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On *GCED* Matrices over *UFDs*

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Abstract. An extension of the *GCED* matrices from the domain of natural integers to the unique factorization domain is given. The structure of these type of matrices defined on both arbitrary sets and *GCED*-closed sets are presented. Moreover, we present exact expressions for the determinant and the inverse of such matrices. The domains of Gaussian integers and polynomials over finite fields are used to illustrate the work.

1. Introduction and Preliminaries

Let $T = \{x_1, x_2, ..., x_m\}$ be a well ordered set of m distinct positive integers with $1 < x_2 < ... < x_m$. The *GCD* matrix on T is defined as $(T)_{m\times m} = (x_i, x_j)$, where (x_i, x_j) is the greatest common divisor of x_i and x_j, and the power *GCD* matrix on T is $(T^r)_{m\times m} = (x_i, x_j)^r$, where r is any real number. A Set $T = \{x_1, x_2, ..., x_m\}$ is said to be factor-closed set if x is an element of T for any divisor x of x_i in T, and it is said to be gcd-closed if (x_i, x_j) is also in T, for all x_i and x_j in T. Smith [15] showed that if $T = \{1, 2, ..., m\}$, then $\det(T) = \prod_{i=1}^{m} \phi(i)$, where ϕ is Euler's totient function and π is a multiplicative function. Moreover, Smith showed that his results are true for factor-closed sets. Beslin and Ligh [3, 4], factorized the *GCD* matrices if T is a gcd-closed set, and computed their determinants. Chun [5] introduced the concept of power *GCD* matrices, and a general formula for their structures, determinants and inverses were given over the domain of natural numbers. Li [13] showed that $\det(T) = \prod_{i=1}^{m} \phi(x_i)$ if and only if $T = \{x_1, x_2, ..., x_m\}$ is a factor closed set of ordered distinct positive integers. Haukkanen and Sillanpaa [10] studied the *GCD* matrices for gcd-closed sets. Haukkanen [9], in his famous paper "On Smith's Determinant" gave a counter example for the conjecture of Bourque-Ligh that the least common multiple matrix, *LCM* matrix, on any gcd-closed set is invertible. Beslin and El-Kassar [2] extended the concept of *GCD* matrices and Smith's determinant to UFDs. El-Kassar et al. [6–8] extended many results concerning *GCD* matrices defined on factor-closed sets to arbitrary principal ideal domains. Hong et al. [11] generalized the power *GCD* matrices defined on factor-closed sets from the standard settings $\mathbb{Z}$ to UFDs.

Raza and Waheed [14], studied the *GCED* matrices defined on a finite set $T = \{x_1, x_2, ..., x_n\}$ of distinct positive integers that are arranged in an increasing order. They defined the *GCED* square matrix (T) having

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$t_{ij} = (x_i, x_j)_e$, the greatest common exponential divisor of x_i and x_j, as it's ij^{th} entry. They gave structure theorems and calculated the determinant of these matrices. Also, they calculated the determinant and the inverse when the matrices are defined on exponential factor-closed sets. It is well known that $(\mathbf{Z}^+ \setminus \{1\}, |_e)$ is a poset under the exponential divisibility relation but not a lattice, since the *GCED* does not always exist. More details are given in the next section. Korkee and Haukkanen [12] embedded this poset in a lattice and studied the *GCED* matrices as an analogue of the *GCD* matrices.

In this paper, we extend the concept of exponential divisors over UFDs. Also, we determine the structure of the *GCED* and the inverse of the *GCED* matrices defined on an arbitrary finite ordered subsets of these domains, as well as their determinant and trace. In addition, some examples in $\mathbf{Z}[i]$ and $\mathbf{Z}_p[x]$, where p is a prime integer, are given in order to describe what have been done.

Why working in UFDs? In a UFD:

- Every non-zero and non-unit element can be written as a product of irreducibles.
- The decomposition of each element is unique up to order and associates.
- Any two elements in a UFD have a greatest common divisor.
- The elements in a UFD can be ordered.

Also, the work done in the literature used the classical domain (domain of natural integers), which is an example of a UFD and hence the previous work is a special case when taking the domain of integers as our UFD. Working in UFDs, many domains can be taken such as $\mathbf{Z}_p[x]$ and $\mathbf{Z}[i]$.

