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Uniqueness of L-Function and Certain Class of Meromorphic Function under Two Weighted Shared Sets of Least Cardinalities

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Abstract. In this article we study the uniqueness problem of an L function belonging to the Selberg class with an arbitrary meromorphic function having finite poles sharing two sets. Actually to answer a question raised by Lin-Lin [Filomat, **30**(2016), 3795-3806], we have significantly improved a recent result [Rend. Del. Math. Palermo, (2020)(published online)] of the authors and that of Chen-Qiu [Acta. Math. Sci., **40B**(4) (2020), 930-980]. Moreover we have also been able to provide the best possible answer of another unsolved question of [Filomat, **30**(2016), 3795-3806] and investigated the results of the same in the light of finite weighted sharing.

1. Introduction

The Riemann zeta function is the function of the complex variable s, defined in the half-plane $Re(s) > 1$ by the absolutely convergent series $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ and in the whole complex plane $\mathbb{C}$ by analytic continuation. Riemann showed $\zeta(s)$ extends to $\mathbb{C}$ as a meromorphic function with only a simple pole at $s = 1$, with residue 1. With the generalization of Riemann hypothesis, the Riemann zeta function has been replaced by the similar, but much more general, global L-functions.

Recently, the value distributions of L-functions have been investigated by many researchers ([5], [6], [9], [10], [16]). The value distribution of an L-function $\mathcal{L}$ is defined as that of meromorphic function. Naturally for some $c \in \mathbb{C} \cup \{\infty\}$ it is about the roots of the equation $\mathcal{L}(s) = c$.

In 1989, a special class of L function known as Selberg class, was introduced by Selberg [15]. Actually, the Selberg class $\mathcal{S}$ of L-functions is the set of all Dirichlet series $\mathcal{L}(s) = \sum_{n=1}^{\infty} a(n)n^{-s}$ of a complex variable s that satisfy the following axioms (see [15]):

(i) Ramanujan hypothesis: $a(n) \ll n^{\epsilon}$ for every $\epsilon > 0$.

(ii) Analytic continuation: There is a non-negative integer k such that $(s-1)^k\mathcal{L}(s)$ is an entire function of finite order.

(iii) Functional equation: $\mathcal{L}$ satisfies a functional equation of type

$$\Lambda_{\mathcal{L}}(s) = \omega\overline{\Lambda_{\mathcal{L}}(1-\overline{s})},$$

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where

$$\Lambda_{\mathcal{L}}(s) = \mathcal{L}(s)Q^s \prod_{j=1}^{K} \Gamma(\lambda_j s + \nu_j)$$

with positive real numbers Q, λ_j and complex numbers ν_j, ω with $Re\nu_j \geq 0$ and $|\omega| = 1$.
(iv) Euler product hypothesis : $\mathcal{L}$ can be written over prime as

$$\mathcal{L}(s) = \prod_p \exp\left(\sum_{k=1}^{\infty} b(p^k)/p^{ks}\right)$$

with suitable coefficients $b(p^k)$ satisfying $b(p^k) \ll p^{k\theta}$ for some $\theta < 1/2$ where the product is taken over all prime numbers p.

Through out this paper, by an L-function we mean a L-function $\mathcal{L}$ in the Selberg class whose degree $d_{\mathcal{L}}$ of an L-function $\mathcal{L}$ is defined to be

$$d_{\mathcal{L}} = 2\sum_{j=1}^{K} \lambda_j,$$

where λ_j and K are respectively the positive real number and the positive integer as in axiom (iii) above.

By the analytic continuation axiom, an L-function can be analytically continued as a meromorphic function in the complex plane $\mathbb{C}$. In the last few years value distribution of L-function has become an interesting area of research.

In this paper we are going to discuss some results in value distribution of Selberg class L-function. Before entering into the detail literature, let us assume $\mathcal{M}(\mathbb{C})$ as the field of meromorphic function over $\mathbb{C}$. To prove the main results we will use Nevanlinna theory. So it is assumed that the readers are familiar with standard notations like the characteristic function $T(r, f)$, the proximity function $m(r, f)$, counting (reduced counting) function $N(r, f)$ ($\overline{N}(r, f)$) that are also explained in [17]. By $S(r, f)$ we mean any quantity that satisfies $S(r, f) = O(\log(rT(r, f)))$ when $r \longrightarrow \infty$, except possibly on a set of finite Lebesgue measure. When f has finite order, then $S(r, f) = O(\log r)$ for all r.

Let us take $f \in \mathcal{M}(\mathbb{C})$, then the order of f is defined as

$$\rho(f) := \limsup_{r\longrightarrow\infty} \frac{\log T(r, f)}{\log r}.$$

In this paper we consider f, a non constant meromorphic function having finitely many poles in $\mathbb{C}$. Clearly $\overline{N}(r, \infty; f) = O(\log r)$.

Before proceeding further, we recall some definitions.

Definition 1.1. *[2] For a non-constant meromorphic function f and* $S \subset \mathbb{C} \cup \{\infty\}$, *let* $E_f(S) = \bigcup_{a\in S}\{(z, p) \in \mathbb{C} \times \mathbb{N} : f(z) = a \text{ with multiplicity } p\}$ $\left(\overline{E}_f(S) = \bigcup_{a\in S}\{(z, 1) \in \mathbb{C} \times \mathbb{N} : f(z) = a\}\right)$. *Then we say f, g share the set S CM(IM) if* $E_f(S) = E_g(S)$ $\left(\overline{E}_f(S) = \overline{E}_g(S)\right)$.

Definition 1.2. *[7] For* $a \in \mathbb{C} \cup \{\infty\}$ *we denote by* $N(r, a; f \mid= 1)$ *the counting function of simple a-points of f. For a positive integer m we denote by* $N(r, a; f| \leq m)(N(r, a; f| \geq m))$ *the counting function of those a-points of f whose multiplicities are not greater(less) than m where each a-point is counted according to its multiplicity.*

$\overline{N}(r, a; f \mid\leq m)(\overline{N}(r, a; f \mid\geq m))$ *are defined similarly, where in counting the a-points of f we ignore the multiplicities.*

Also $N(r, a; f \mid< m), N(r, a; f \mid> m), \overline{N}(r, a; f \mid< m)$ *and* $\overline{N}(r, a; f \mid> m)$ *are defined analogously.*

Definition 1.3. *[8] Let k be a non-negetive integer or infinity. For* $a \in \mathbb{C} \cup \{\infty\}$ *we denote by* $E_k(a; f)$ *the set of all zeros of* $f - a$ *where a zero of multiplicity m is counted m times if* $m \leq k$ *and* $k + 1$ *times if* $m > k$. *If* $E_k(a, f) = E_k(a; g)$ *then we say that f and g share the value a with weight k and we write it as f, g share* (a, k).

Definition 1.4. *[8] Let S be a set of distinct elements of $\mathbb{C} \cup \{\infty\}$ and k be a non-negative integer or ∞. We denote by $E_f(S,k)$ the set $\cup_{a\in S} E_k(a;f)$. Clearly $E_f(S) = E_f(S,\infty)$ and $\overline{E}_f(S) = E_f(S,0)$.*

If $E_f(S,k) = E_g(S,k)$, then we say that f, g share the set S with weight k and write it as f, g share (S,k).

