambda}(\cdot))$ holds. Hence,

$$F(\lambda(\cdot)) \leq F(\hat{\lambda}(\cdot)) \ \forall \lambda(\cdot) \in \Omega_D.$$

Thus, $\hat{\lambda}(\cdot)$ is an optimal solution for (DCTP). Since $\hat{\lambda}_i(t) f_i(t, \hat{x}(t)) = 0$, $i \in I$, a.e. in $[0,T]$, we obtain

$$F(\hat{\lambda}(\cdot)) = \int_0^T \left(f_0(t, \hat{x}(t)) + \sum_{i \in I} \hat{\lambda}_i(t) f_i(t, \hat{x}(t)) \right) dt = J_0(\hat{x}(\cdot)).$$

Thus, the proof is complete. □

As an illustration, we will consider the following simple example of strong duality.

Example 4.7. *Consider the following primal problem* (P) *from Example 3.5 and the corresponding dual problem* (D)*:*

$$\begin{aligned} F(\lambda(\cdot)) &\to \sup; \\ \lambda_1(t) &\geq 0 \text{ a.e. in } [0,1], \\ \lambda_2(t) &\geq 0 \text{ a.e. in } [0,1], \\ \lambda_i(\cdot) &\in L_\infty([0,1];\mathbb{R}), \ i = 1,2, \end{aligned} \tag{D}$$

where

$$F(\lambda(\cdot)) = \inf_{x(\cdot) \in L_\infty([0,1];\mathbb{R})} \int_0^1 \Big(|x(t) - t| + x^2(t) - 2tx(t) - \lambda_1(t)x(t) + \lambda_2(t)(e^{x(t)-t} - 1) + t^2 + 1 \Big) dt.$$

It can be verified easily that $\hat{x}(t) = t$ a.e. in $[0,1]$ is an optimal solution of the problem (P) and there exists the multiplier $\hat{\lambda}(\cdot) \in L_\infty([0,1];\mathbb{R}^2)$, $\hat{\lambda}(t) = (\hat{\lambda}_1(t), \hat{\lambda}_2(t)) = (0,1)$ a.e. in $[0,1]$, such that $(\hat{x}(\cdot), \hat{\lambda}_1(\cdot), \hat{\lambda}_2(\cdot))$ is a Karush-Kuhn-Tucker saddle point for (P) and $\hat{\lambda}_i(t) f_i(t, \hat{x}(t)) = 0$, $i = 1,2$, a.e. in $[0,1]$. Also (SQ) and (RC) qaulifications are satisfied. Hence, $\hat{\lambda}(t) = (0,1)$ a.e. in $[0,1]$ is an optimal solution of the problem (D) and $F(\hat{\lambda}(\cdot)) = 1$. It is obvious that optimal values of dual problem (D) and primal problem (P) are equal. Thus, there is no duality gap.

5. Conclusion

This paper addressed the convex continuous-time programming problem. The results were formulated without using differentiability. Saddle point type necessary optimality conditions were presented under Slater constraint qualification, using a new theorem of the alternative with the additional regularity assumption. Sufficient saddle point optimality conditions were given and the weak and strong duality theorems were stated. It would be of interest to see how the similar approach can be extended to examine optimality conditions for nonsmooth fractional continuous-time programming and smooth quadratic continuous-time programming, where the objective function is symmetric positive definite matrix which elements are measurable functions on $[0,T]$.

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