Throughout this paper,

- D is a UFD.
- p_i is a prime element in D.
- a_i, b_i and c_i are positive integers.
- $z \sim w$ means z and w are two associates.
- $T = \{x_1, x_2, \ldots, x_n\}$ is a finite ordered set (increasing order) of nonzero, non-unit and non-associate elements in D.

2. Exponential Divisors in UFDs

In this section, we introduce the concept of the exponential divisors over D.

Definition 2.1. *A nonzero element* $d = \prod_{i=1}^{r} p_i^{a_i}$ *in* D *is an exponential divisor of* $a = \prod_{i=1}^{r} p_i^{c_i}$ *if* $a_i \mid c_i$ *for every* $1 \le i \le r$, *denoted by* $d \mid_e a$.

A unit u in D is not an exponential divisor for any nonzero, non unit element a in D and by convention $u \mid_e v$ for any unit v in D. Two elements in D have a common exponential divisor if and only if they have the same prime factors. We denote the *GCED* of a and b by $(a, b)_e$ or $GCED(a, b)$. By convention, $(u, v)_e = 1$ and $(u, a)_e$ does not exist for any nonzero, non-unit element a in D. Two elements $a = \prod_{i=1}^{r} p_i^{b_i}$ and $b = \prod_{i=1}^{r} p_i^{c_i}$ in D are exponentially coprime if $\gcd(b_i, c_i) = 1$, for every $1 \le i \le r$.

A subset $T = \{x_1, x_2, \ldots, x_n\}$ of D is a *GCED* closed set if $(x_i, x_j)_e$ is also an element of T for all x_i, x_j in T, where $1 \le i, j \le n$. For example, the subset $T = \{1 + 3i, -1 + 7i, -8 + 6i\}$ of $\mathbf{Z}[i]$ is a *GCED* closed set while the set $T = 2 + 4i, -1 + 7i, -8 + 6i$ is not.

Definition 2.2. *Given two functions f and g defined on D. Define the exponential convolution of f and g of a nonzero element $a = \prod_{i=1}^{r} p_i^{c_i}$ in D as:*

$$(f \odot g)(a) = \sum_{a_1 b_1 = c_1} \dots \sum_{a_r b_r = c_r} f(p_1^{a_1} p_2^{a_2} \dots p_r^{a_r}) g(p_1^{b_1} p_2^{b_2} \dots p_r^{b_r}).$$

Using the Möbius inversion exponential formula, $g(a) = \sum_{d|_e a} f(d)\mu^{(e)}(\frac{a}{d})$ if $f(a) = \sum_{d|_e a} g(d)$, where $\mu^{(e)}(u) = 1$ and $\mu^{(e)}(a) = \mu(c_1)\mu(c_2)\dots\mu(c_r)$.

3. Ordering in Special UFDs

The domains of Gaussian integers $\mathbf{Z}[i]$ and polynomials over finite fields $\mathbf{Z}_p[x]$ are not ordered. We use a well-defined linear ordering defined on these domains so that any two elements are comparable. The ordering in these domains is given in the following two definitions.

Definition 3.1. *(Ordering in the Set of Gaussian Integers) Let $T = \{z_1, z_2, \dots, z_n\}$ be a subset of $\mathbf{Z}[i]$. Define an ordering on T as follows: If $q(z_i) < q(z_j)$, then $z_i < z_j$. If $q(z_i) = q(z_j)$, where $z_i \sim a + ib$ and $z_j \sim c + id$, such that $a, b, c, d \geq 0$, then $z_i < z_j$ if $b < d$. The valuation function q is defined as: $q(a + ib) = a^2 + b^2$. The relation $<$ is a well-defined linear ordering on T.*

Example 3.2. *$T = \{-2 + 3i, -2 - 3i, 4 + 5i\}$ is ordered set in $Z[i]$. $z_1 = i(3+2i) \approx 3+2i$ and $z_2 = -(2+3i) \approx 2+3i$, so $z_1 < z_2 < z_3$.*