Definition 1.5. *[1] Let $a, b \in \mathbb{C} \cup \{\infty\}$. We denote by $\overline{N}(r,a;f \mid g = b)$ the reduced counting function of those a-points of f, counted ignoring its multiplicity, which are b-points of g.*

We use $\#(S)$ to denote the cardinality of the set S.

In 2007, Steuding [p. 152, [16]] showed under certain restriction namely $a(1) = 1$, two L-functions become identical when they share a value $c \in \mathbb{C}$ (c,∞). Later Hu-Li [6] pointed out that when $c = 1$, Steuding's [16] result cease to hold.

We know L-functions possess meromorphic continuations, so it is natural to conjecture that an L-function may become identical with an arbitrary meromorphic function. But, in 2010, Li [9] showed by the following example that uniqueness result do not in general hold for an L-function and a meromorphic function.

Example 1.6. *For an entire function g, the functions ζ and ζe^g share $(0,\infty)$, but $\zeta \neq \zeta e^g$.*

So it is natural to investigate how many distinct complex values are sufficent to determine an L function. In this respect, Li [9] proved the following uniqueness result.

Theorem 1.7. *[9] Let f be a meromorphic function in $\mathbb{C}$ having finitely many poles and let a and b be any two distinct finite complex values. If f and a non constant L-function $\mathcal{L}$ share (a,∞) and $(b,0)$, then $f \equiv \mathcal{L}$.*

In 2011, Garunkstis-Grahl-Steuding [5] showed that, when the shared values are taken from $\mathbb{C} \cup \{\infty\}$, the condition"finitely many poles" can be dropped for the case of following 2 CM and 1 IM shared values result. The result in [5] is as follows:

Theorem 1.8. *[5] Let f be a meromorphic function in $\mathbb{C}$ and let a, b and c be three distinct values in $\mathbb{C}$. If f shares (a,∞), (b,∞) and $(c,0)$ with a non-constant L-function $\mathcal{L}$, then $\mathcal{L} \equiv f$.*

For three IM shared values, Li-Yi [10] obtained the following theorem.

Theorem 1.9. *[10] Let f be a transcendental meromorphic function in C having finitely many poles in $\mathbb{C}$, and let b_1, b_2, b_3 be three distinct finite complex values. If f shares $(b_1,0)$, $(b_2,0)$, $(b_3,0)$ with a non-constant L-function $\mathcal{L}$ then $\mathcal{L} \equiv f$.*

In 2016, Lin-Lin [11] considered the set sharing problem instead of value sharing and established the following theorem.

Theorem 1.10. *[11] Let f be a meromorphic function in $\mathbb{C}$ with finitely many poles, $S_1, S_2 \subset \mathbb{C}$ be two distinct sets such that $S_1 \cap S_2 = \phi$ and $\#(S_i) \leq 2$, $i = 1, 2$. Suppose for a finite set $S = \{\alpha_i \mid i = 1, 2, \ldots, n\}$, $C(S)$ is defined by $C(S) = \frac{1}{n}\sum_{i=1}^{n} \alpha_i$. If f and a non-constant L-function $\mathcal{L}$ share (S_1,∞) and $(S_2,0)$, then (i) $\mathcal{L} = f$ when $C(S_1) \neq C(S_2)$ and (ii) $\mathcal{L} = f$ or $\mathcal{L} + f = 2C(S_1)$ when $C(S_1) = C(S_2)$.*

In [11], the authors asked the following question:

Question 1.1 (see Q.1.17, [11]). *What can be said about the conclusions of Theorem 1.10 if max $\{\#(S_1), \#(S_2)\} \geq 3$?*

In the mean time, to prove a uniqueness relation for a special class of meromorphic functions, Chen [3] first resorted to the following condition

$$(\beta_1 - \alpha_1)^2(\beta_1 - \alpha_2)^2 \ldots (\beta_1 - \alpha_m)^2 \neq (\beta_2 - \alpha_1)^2(\beta_2 - \alpha_2)^2 \ldots (\beta_2 - \alpha_m)^2, \tag{1.1}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_m, \beta_1, \beta_2$ are $m + 2$ distinct complex numbers.

In order to provide an answer to *Question 1.1*, utilizing (1.1), Sahoo-Halder [13], proved two theorems, among which we recall their second one.

Theorem 1.11. *[13] Let f be a meromorphic function in $\mathbb{C}$ with finitely many poles and $m(\geq 3)$ be a positive integer. Suppose that $S_1 = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$, $S_2 = \{\beta_1, \beta_2\}$ be two subsets of $\mathbb{C}$ such that $S_1 \cap S_2 = \phi$ and (1.1) holds. If f and a non-constant L-function $\mathcal{L}$ share $(S_1, 0)$ and (S_2, ∞), then $\mathcal{L} = f$.*

Very recently, the present authors [12] have pointed out a major gap in the proof of *Theorem 1.11* and proved the corrected form of the same theorem with some restrictions on the set S_1. To state our result we require the following definition: Let $P(z)$ be defined as

$$\begin{aligned} P(z) &= \frac{(z-\alpha_1)(z-\alpha_2)\ldots(z-\alpha_m) - (-1)^m \alpha_1\alpha_2\ldots\alpha_m}{(-1)^{m+1}\alpha_1\alpha_2\ldots\alpha_m} \\ &= \frac{z^m - (\sum \alpha_i) z^{m-1} + \ldots + (-1)^{m-1}(\sum \alpha_{i_1}\alpha_{i_2}\ldots\alpha_{i_{m-1}})z}{(-1)^{m+1}\alpha_1\alpha_2\ldots\alpha_m} \end{aligned}$$

where $\alpha_i(\neq 0) \in S_1$ for $i = 1, 2, \ldots, m$. Let m_1 and m_2 denote respectively the number of simple and multiple zeros of $P(z)$.

In [12] we have circumstantially inspect **Question 1.1** and observe that it is actually combination of four questions. To solve (**ii**) (Remark 1.2, p. 5, [12]) and in order to rectify *Theorem 1.11* we proved the following theorem.

Theorem 1.12. *[12] Under the same situation as in* Theorem 1.11, *if $\alpha_i \neq 0; i = 1, \ldots, m$, and $m > 2m_1 + 4m_2 + 3$ with $m_1 + m_2 > 1$, then $\mathcal{L} \equiv f$.*

Note 1.1. *Recently, Sahoo-Halder claimed in [14] that they have pointed out the errors of Theorem 1.11 [13] and rectified the same. But if anyone observe minutely the paper [14], it will be revealed that long before the date of submission of the article [14] in the journal, the paper [12] was appeared. So by no means the authors can claim that they have pointed and rectified the errors in Theorem 1.11. In fact, Theorem 1.2 in [12] is a better result than that of Theorem 2 (i) in [14] as far as lower bound of n is concerned. Only the authors of [14] could claim that the result Theorem 2 (ii) is the original one and we wonder, whether in that case, there would have been any significance of the correction [14]. Not only that the Remark 2 of [14] has somehow been motivated from Example 1.2 of [12]. So these are nothing but suppression of facts. From the academic point of view, these types of unfortunate incidents are not at all desirable.*

Recently, with the aid of an extra supposition, under almost the same situations as in *Theorem 1.11*, Chen-Qiu [4] proved the following theorem.