Definition 3.3. *(Ordering in polynomial rings over a field) Let $T = \{f_1, f_2, \dots, f_n\}$ be a subset of $\mathbf{Z}_p[x]$, where p is a prime integer. Define an ordering on T as follows: If $\deg(f_i) < \deg(f_j)$, then $f_i < f_j$. If $\deg(f_i) = \deg(f_j)$ with $f_i \sim x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$ and $f_j \sim x^n + b_{n-1}x^{n-1} + \dots + b_1 x + b_0$ with $0 \leq a_j, b_j \leq p - 1$, then $f_i(x) < f_j(x)$ if $a_{j_0} < b_{j_0}$, where j_0 is the smallest index j such that $a_j \neq b_j$. Again, the relation $<$ is a well-defined linear ordering on T.*

Example 3.4. *$T = \left\{x^2 + 2x + 1, x^2 + 3x + 1, x^4 + x^2 + 1\right\}$ is an ordered set in $Z_4[x]$. $a_1 = 2$ and $b_1 = 3$, so $f_1 < f_2 < f_3$.*

Definition 3.5. *(Positive Elements in UFDs) An nonzero element n in D is positive if $n > 0$, the zero element in D and $>$ is the ordering defined on D.*

4. GCED Matrices in UFDs

In this section, we introduce the concept of *GCED* matrices defined on *GCED*-closed and *GCED* non-closed sets over UFDs. Complete characterization for the factorization, determinant, trace and inverse of such matrices is given. Moreover, examples in $\mathbf{Z}[i]$ and in $\mathbf{Z}_p[x]$ are presented.

4.1. Structures and Determinants of the GCED Matrices

Let $T = \{x_1, x_2, \dots, x_n\}$ be a subset of D. The *GCED* matrix (T_e) defined on T is the $n \times n$ matrix whose ij^{th} entry is $(x_{ij})_{(e)} = (x_i, x_j)_e$, the greatest common exponential divisor of x_i and x_j.

Let $R = \{y_1, y_2, \dots, y_m\}$ be the minimal *GCED*-closed set containing T (*GCED* closure of T), such that $y_1 < y_2 < \dots < y_m$. Define the function $g(m)$ as follows:

$$g(m) = \sum_{a_1 b_1 = c_1} \dots \sum_{a_r b_r = c_r} (p_1^{a_1} p_2^{a_2} \dots p_r^{a_r}) \mu^{(e)}(p_1^{b_1} p_2^{b_2} \dots p_r^{b_r}).$$

where $m = p_1^{c_1} p_2^{c_2} \dots p_r^{c_r}$ is an element in D.

Theorem 4.1. *Let $T = \{x_1, x_2, \ldots, x_n\}$ be a GCED-closed set in D. Then,*

$$\sum_{x_k|_e(x_i,x_j)_e}\left(\sum_{\substack{d|_e x_k\\ d\nmid_e x_r\\ x_r<x_k}} g(d)\right) = \sum_{d|_e(x_i,x_j)_e} g(d).$$

Proof. Let $d \mid_e (x_i, x_j)_e$ and let $S = \{x_{k_1}, x_{k_2}, ..., x_{k_r}\}$ be an ordered subset of T such that $x_{k_m} \mid_e \left(x_i, x_j\right)_e$ and $d \mid_e x_{k_m}$ for every $1 \leq m \leq r$. Then $d \mid_e (x_{k_1}, x_{k_2}, ..., x_{k_r})_e$ which is an element in T as T is a *GCED*-closed set. Since T is an ordered set, then $(x_{k_1}, x_{k_2}, ..., x_{k_r})_e = x_{k_1}$. But $d \mid x_{k_1}$and $d \nmid_e x_r$ whenever $x_r < x_{k_1}$ as x_{k_1} is the minimal element in S. So, each divisor of $\left(x_i, x_j\right)_e$ is found once in the sum. Hence,

$$\sum_{x_k|_e(x_i,x_j)_e}\left(\sum_{\substack{d|_e x_k\\ d\nmid_e x_r\\ x_r<x_k}} g(d)\right) = \sum_{d|_e(x_i,x_j)_e} g(d).$$

□

Let $R = \{y_1, y_2, \ldots, y_m\}$ be the *GCED*-closure of $T = \{x_1, x_2, \ldots, x_n\}$, where $y_1 < y_2 < \cdots < y_m$ and $x_1 < x_2 < \cdots < x_n$.

Theorem 4.2. $(T_e) = C\psi C^t$*, where the $n \times m$ matrix $C = (c_{ij})$ is defined as:*

$$c_{ij} = \begin{cases} 1, & y_j \mid_e x_i \\ 0, & else \end{cases}$$

and ψ is an $m \times m$ diagonal matrix defined as:

$$\psi = diag\left(\sum_{d|_e y_1} g(d), \sum_{\substack{d|_e y_2\\ d\nmid_e y_1}} g(d), ..., \sum_{\substack{d|_e y_m\\ d\nmid_e y_r\\ y_r<y_m}} g(d)\right).$$

Proof. The ij^{th} entry of $C\psi C^t$ is

$$\begin{aligned}\left(C\psi C^t\right)_{ij} &= \sum_{k=1}^{m} c_{ik}\left(\sum_{\substack{d|_e y_k\\ d\nmid_e y_r\\ y_r<y_k}} g(d)\right) c_{jk} = \sum_{\substack{y_k|_e x_i\\ y_k|_e x_j}}\left(\sum_{\substack{d|_e y_k\\ d\nmid_e y_r\\ y_r<y_k}} g(d)\right) \\ &= \sum_{y_k|_e(x_i,x_j)_e}\left(\sum_{\substack{d|_e y_k\\ d\nmid_e y_r\\ y_r<y_k}} g(d)\right) = \sum_{d|_e(x_i,x_j)_e} g(d).\end{aligned}$$

By the Möbius inversion exponential formula, it follows that

$$\sum_{d|_e m} g(d) = m.$$

Hence,

$$\left(C\psi C^t\right)_{ij} = (x_i, x_j)_e = ((T_e))_{ij}.$$

□

Theorem 4.3. *$det(T_e) = \sum_{1\le k_1<k_2<\ldots<k_n\le m} (detC_{(k_1,k_2,\ldots,k_n)})^2 \prod_{i=1}^{n} \left(\sum_{\substack{d|_e y_{k_i} \\ d\nmid_e y_{k_r} \\ y_{k_r}<y_{k_i}}} g(d) \right)$, where $C_{(k_1,k_2,\ldots,k_n)}$ is the submatrix of C consisting of $k_1^{th}, k_2^{th}, \ldots, k_n^{th}$ columns of C.*

Proof. Let D_e be an extension field of $D(x)$, the field of fractions of D, in which $\sqrt{\sum_{\substack{d|_e y_{k_i} \\ d\nmid_e y_{k_r} \\ y_{k_r}<y_{k_i}}} g(d)}$ exists. $(T_e) = C\psi C^t = AA^t$, where $A = C\psi^{\frac{1}{2}}$. Apply the Cauchy-Binet formula to get

$$\begin{aligned} det(T_e) &= \sum_{1\le k_1<\ldots<k_n\le m} (detA_{(k_1,k_2,\ldots,k_n)})(detA^t_{(k_1,k_2,\ldots,k_n)}) \\ &= \sum_{1\le k_1<\ldots<k_n\le m} (detA_{(k_1,k_2,\ldots,k_n)})^2, \end{aligned}$$

where $A_{(k_1,k_2,\ldots,k_n)}$ is the submatrix of A consisting of $k_1^{th}, k_2^{th}, \ldots, k_n^{th}$ columns of A. Moreover, $detA_{(k_1,k_2,\ldots,k_n)} = detC_{(k_1,k_2,\ldots,k_n)} \sqrt{\prod_{i=1}^{n} \left(\sum_{\substack{d|_e y_{k_i} \\ d\nmid_e y_{k_r} \\ y_{k_r}<y_{k_i}}} g(d) \right)}$. Hence,

$$det(T_e) = \sum_{1\le k_1<k_2<\ldots<k_n\le m} (detC_{(k_1,k_2,\ldots,k_n)})^2 \prod_{i=1}^{n} \left(\sum_{\substack{d|_e y_{k_i} \\ d\nmid_e y_{k_r} \\ y_{k_r}<y_{k_i}}} g(d) \right).$$