Theorem 1.13. *[4] Fix a positive integer m and take two subsets $S_1 = \{\alpha_1, \alpha_2, \ldots, \alpha_m\} \subseteq \mathbb{C}$, $S_2 = \{\beta_1, \beta_2\} \subseteq \mathbb{C}$, such that $S_1 \cap S_2 = \phi$. Let $\mathcal{L}_1 = (\mathcal{L} - \alpha_1)(\mathcal{L} - \alpha_2)\ldots(\mathcal{L} - \alpha_m)$ and $f_1 = (f - \alpha_1)(f - \alpha_2)\ldots(f - \alpha_m)$ and $c_1 = (\beta_1 - \alpha_1)(\beta_1 - \alpha_2)\ldots(\beta_1 - \alpha_m)$, $c_2 = (\beta_2 - \alpha_1)(\beta_2 - \alpha_2)\ldots(\beta_2 - \alpha_m)$ and $c_1^2 \neq c_2^2$. If a non-constant L-function $\mathcal{L}$ and a meromorphic function f with finite number of poles share $(S_1, 0)$, (S_2, ∞) and $\mathcal{L}_1$, f_1 share (S', ∞) where $S' = \{c_1, c_2\}$, then $f \equiv \mathcal{L}$.*

Remark 1.1. *A close inspection into the statement of* Theorem 1.13, *will reveal that inclusion of the extra condition over the sharing of the set $\{c_1, c_2\}$ actully make the suppositions of the same theorem more complicated and convert it a problem of three set sharing under certian constraints. On the otherhand, though* Theorem 1.12 *is a new as well as corrected version of* Theorem 1.11, *but we must have $\#(S_1) \geq 10$. So it is very much desirable to re-investigate the corrected form* Theorem 1.11 *keeping the cardinalities of the sets S_i i = 1, 2, intact and at the same time, not alternating the sharing hypothesis on the two range sets.*

One of the purposes of writing this paper is to resolve the issue addressed in *Remark 1.1*. In fact, in our first theorem, we will show that under the same hypothesis, the corrected form of *Theorem 1.11* is acheviable, if the condition (1.1) is relpaced by a another one. As in *Question 1.1* no restrictions was there for the sufficient conditions, our following result gives the best possible answer of the same question, improving both *Theorems 1.12* and *1.13*.

Theorem 1.14. *Let f be a non-constant meromorphic function with finite number of poles, and $\mathcal{L}$ be a non-constant L-function. Also consider $S_1 = \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ $(k \geq 2)$ and $S_2 = \{\beta_1, \beta_2\}$, where $\alpha_1, \ldots, \alpha_k, \beta_1, \beta_2$ are $k+2$ distinct finite complex numbers satisfying $\beta_1 + \beta_2 \neq \alpha_i + \alpha_j$, for $1 \leq i, j \leq k$. If f and $\mathcal{L}$ share $(S_1, 0)$ and (S_2, ∞) then $f \equiv \mathcal{L}$.*

The following examples show the sharpness of the given condition in *Theorem 1.14*.

Example 1.15. *Let $S_1 = \{0, a, -a\}$ and $S_2 = \{1, -1\}$, where a is some finite complex number. Considering $f = -\zeta$ and $\mathcal{L} = \zeta$, it is clear that f and $\mathcal{L}$ share (S_i, ∞) for $i = 1, 2$. Here we can see $\beta_1 + \beta_2 = \alpha_i + \alpha_j$ for some $1 \leq i, j \leq 3$ and $\mathcal{L} \not\equiv f$.*

Example 1.16. *Let $S_1 = \{b, -b\}$ and $S_2 = \{i, -i\}$, where b is some finite complex number. Considering $f = -\zeta$ and $\mathcal{L} = \zeta$, it is clear that f and $\mathcal{L}$ share (S_i, ∞) for $i = 1, 2$. Here we can see $\beta_1 + \beta_2 = \alpha_i + \alpha_j$ for some $1 \leq i, j \leq 3$ and $\mathcal{L} \not\equiv f$.*

Next example shows that the condition f has finitely many poles can not be removed in *Theorem 1.14*.

Example 1.17. *Let $S_1 = \{1, \sqrt{2}+1, \sqrt{2}-1\}$ and $S_2 = \{i, -i\}$. Considering $f = \frac{1}{\zeta}$ and $\mathcal{L} = \zeta$, it is clear that f and $\mathcal{L}$ share (S_i, ∞) for $i = 1, 2$. Also $\beta_1 + \beta_2 \neq \alpha_i + \alpha_j$ for any choice of i and j, but $\mathcal{L} \not\equiv f$.*

In the main theorem of [11], Lin-Lin discussed four cases with $\#(S_i) \leq 2$ $(i = 1, 2)$, where S_1 is always being shared CM and S_2 is being shared IM and established a uniqueness relation between f and $\mathcal{L}$.

In [11], concerening IM sharing of both the sets Lin-Lin raised the following question

Question 1.2 (see Q.1.16 in [11]). *Can CM shared set S_1 be replaced by an IM shared set in Theorem 1.10?*

Remark 1.2. *If we consider Question 1.2 meticulously, we see that, it is practically the combinations of three parts as follows:*
Sharing of S_1 IM and S_2 IM,
i) with $\#(S_1) = 1$ and $\#(S_2) = 1$.
ii)with $\#(S_1) = 1$ and $\#(S_2) = 2$.
iii)with $\#(S_1) = 2$ and $\#(S_2) = 2$.

Taking into account *Question 1.2*, in the present paper, we have been able to settle the issue for $\min\{\#(S_i)\} = 1$ $(i = 1, 2)$ i.e., the case (i), (ii). But the remainig case (iii) i.e., for $\min\{\#(S_i)\} = 2$ $(i = 1, 2)$ is still unsolved and require further investigations. In the next theorems, we will show that under some supposition the problem of sharing two sets IM can be resolved.

For the sake of next two theorems, we require the following notations. Let a meromorphic function f and an L function $\mathcal{L}$ share $S_1 = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ and $S_2 = \{\beta_1, \beta_2, \ldots, \beta_m\}$ IM, by $\overline{N}_0(r, 0; f')$ $(\overline{N}_0(r, 0; \mathcal{L}'))$ we mean the reduce counting function of those zeros of f' $(\mathcal{L}')$ where $f \neq \alpha_1, , \alpha_2, \ldots, \alpha_n, \beta_1, \beta_2, \ldots, \beta_m$ $(\mathcal{L} \neq \alpha_1, , \alpha_2, \ldots, \alpha_n, \beta_1, \beta_2, \ldots, \beta_m)$.

Theorem 1.18. *Let f be a non-constant meromorphic function with finite number of poles and $\mathcal{L}$ be a non-constant L-function. Also let $S_1 = \{\alpha, \beta\}$ and $S_2 = \{\gamma\}$, where α, β and γ be 3 distinct finite complex numbers and $2\gamma \neq \alpha + \beta$. If f and $\mathcal{L}$ share $(S_i, 0)$ for $i = 1, 2$ and $\overline{N}_0(r, 0; f') = \overline{N}_0(r, 0; \mathcal{L}') = O(\log r)$, then $f \equiv \mathcal{L}$.*

Theorem 1.19. *Let f be a non-constant meromorphic function with finite number of poles and $\mathcal{L}$ be a non-constant L-function. Also let $S_1 = \{\alpha\}$ and $S_2 = \{\beta\}$, where α, β be 2 distinct finite complex numbers. If f and $\mathcal{L}$ share $(S_i, 0)$ for $i = 1, 2$ and $\overline{N}_0(r, 0; f') = \overline{N}_0(r, 0; \mathcal{L}') = O(\log r)$, then $f \equiv \mathcal{L}$.*

Our next example shows that *Theorem 1.18* cease to hold when L-function is replaced by meromorphic function with finite number of poles.

Example 1.20. *Let $S_1 = \{1\}$ and $S_2 = \{-1\}$. Clearly $f = e^z$ and $g = e^{-z}$ share S_i $(i = 1, 2)$ and $\overline{N}_0(r, 0; f') = \overline{N}_0(r, 0; g') = O(\log r)$ but $f \neq g$.*

Example 1.21. *$f = e^z$ and $g = e^{-z}$ share $\{1\}$ and $\{i, -i\}$ CM also $i - i \neq 2$ and $\overline{N}_0(r, 0; f') = S(r, f)$, $\overline{N}_0(r, 0; g') = S(r, g)$ but $f \neq g$.*

Remark 1.3. *Now from the bove two theorems one question is obvious i.e., " is it possible to reduce the CM sharing in Theorem 1.10 with out asuuming $\overline{N}_0(r, 0; f') = O(\log r) = \overline{N}_0(r, 0; \mathcal{L}')$?". Also if it is possible then " what will be the least weight of sharing ?".*

With regarding this issue, in the next theorem we have tried to relaxe the CM sharing to weighted sharing in *Theorem 1.10*.

Theorem 1.22. *Let f be a meromorphic function in $\mathbb{C}$ with finitely many poles, $\mathcal{L}$ be a non-constant L-function and $S_1, S_2 \subset \mathbb{C}$, $C(S)$ be defined as in Theorem 1.10. If f and a non-constant L-function $\mathcal{L}$ share (S_1, m_1) and (S_2, m_2) where $m_1.m_2 \geq 2$ then (i) $\mathcal{L} = f$ when $C(S_1) \neq C(S_2)$ and (ii) $\mathcal{L} = f$ or $\mathcal{L} + f = 2C(S_1)$ when $C(S_1) = C(S_2)$.*

2. Lemma

Lemma 2.1. *Let f be a meromorphic function having finitely many poles in $\mathbb{C}$ and S_1, S_2 be defined as in Theorem 1.14. If f and a non constant L-function $\mathcal{L}$ share the set S_1, S_2 with weight 0, then $\rho(f) = \rho(\mathcal{L}) = 1$.*

Proof. We omit the proof as the same can be found out in the proof of Lemma 3, in [13]. □

Lemma 2.2. *[11] If $\mathcal{L}$ is a non-constant L-function, then there is no generalized Picard exceptional value of $\mathcal{L}$ in the complex plane.*

Let us consider f, g be two non-constant meromorphic functions and 'α_1' be a value in the extended complex plane shared IM both by f and g. By z_0, a (m, n) fold 'α_1' point of f (g) we mean at z_0, $f - \alpha_1$ ($g - \alpha_1$) has a zero of order m whereas, $g - \alpha_1$ ($f - \alpha_1$) has zero of order n. We denote by $\overline{N}_{(m,n)}(r, \alpha_1; f)$ ($\overline{N}_{(m,n)}(r, \alpha_1; g)$), the reduced counting function of all (m, n) fold $\{\alpha_1\}$-points of f (g). Clearly here $\overline{N}_{(m,n)}(r, \alpha_1; f) = \overline{N}_{(n,m)}(r, \alpha_1; g)$.

Next let f and g share $\{\alpha_2, \alpha_3\}$ IM and z_0 be a (p, q) fold α_2 point of f. Now a 'α_2' point of f is either a 'α_2' or a 'α_3' point of g. Here by z_0 a (p, q) fold 'α_2' point of f we want to mean that, at z_0 $f - \alpha_2$ has a zero of order p and at z_0, $g - \alpha_2$ or $g - \alpha_3$ has zero of order q, i.e, $(f - \alpha_2) = (z - z_0)^p \phi(z)$ and $(g - \alpha_2)(g - \alpha_3) = (z - z_0)^q \psi(z)$ for some function ϕ and ψ ($\phi(z_0) \neq 0$, $\psi(z_0) \neq 0$). Also by $\overline{N}_{(p,q)}(r, \alpha_2; f)$ we denote the reduce counting function for all (p, q) -α_2 point of f. Clearly here $\overline{N}_{(m,n)}(r, \alpha_2; f) + \overline{N}_{(m,n)}(r, \alpha_3; f) = \overline{N}_{(n,m)}(r, \alpha_2; g) + \overline{N}_{(n,m)}(r, \alpha_3; g)$.

By $n_\alpha(t, f)$ ($\overline{n}_\alpha(t, f)$) we count the poles of f in $|z| \leq t$, which appear due to 'α' points and counted according to it's multiplicity (ignoring multiplicity).

Also by $\overset{3}{\overline{N}}_{(m,n)}(r, \alpha_j; f)$ we denote the reduced counting function for all those (m, n) fold 'α_j' points of f where $\min\{m, n\} \geq 3$.

Lemma 2.3. *Let f and g be two distinct nonconstant meromorphic functions with finite number of poles and let α_1, α_2 be two distinct finite complex values. If f and g share $(\alpha_1, 0)$, $(\alpha_2, 0)$, then*

$$\begin{aligned}\frac{1}{19}T(r, f) \leq\ & 2\sum_{j=1}^{2}\sum_{k=1}^{18}\overline{N}_{(1,k)}(r, \alpha_j; f) + 2\sum_{j=1}^{2}\sum_{k=2}^{18}\overline{N}_{(k,1)}(r, \alpha_j; f) + \sum_{j=1}^{2}\sum_{l=2}^{18}\overline{N}_{(2,l)}(r, \alpha_j; f) \\ & + \sum_{j=1}^{2}\sum_{l=3}^{18}\overline{N}_{(l,2)}(r, \alpha_j; f) + S(r, f) + S(r, g) + O(\log r).\end{aligned}$$

as $r \longrightarrow \infty$, outside of a possible exceptional set of finite linear measure.

Proof. Let us consider the auxilary function

$$\Phi = \frac{f'g'(f-g)}{(f-\alpha_1)(f-\alpha_2)(g-\alpha_1)(g-\alpha_2)}. \tag{2.1}$$

Clearly $\Phi \not\equiv 0$. From the lemma of logarithmic derivative we will get $m(r,\Phi) = S(r,f) + S(r,g)$.

Also the poles of Φ come from the poles of f and g which are finitely many and from the $(k,1)$, $(1,k)$ folds α_j, $j = 1,2$ points of f and g where $k \geq 1$.