□

Remark 4.4. *If $<$ is the ordering defined on D, then $\sum_{\substack{d|_e y_{k_i} \\ d\nmid_e y_{k_r} \\ y_{k_r}<y_{k_i}}} g(d) > 0$.*

Example 4.5. *Let $T = \{-2+4i, -1+7i, -12-16i\}$ which is not a GCED-closed set in $\mathbf{Z}[i]$. Its GCED-closure is $R = \{1+3i, -2+4i, -1+7i, -12-16i\}$. The GCED matrix (T_e) defined on T is:*

$$(T_e) = \begin{bmatrix} -2+4i & 1+3i & -2+4i \\ 1+3i & -1+7i & -1+7i \\ -2+4i & -1+7i & -12-16i \end{bmatrix}.$$

And

$$C\psi C^t = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1+3i & 0 & 0 & 0 \\ 0 & -3+i & 0 & 0 \\ 0 & 0 & -2+4i & 0 \\ 0 & 0 & 0 & -8-24i \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = (T_e),$$

$$\begin{aligned} det(T_e) &= \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix}^2 \sum_{d|_e y_1} g(d) \sum_{\substack{d|_e y_2 \\ d\nmid_e y_1}} g(d) \sum_{\substack{d|_e y_3 \\ d\nmid_e y_r \\ y_r<y_3}} g(d) \\ &+ \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{vmatrix}^2 \sum_{d|_e y_1} g(d) \sum_{\substack{d|_e y_2 \\ d\nmid_e y_1}} g(d) \sum_{\substack{d|_e y_4 \\ d\nmid_e y_r \\ y_r<y_4}} g(d) \\ &+ \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix}^2 \sum_{d|_e y_1} g(d) \sum_{\substack{d|_e y_3 \\ d\nmid_e y_r \\ y_r<y_3}} g(d) \sum_{\substack{d|_e y_4 \\ d\nmid_e y_r \\ y_r<y_4}} g(d) \\ &+ \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix}^2 \sum_{\substack{d|_e y_2 \\ d\nmid_e y_1}} g(d) \sum_{\substack{d|_e y_3 \\ d\nmid_e y_r \\ y_r<y_3}} g(d) \sum_{\substack{d|_e y_4 \\ d\nmid_e y_r \\ y_r<y_4}} g(d) \\ &= -388+616i. \end{aligned}$$

Corollary 4.6. *Let $T = \{x_1, x_2, \ldots, x_n\}$ be a GCED-closed subset of D. Then,*

$$det(T_e) = \prod_{k=1}^{n} \left(\sum_{\substack{d|_e x_k \\ d\nmid_e x_r \\ x_r<x_k}} g(d) \right).$$

Proof. The matrix C is a lower triangular with main diagonal $(1,1,...,1)_n$ since T is a *GCED*-closed set and

$$det(T_e) = \prod_{k=1}^{n} \left(\sum_{\substack{d|_e x_k \\ d\nmid_e x_r \\ x_r<x_k}} g(d) \right).$$

□

Corollary 4.7. *Let $T = \{x_1, x_2, \ldots, x_n\}$ be a subset of D, then*

$$tr\left((T_e)\right) = \sum_{i=1}^{n} x_i.$$

Theorem 4.8. *Let $T = \{x_1, x_2, \ldots, x_n\}$ be a subset of D. Then, $det(T_e) = \prod_{k=1}^{n}\left(\sum_{\substack{d|_e x_k\\ d\nmid_e x_r\\ x_r<x_k}} g(d)\right)$ if and only if T is GCED-closed.*

Proof. The necessary condition follows from corollary 4.6. Now, assume that T is not a $GCED$-closed set and the equality holds. Theorem 4.3 gives

$$det(T_e) = \sum_{1\le k_1<k_2<\ldots<k_n\le m} (detC_{(k_1,k_2,\ldots,k_n)})^2 \prod_{i=1}^{n}\left(\sum_{\substack{d|_e y_{k_i}\\ d\nmid_e y_{k_r}\\ y_{k_r}<y_{k_i}}} g(d)\right).$$