Now, since $n_{\alpha_j}(t,\Phi) = \overline{n}_{\alpha_j}(t,\Phi)$, we have

$$\begin{aligned}\sum_{j=1}^{2}\overline{N}^{3}_{(m,n)}(r,\alpha_j;f) &\leq \overline{N}(r,0;\Phi) \leq T(r,\Phi)+O(1) \leq N(r,\infty;\Phi)+S(r,f)+S(r,g)\\ &\leq \sum_{j=1}^{2}\sum_{k=1}^{\infty}\overline{N}_{(1,k)}(r,\alpha_j;f)+\sum_{j=1}^{2}\sum_{k=2}^{\infty}\overline{N}_{(k,1)}(r,\alpha_j;f)+O(\log r)+S(r,f)+S(r,g). \end{aligned}\tag{2.2}$$

We note that

$$\begin{aligned}\sum_{j=1}^{2}\overline{N}(r,\alpha_j;f) &\leq \sum_{j=1}^{2}\sum_{k=1}^{\infty}\overline{N}_{(1,k)}(r,\alpha_j;f)+\sum_{j=1}^{2}\sum_{k=2}^{\infty}\overline{N}_{(k,1)}(r,\alpha_j;f)+\sum_{j=1}^{2}\sum_{l=2}^{\infty}\overline{N}_{(2,l)}(r,\alpha_j;f)\\ &\quad+\sum_{j=1}^{2}\sum_{l=3}^{\infty}\overline{N}_{(l,2)}(r,\alpha_j;f)+\sum_{j=1}^{2}\sum_{p\geq 3,q\geq 3}^{\infty}\overline{N}_{(p,q)}(r,\alpha_j;f).\end{aligned}$$

So using (2.2) we get

$$\begin{aligned}\sum_{j=1}^{2}\overline{N}(r,\alpha_j;f) &\leq 2\left(\sum_{j=1}^{2}\sum_{k=1}^{\infty}\overline{N}_{(1,k)}(r,\alpha_j;f)+\sum_{j=1}^{2}\sum_{k=2}^{\infty}\overline{N}_{(k,1)}(r,\alpha_j;f)\right)+\sum_{j=1}^{2}\sum_{l=2}^{18}\overline{N}_{(2,l)}(r,\alpha_j;f)\\ &\quad+\sum_{j=1}^{2}\sum_{l=3}^{18}\overline{N}_{(l,2)}(r,\alpha_j;f)+\sum_{j=1}^{2}\sum_{l=19}^{\infty}\overline{N}_{(l,2)}(r,\alpha_j;f)+\sum_{j=1}^{2}\sum_{l=19}^{\infty}\overline{N}_{(2,l)}(r,\alpha_j;f)+O(\log r)\\ &\quad+S(r,f)+S(r,g)\\ &\leq 2\left(\sum_{j=1}^{2}\sum_{k=1}^{\infty}\overline{N}_{(1,k)}(r,\alpha_j;f)+\sum_{j=1}^{2}\sum_{k=2}^{\infty}\overline{N}_{(k,1)}(r,\alpha_j;f)\right)+\sum_{j=1}^{2}\sum_{l=2}^{18}\overline{N}_{(2,l)}(r,\alpha_j;f)\\ &\quad+\sum_{j=1}^{2}\sum_{l=3}^{18}\overline{N}_{(l,2)}(r,\alpha_j;f)+\sum_{j=1}^{2}\overline{N}(r,\alpha_j;f\mid\geq 19)+\sum_{j=1}^{2}\overline{N}(r,\alpha_j;g\mid\geq 19)+O(\log r)\\ &\quad+S(r,f)+S(r,g).\end{aligned}$$

i.e.,

$$\begin{aligned}\sum_{j=1}^{2}\overline{N}(r,\alpha_j;f) &\leq 2\left(\sum_{j=1}^{2}\sum_{k=1}^{\infty}\overline{N}_{(1,k)}(r,\alpha_j;f)+\sum_{j=1}^{2}\sum_{k=2}^{\infty}\overline{N}_{(k,1)}(r,\alpha_j;f)\right)+\sum_{j=1}^{2}\sum_{l=2}^{18}\overline{N}_{(2,l)}(r,\alpha_j;f)\\ &\quad+\sum_{j=1}^{2}\sum_{l=3}^{18}\overline{N}_{(l,2)}(r,\alpha_j;f)+\frac{2}{19}T(r,f)+\frac{2}{19}T(r,g)+O(\log r)+S(r,f)+S(r,g)\end{aligned}$$

$$\leq 2\left(\sum_{j=1}^{2}\sum_{k=1}^{18}\overline{N}_{(1,k)}(r,\alpha_j;f)+\sum_{j=1}^{2}\sum_{k=2}^{18}\overline{N}_{(k,1)}(r,\alpha_j;f)\right)+\sum_{j=1}^{2}\sum_{l=2}^{18}\overline{N}_{(2,l)}(r,\alpha_j;f)$$
$$+\sum_{j=1}^{2}\sum_{l=3}^{18}\overline{N}_{(l,2)}(r,\alpha_j;f)+\frac{6}{19}T(r,f)+\frac{6}{19}T(r,g)+O(\log r)+S(r,f)+S(r,g). \tag{2.3}$$

Again,

$$\begin{aligned} T(r,g) &\leq \sum_{j=1}^{2}\overline{N}(r,\alpha_j;g)+\overline{N}(r,\infty;g)+S(r,g)\\ &\leq \sum_{j=1}^{2}\overline{N}(r,\alpha_j;f)+O(\log r)+S(r,g)\\ &\leq 2T(r,f)+O(\log r)+S(r,f)+S(r,g). \end{aligned}$$

Using the above equation in (2.3) and then by the Second Fundamental Theorem we have,

$$\begin{aligned} T(r,f) &\leq \sum_{j=1}^{2}\overline{N}(r,\alpha_j;f)+\overline{N}(r,\infty;f)+S(r,f)\\ &\leq 2\left(\sum_{j=1}^{2}\sum_{k=1}^{18}\overline{N}_{(1,k)}(r,\alpha_j;f)+\sum_{j=1}^{2}\sum_{k=2}^{18}\overline{N}_{(k,1)}(r,\alpha_j;f)\right)+\sum_{j=1}^{2}\sum_{l=2}^{18}\overline{N}_{(2,l)}(r,\alpha_j;f)\\ &\quad+\sum_{j=1}^{2}\sum_{l=3}^{18}\overline{N}_{(l,2)}(r,\alpha_j;f)+\frac{18}{19}T(r,f)+O(\log r)+S(r,f)+S(r,g), \end{aligned} \tag{2.4}$$

as $r \longrightarrow \infty$ out side the of a possible exceptional set of linear measure, which reveals the conclusion of this lemma.

□

Lemma 2.4. *Let f and g be two distinct nonconstant meromorphic functions with finite number of poles and let α_1, α_2 ,α_3 be three distinct finite complex values and $f+g\not\equiv\alpha_2+\alpha_3$. If f and g share $(\alpha_1,0)$, $(\{\alpha_2,\alpha_3\},0)$, then*

$$\begin{aligned} \frac{1}{32}T(r,f) &\leq 2\sum_{j=1}^{3}\sum_{k=1}^{15}\overline{N}_{(1,k)}(r,\alpha_j;f)+2\sum_{j=1}^{3}\sum_{k=2}^{15}\overline{N}_{(k,1)}(r,\alpha_j;f)+\sum_{j=1}^{3}\sum_{l=2}^{15}\overline{N}_{(2,l)}(r,\alpha_j;f)\\ &\quad+\sum_{j=1}^{3}\sum_{l=3}^{15}\overline{N}_{(l,2)}(r,\alpha_j;f)+S(r,f)+S(r,g)+O(\log r). \end{aligned}$$

as $r \longrightarrow \infty$, outside of a possible exceptional set of finite linear measure.

Proof. Let us consider the auxilary function

$$\Phi_o=\frac{f'g'(f-g)(f+g-\alpha_2-\alpha_3)}{(f-\alpha_1)(f-\alpha_2)(f-\alpha_3)(g-\alpha_1)(g-\alpha_2)(g-\alpha_3)}. \tag{2.5}$$

Clearly $\Phi_o\neq 0$. From the lemma of logarithmic derivative we will get $m(r,\Phi_o)=S(r,f)+S(r,g)$. Also the poles of Φ_o come from the poles of f and g which are finitely many and from the $(k,1)$, $(1,k)$ folds α_j, $j=1,2,3$ points of f and g where $k\geq 1$.

Since $n_{\alpha_j}(t,\Phi_o)=\overline{n}_{\alpha_j}(t,\Phi_o)$,

$$\begin{aligned}\sum_{j=1}^{3}\overline{N}^{3}_{(m,n)}(r,\alpha_j;f) &\leq \overline{N}(r,0;\Phi_o)\leq T(r,\Phi_o)+O(1)\\ &\leq N(r,\infty;\Phi_o)+S(r,f)+S(r,g)\\ &\leq \sum_{j=1}^{3}\sum_{k=1}^{\infty}\overline{N}_{(1,k)}(r,\alpha_j;f)+\sum_{j=1}^{3}\sum_{k=2}^{\infty}\overline{N}_{(k,1)}(r,\alpha_j;f)+O(\log r)+S(r,f)+S(r,g). \quad (2.6)\end{aligned}$$

Now using (2.6),

$$\begin{aligned}&\sum_{j=1}^{3}\overline{N}(r,\alpha_j;f)\\ \leq\ & \sum_{j=1}^{3}\sum_{k=1}^{\infty}\overline{N}_{(1,k)}(r,\alpha_j;f)+\sum_{j=1}^{3}\sum_{k=2}^{\infty}\overline{N}_{(k,1)}(r,\alpha_j;f)+\sum_{j=1}^{3}\sum_{l=2}^{\infty}\overline{N}_{(2,l)}(r,\alpha_j;f)\\ &+\sum_{j=1}^{3}\sum_{l=3}^{\infty}\overline{N}_{(l,2)}(r,\alpha_j;f)+\sum_{j=1}^{3}\sum_{p\geq 3,q\geq 3}^{\infty}\overline{N}_{(p,q)}(r,\alpha_j;f)\\ \leq\ & 2\left(\sum_{j=1}^{3}\sum_{k=1}^{\infty}\overline{N}_{(1,k)}(r,\alpha_j;f)+\sum_{j=1}^{3}\sum_{k=2}^{\infty}\overline{N}_{(k,1)}(r,\alpha_j;f)\right)+\sum_{j=1}^{3}\sum_{l=2}^{15}\overline{N}_{(2,l)}(r,\alpha_j;f)\\ &+\sum_{j=1}^{3}\sum_{l=3}^{15}\overline{N}_{(l,2)}(r,\alpha_j;f)+\sum_{j=1}^{3}\sum_{l=16}^{\infty}\overline{N}_{(2,l)}(r,\alpha_j;f)+\sum_{j=1}^{3}\sum_{l=16}^{\infty}\overline{N}_{(l,2)}(r,\alpha_j;f)+O(\log r)\\ &+S(r,f)+S(r,g). \quad (2.7)\end{aligned}$$

Since f and g share $\{\alpha_2,\alpha_3\}$ IM and note that for $i=2,3$,

$$\overline{N}_{(1,k)}(r,\alpha_i;f)\leq \overline{N}(r,\alpha_2;g\mid\geq k)+\overline{N}(r,\alpha_3;g\mid\geq k)\leq \frac{2}{k}T(r,g)+S(r,g);$$

from above we get

$$\begin{aligned}&\sum_{j=1}^{3}\overline{N}_3(r,\alpha_j;f)\\ \leq\ & 2\left(\sum_{j=1}^{3}\sum_{k=1}^{15}\overline{N}_{(1,k)}(r,\alpha_j;f)+\sum_{j=1}^{3}\sum_{k=2}^{15}\overline{N}_{(k,1)}(r,\alpha_j;f)\right)+\sum_{j=1}^{3}\sum_{l=2}^{15}\overline{N}_{(2,l)}(r,\alpha_j;f)\\ &+\sum_{j=1}^{3}\sum_{l=3}^{15}\overline{N}_{(l,2)}(r,\alpha_j;f)+\frac{9}{16}T(r,f)+\frac{15}{16}T(r,g)+O(\log r)+S(r,f)+S(r,g). \quad (2.8)\end{aligned}$$

Now

$$2T(r,g)\leq 3T(r,f)+O(\log r)+S(r,g) \quad (2.9)$$

Using the Second Fundamental Theorem we have,

$$2T(r,f)\ \leq\ \sum_{j=1}^{3}\overline{N}(r,\alpha_j;f)+\overline{N}(r,\infty;f)+S(r,f). \quad (2.10)$$

Now with the help of (2.8) an (2.9) from (2.10) we will get the conclusion of this lemma. □

Lemma 2.5. *(see [17], Theorem 1.14) Let $f(z)$, $g(z) \in M(\mathbb{C})$. If the order of f and g, $\rho(f)$ and $\rho(g)$ respectively. Then*

$$\rho(f.g) \leq \max\{\rho(f), \rho(g)\};$$
$$\rho(f+g) \leq \max\{\rho(f), \rho(g)\}.$$

3. Proofs of the theorems

Proof. [Proof of Theorem 1.14] Since f and $\mathcal{L}$ share S_1 and S_2 then from *Lemma 2.1*, we have $\rho(f) = \rho(\mathcal{L}) = 1$ and hence $S(r, f) = S(r, \mathcal{L}) = O(\log r)$. Since f has fintely many poles and $\mathcal{L}$ has only one pole at $z = 1$, therefore $\overline{N}(r, \infty; f) = \overline{N}(r, \infty; \mathcal{L}) = O(\log r)$.

Now let us first consider the following auxiliary function:

$$\mathcal{G} = \frac{\mathcal{U}(f-\beta_1)(f-\beta_2)}{(\mathcal{L}-\beta_1)(\mathcal{L}-\beta_2)}, \tag{3.1}$$

where $\mathcal{U}$ is a rational function such that $\mathcal{G}$ has neither a pole nor a zero in $\mathbb{C}$. It is evident that such a function $\mathcal{U}$ does exist since f has finitely many poles and only possible pole of $\mathcal{L}$ occurs at $z = 1$ and a possible zero or pole of $\mathcal{G}$ may only come from a pole of $\mathcal{L}$ or f, in view of the condition that f and $\mathcal{L}$ share the set (S_2, ∞). Since $\mathcal{G}$ is an entire function with no zero and no pole then we can write it

$$\mathcal{G} = \frac{\mathcal{U}(f-\beta_1)(f-\beta_2)}{(\mathcal{L}-\beta_1)(\mathcal{L}-\beta_2)} = e^{\phi}, \tag{3.2}$$

for some entire function ϕ, s.t $\rho(e^{\phi}) \leq 1$.