This sum runs over the all combinations of the k_i^{th} columns of the matrix C, where $1 \le i \le n$. In each combination we get a new term in this sum, as y_{k_i} related to the chosen column k_i. Since T is a subset of R, then $det(T_e) = \prod_{k=1}^{n}\left(\sum_{\substack{d|_e x_k\\ d\nmid_e x_r\\ x_r<x_k}} g(d)\right) + s$, where $s > 0$. Consequently, $det(T_e) > \prod_{k=1}^{n}\left(\sum_{\substack{d|_e x_k\\ d\nmid_e x_r\\ x_r<x_k}} g(d)\right)$ which contradicts the necessary condition that the equality holds. □

4.2. ***Inverse of the GCED Matrix***

Let $T = \{x_1, x_2, \ldots, x_n\}$ be a $GCED$-closed subset of D. We have defined the $n \times n$ matrix $C = (c_{ij})$ as:

$$c_{ij} = \begin{cases} 1, & y_j \mid_e x_i \\ 0, & \text{else.} \end{cases}.$$

Theorem 4.9. *The inverse of C is the $n \times n$ matrix $W = (w_{ij})$ which is defined as:*

$$w_{ij} = \begin{cases} \displaystyle\sum_{\substack{d|_e \frac{x_i}{x_j}\\ d\nmid_e \frac{x_r}{x_j},\ x_r<x_i}} \mu^{(e)}(d), & \text{if } x_j \mid_e x_i \\ 0, & \text{otherwise.} \end{cases}.$$

Proof. The ij^{th} entry of CW is given by

$$(cw)_{ij} = \sum_{k=1}^{n} c_{ik} w_{kj} = \sum_{\substack{x_k|_e x_i\\ x_j|_e x_k}} \left(\sum_{\substack{d|_e \frac{x_k}{x_j}\\ d\nmid_e \frac{x_r}{x_j}\\ x_r<x_k}} \mu^{(e)}(d)\right) = \sum_{\frac{x_k}{x_j}|_e \frac{x_i}{x_j}} \left(\sum_{\substack{d|_e \frac{x_k}{x_j}\\ d\nmid_e \frac{x_r}{x_j}\\ x_r<x_k}} \mu^{(e)}(d)\right).$$

By a similar argument to that given in theorem 1, we have

$$\sum_{\frac{x_k}{x_j}|_e \frac{x_i}{x_j}} \left(\sum_{\substack{d|_e \frac{x_k}{x_j} \\ d\nmid_e \frac{x_r}{x_j} \\ x_r<x_k}} \mu^{(e)}(d) \right) = \sum_{d|_e \frac{x_i}{x_j}} \mu^{(e)}(d) = \mu^2\left(\frac{x_i}{x_j}\right) = \begin{cases} 1 & \text{if } x_i = x_j \\ 0 & \text{otherwise} \end{cases}.$$

□

Theorem 4.10. *The inverse of the $n \times n$ GCED matrix (T_e) is the matrix $M_{(e)} = (m_{ij})_{(e)}$ where*

$$\left(m_{ij}\right) = \sum_{\substack{x_i|_e x_k \\ x_j|_e x_k}} \left(\sum_{\substack{d|_e \frac{x_k}{x_i} \\ d\nmid_e \frac{x_r}{x_i} \\ x_r<x_k}} \mu^{(e)}(d) \frac{1}{\sum_{\substack{d|_e x_k \\ d\nmid_e x_r \\ x_r<x_k}} g(d)} \sum_{\substack{d|_e \frac{x_k}{x_j} \\ d\nmid_e \frac{x_r}{x_j} \\ x_r<x_k}} \mu^{(e)}(d) \right).$$