Next consider a function

$$\hat{\Phi} = \left(\frac{e^{\phi}}{\mathcal{U}} - 1\right) \prod_{i \neq j; i,j=1}^{k} \left(\frac{e^{\phi}}{\mathcal{U}} - \frac{(\alpha_j-\beta_1)(\alpha_j-\beta_2)}{(\alpha_i-\beta_1)(\alpha_i-\beta_2)}\right).$$

Now we claim that $\hat{\Phi} \equiv 0$, otherwise from the construction of $\hat{\Phi}$ we get

$$\begin{aligned} \sum_{i=1}^{k} \overline{N}(r, \alpha_i; f) = \sum_{i=1}^{k} \overline{N}(r, \alpha_i; \mathcal{L}) &\leq \overline{N}(r, 0; \hat{\Phi}) \\ &\leq T(r, \hat{\Phi}) + O(1) \\ &\leq O\left(T\left(r, \frac{e^{\phi}}{\mathcal{U}}\right)\right) \leq O(r). \end{aligned} \tag{3.3}$$

Using the Second Fundamental Theorem and (3.3) we have

$$(k-1)T(r, \mathcal{L}) \leq \sum_{i=1}^{k} \overline{N}(r, \alpha_i; \mathcal{L}) + \overline{N}(r, \infty; \mathcal{L}) + O(\log r) \leq O(r), \tag{3.4}$$

which implies the degree of $\mathcal{L}$ is zero and hence $\mathcal{L}$ is constant. Therefore our claim is proved.

Since we get $\hat{\Phi} \equiv 0$. Then atleast one of the factors will be identically zero. Let us consider the following two cases:

Case I. Suppose that for some i and j $(i \neq j)$ we have

$$\frac{(f-\beta_1)(f-\beta_2)}{(\mathcal{L}-\beta_1)(\mathcal{L}-\beta_2)} \equiv \frac{(\alpha_j-\beta_1)(\alpha_j-\beta_2)}{(\alpha_i-\beta_1)(\alpha_i-\beta_2)}. \tag{3.5}$$

Since $\mathcal{L}$ has no generelized picard exceptional value then we can find some z_0 s.t $\mathcal{L}(z_0) = \alpha_i$. Now let us assume $f(z_0) = \alpha_r$ for some α_r $(\neq \alpha_j)$. Then from (3.5) we have

$$\beta_1 + \beta_2 = \alpha_r + \alpha_j,$$

a contradiction. Hence we must have $f(z_0) = \alpha_j$ which implies $\overline{E}(\mathcal{L}, \alpha_i) \subseteq \overline{E}(f, \alpha_j)$. Proceeding similarly we can get $\overline{E}(f, \alpha_j) \subseteq \overline{E}(\mathcal{L}, \alpha_i)$ and hence we obtain $\overline{E}(\mathcal{L}, \alpha_i) = \overline{E}(f, \alpha_j)$, here $\overline{E}(f, \alpha_j)$ $(E(f, \alpha_j))$ is the collection of all α_j points of f and each point is counted ignoring (counting) it's multiplicity.

Now from (3.5) we get

$$\frac{(\alpha_i - \beta_1)(\alpha_i - \beta_2)(f - \beta_1)(f - \beta_2) - (\alpha_i - \beta_1)(\alpha_i - \beta_2)(\alpha_j - \beta_1)(\alpha_j - \beta_2)}{(\alpha_j - \beta_1)(\alpha_j - \beta_2)(\mathcal{L} - \beta_1)(\mathcal{L} - \beta_2) - (\alpha_i - \beta_1)(\alpha_i - \beta_2)(\alpha_j - \beta_1)(\alpha_j - \beta_2)} \equiv 1$$

$$\Longrightarrow \frac{(f - \beta_1)(f - \beta_2) - (\alpha_j - \beta_1)(\alpha_j - \beta_2)}{(\mathcal{L} - \beta_1)(\mathcal{L} - \beta_2) - (\alpha_i - \beta_1)(\alpha_i - \beta_2)} \equiv \frac{(\alpha_j - \beta_1)(\alpha_j - \beta_2)}{(\alpha_i - \beta_1)(\alpha_i - \beta_2)}$$

$$\Longrightarrow \frac{(f - \alpha_j)(f + \alpha_j - \beta_1 - \beta_2)}{(\mathcal{L} - \alpha_i)(\mathcal{L} + \alpha_i - \beta_1 - \beta_2)} \equiv \frac{(\alpha_j - \beta_1)(\alpha_j - \beta_2)}{(\alpha_i - \beta_1)(\alpha_i - \beta_2)}. \tag{3.6}$$

Since $\beta_1 + \beta_2 \neq \alpha_i + \alpha_j$ for any $1 \leq i, j \leq k$ and $\overline{E}(f, \alpha_j) = \overline{E}(\mathcal{L}, \alpha_i)$ then from (3.6) we obtain

$$E(f, \alpha_j) = E(\mathcal{L}, \alpha_i). \tag{3.7}$$

Let us consider the following function

$$\mathcal{G}, = \frac{Q(f - \alpha_j)}{(\mathcal{L} - \alpha_i)},$$

where Q is a rational function such that $\mathcal{G},$ has neither a pole nor a zero in $\mathbb{C}$. It is evident that such a function Q does exist since $E(f, \alpha_j) = E(\mathcal{L}, \alpha_i)$ and f and $\mathcal{L}$ has finitely many poles. Therefore $\mathcal{G},$ is a zero free entire function, we can write it

$$\mathcal{G}, = \frac{Q(f - \alpha_j)}{(\mathcal{L} - \alpha_i)} = e^{\chi}, \tag{3.8}$$

for some entire function χ.

Also from *Lemma 2.5* we have

$$\rho(e^{\chi}) \leq \rho(f) = \rho(\mathcal{L}) = 1,$$

hence χ is a polynomial of degree atmost one.

Again consider

$$\hat{\chi} = \prod_{l,s=1}^{2} \left(\frac{e^{\chi}}{Q} - \frac{\beta_l - \alpha_j}{\beta_s - \alpha_i} \right).$$

It is easy to verify for any $1 \leq i, j \leq k$, when $\beta_1 + \beta_2 \neq \alpha_i + \alpha_j$, then $\hat{\chi} \not\equiv 0$.

Clearly from (3.8) we have

$$\sum_{i=1}^{2} \overline{N}(r, \beta_i; f) = \sum_{i=1}^{2} \overline{N}(r, \beta_i; \mathcal{L}) \leq \overline{N}(r, 0; \hat{\chi}) \leq T(r, \hat{\chi}) + O(1) \leq O(r). \tag{3.9}$$

Using the Second Fundamental Theorem and (3.9) we have

$$T(r, \mathcal{L}) \leq \sum_{i=1}^{2} \overline{N}(r, \beta_i; \mathcal{L}) + \overline{N}(r, \infty; \mathcal{L}) + O(\log r) \leq O(r), \tag{3.10}$$

a contradiction.

Therefore construction of such function $\mathcal{G},$ is not possible. As $\mathcal{L}$ has no generalized Picard exceptional value, we can say $E(f, \alpha_j) \neq E(\mathcal{L}, \alpha_i)$, which contradicts (3.7) and hence (3.6), (3.5) respectively.