Proof. $M_{(e)} = T_{(e)}^{-1} = (C\psi C^t)^{-1} = W^t\psi^{-1}W$, where $W = C^{-1}$ and $\psi^{-1} = diag\left(\frac{1}{\sum_{d|_e x_1} g(d)}, \frac{1}{\sum_{\substack{d|_e x_2 \\ d\nmid_e x_1}} g(d)}, \dots, \frac{1}{\sum_{\substack{d|_e x_n \\ d\nmid_e x_r \\ x_r<x_n}} g(d)} \right)$. So,

$$\begin{aligned} m_{ij} &= (W^t\psi^{-1}W)_{ij} \\ &= \sum_{k=1}^{n} w_{ki} \frac{1}{\sum_{\substack{d|_e x_k \\ d\nmid_e x_r \\ x_r<x_k}} g(d)} w_{kj} \\ &= \sum_{\substack{x_i|_e x_k \\ x_j|_e x_k}} \left(\sum_{\substack{d|_e \frac{x_k}{x_i} \\ d\nmid_e \frac{x_r}{x_i} \\ x_r<x_k}} \mu^{(e)}(d) \frac{1}{\sum_{\substack{d|_e x_k \\ d\nmid_e x_r \\ x_r<x_k}} g(d)} \sum_{\substack{d|_e \frac{x_k}{x_j} \\ d\nmid_e \frac{x_r}{x_j} \\ x_r<x_k}} \mu^{(e)}(d) \right). \end{aligned}$$

□

Example 4.11. *Let $T = \left\{x^2 + 2, x^3 + 2x^2 + 2x + 1, x^4 + x^2 + 1\right\}$ which is GCED-closed set in $\mathbf{Z}_3[x]$. The GCED matrix defined on T is:*

$$(T_e) = \begin{bmatrix} x^2+2 & x^2+2 & x^2+2 \\ x^2+2 & x^3+2x^2+2x+1 & x^3+2x^2+2x+1 \\ x^2+2 & x^3+2x^2+2x+1 & x^4+x^2+1 \end{bmatrix}.$$

Then,

$$
\begin{aligned}
m_{11} &= \mu^{(e)}(x^2+2)\frac{1}{g(x^2+2)}\mu^{(e)}(x^2+2)\\
&+ \mu^{(e)}(x^3+2x^2+2x+1)\frac{1}{g(x^3+2x^2+2x+1)}\mu^{(e)}(x^3+2x^2+2x+1)\\
&+ [\mu^{(e)}(x^3+x^2+2x+2)+\mu^{(e)}(x^4+x^2+1)]\times\\
&\frac{1}{g(x^3+x^2+2x+2)+g(x^4+x^2+1)}\\
&\times [\mu^{(e)}(x^3+x^2+2x+2)+\mu^{(e)}(x^4+x^2+1)]\\
&= \frac{1}{x^2+2}+\frac{1}{x^3+x^2+2x+2}=\frac{1}{(x+1)^2}.
\end{aligned}
$$

$$
\begin{aligned}
m_{12} &= \mu^{(e)}(x^3+2x^2+2x+1)\frac{1}{g(x^3+2x^2+2x+1)}\mu^{(e)}(x^2+2)\\
&+ [\mu^{(e)}(x^3+x^2+2x+2)+\mu^{(e)}(x^4+x^2+1)]\times\\
&\frac{1}{g(x^3+x^2+2x+2)+g(x^4+x^2+1)}\mu^{(e)}(x^3+x^2+2x+2)\\
&= -\frac{1}{x^3+x^2+2x+2}.
\end{aligned}
$$

$$
\begin{aligned}
m_{13} &= [\mu^{(e)}(x^3+x^2+2x+2)\\
&+ \mu^{(e)}(x^4+x^2+1)]\frac{1}{g(x^3+x^2+2x+2)+g(x^4+x^2+1)}\mu^{(e)}(x^2+2)\\
&= 0.
\end{aligned}
$$

Completing the computation, we get

$$
M_{(e)} = \begin{bmatrix}
\frac{1}{(x+1)^2} & -\frac{1}{x^3+x^2+2x+2} & 0\\
-\frac{1}{x^3+x^2+2x+2} & -\frac{x^4+2}{2x^7+x} & \frac{1}{2x^4+x^3+x^2+2x}\\
0 & \frac{1}{2x^4+x^3+x^2+2x} & -\frac{1}{2x^4+x^3+x^2+2x}
\end{bmatrix}.
$$

5. Conclusion

We considered the *GCED* matrices defined on *GCED* closed and non-*GCED* closed sets over a unique factorization domain D. We gave a complete characterization of their structure, determinant, trace, and inverse.

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