Therefore (3.7) is not possible and hence our assumption is wrong. Thus **Case I**. is not valid.

Case II. When $i = j$, then (3.5) becomes

$$\frac{(f-\beta_1)(f-\beta_2)}{(\mathcal{L}-\beta_1)(\mathcal{L}-\beta_2)} \equiv 1. \tag{3.11}$$

Since $\beta_1 + \beta_2 \neq \alpha_i + \alpha_j$, for any $1 \leq i, j \leq k$ we get from (3.11) $f \equiv \mathcal{L}$. □

Proof. [Proof of Theorem 1.18] First suppose that $f \not\equiv \mathcal{L}$. It is given f and $\mathcal{L}$ share $\{\alpha,\ \beta\}$ and $\{\gamma\}$ IM, now using the Second Fundamental Theorem we have

$$\begin{aligned} 2T(r,f) &\leq \overline{N}(r,\gamma;f) + \overline{N}(r,\alpha;f) + \overline{N}(r,\beta;f) + \overline{N}(r,\infty;f) + S(r,f) \\ &\leq \overline{N}(r,\gamma;\mathcal{L}) + \overline{N}(r,\alpha;\mathcal{L}) + \overline{N}(r,\beta;\mathcal{L}) + O(\log r) + S(r,f) \\ &\leq 3T(r,\mathcal{L}) + O(\log r) + S(r,f), \end{aligned} \tag{3.12}$$

as $r \longrightarrow \infty$, outside of a possible exceptional set of finite linear measure.

By similar arguments we have

$$2T(r,\mathcal{L}) \leq 3T(r,f) + O(\log r) + S(r,\mathcal{L}). \tag{3.13}$$

Clearly from (3.12) and (3.13) we have $\rho(f) = \rho(\mathcal{L})$ $(= 1)$ and therefore we have $S(r,f) = S(r,\mathcal{L}) = O(\log r)$. Now let us consider the following function

$$\Psi_o = \frac{f''}{f'} - \frac{f'}{f-\gamma} - \frac{f'}{f-\alpha} - \frac{f'}{f-\beta} - \left(\frac{\mathcal{L}''}{\mathcal{L}'} - \frac{\mathcal{L}'}{\mathcal{L}-\gamma} - \frac{\mathcal{L}'}{\mathcal{L}-\alpha} - \frac{\mathcal{L}'}{\mathcal{L}-\beta}\right). \tag{3.14}$$

Since f and $\mathcal{L}$ share $(\gamma, 0)$, $(\{\alpha,\beta\}, 0)$ then clearly the only poles of Ψ_o come from the infinity points of f and $\mathcal{L}$ and from the zeros of f', $\mathcal{L}'$ where $f \neq \alpha,\beta,\gamma$ $(\mathcal{L} \neq \alpha,\beta,\gamma)$.

Also $S(r,f) = S(r,\mathcal{L}) = O(\log r)$, it is easy to verify that

$$m(r,\Psi_o) = O(\log r)$$

whereas from the given condition we have

$$\overline{N}(r,\infty;\Psi_o) = O(\log r). \tag{3.15}$$

Therefore Ψ_o is a rational function with finite number of poles.
Hence we can write it

$$\Psi_o = P(z) + \sum_{i=1}^{q} \frac{m_i}{z - z_i}, \tag{3.16}$$

where $P(z)$ is a polynomial and $m_1,\ m_2, \ldots, m_q$ are integers and $z_1,\ z_2, \ldots, z_q$ are zeros of $f',\ \mathcal{L}'$ where $f \neq \alpha,\ \beta,\ \gamma$ and poles of $f, \mathcal{L}$.

Integrating both sides of (3.16) and using (3.14) we get,

$$\frac{f'(\mathcal{L}-\gamma)(\mathcal{L}-\alpha)(\mathcal{L}-\beta)}{\mathcal{L}'(f-\gamma)(f-\alpha)(f-\beta)} = Q_o(z)e^{\int P(z)dz}, \tag{3.17}$$

where $Q_o(z) = c\prod_{i=1}^{q}(z-z_i)^{m_i}$, c is some constant.

Let us denote

$$\hat{\Psi}_o = \frac{f'(\mathcal{L}-\gamma)(\mathcal{L}-\alpha)(\mathcal{L}-\beta)}{\mathcal{L}'(f-\gamma)(f-\alpha)(f-\beta)} = Q_o.e^{\eta}, \tag{3.18}$$

where $\eta = \int P dz$.

Using *Lemma 2.5* we have

$$\rho(e^{\eta}) \leq \rho(f) = \rho(\mathcal{L}), \tag{3.19}$$

since $\rho(Q_o) = 0$. Therefore degree of η is atmost one.

Next we claim $\hat{\Psi}_o$ is a constant function. Otherwise, suppose that $\hat{\Psi}_o$ is non-constant. Then from *Lemma 2.4* we get

$$T(r, f) \leq O(Q_o.e^{\eta}) \leq O(r),$$

a contradiction. Hence our claim is proved.

Now from *Lemma 2.4* at least one of the following holds

$$\sum_{k=1}^{15} \overline{N}_{(k,1)}(r, \gamma; f) + \sum_{k=2}^{15} \overline{N}_{(1,k)}(r, \gamma; f) + \sum_{l=2}^{15} \overline{N}_{(l,2)}(r, \gamma; f) + \sum_{l=3}^{15} \overline{N}_{(2,l)}(r, \gamma; f) \neq S(r, f), \tag{3.20}$$

$$\sum_{k=1}^{15} \overline{N}_{(k,1)}(r, \alpha; f) + \sum_{k=2}^{15} \overline{N}_{(1,k)}(r, \alpha; f) + \sum_{l=2}^{15} \overline{N}_{(l,2)}(r, \alpha; f) + \sum_{l=3}^{15} \overline{N}_{(2,l)}(r, \alpha; f) \neq S(r, f), \tag{3.21}$$

and

$$\sum_{k=1}^{15} \overline{N}_{(k,1)}(r, \beta; f) + \sum_{k=2}^{15} \overline{N}_{(1,k)}(r, \beta; f) + \sum_{l=2}^{15} \overline{N}_{(l,2)}(r, \beta; f) + \sum_{l=3}^{15} \overline{N}_{(2,l)}(r, \beta; f) \neq S(r, f), \tag{3.22}$$

holds.

Now at first we consider (3.20) holds.

If $\overline{N}_{(1,1)}(r, \gamma; f) \neq S(r, f)$.

Then we will get $\hat{\Psi}_o = 1$, i.e.,

$$\frac{f'}{(f - \gamma)(f - \alpha)(f - \beta)} = \frac{\mathcal{L}'}{(\mathcal{L} - \gamma)(\mathcal{L} - \alpha)(\mathcal{L} - \beta)}, \tag{3.23}$$

clearly then f and $\mathcal{L}$ share $\{\gamma\}$ CM and $\{\alpha,\ \beta\}$ IM, hence using the given condition from *Theorem 1.10* we will get $f \equiv \mathcal{L}$.

Next suppose for some $2 \leq k \leq 15$, $N_{(k,1)}(r, \gamma; f) \neq S(r, f)$, then we will get $\hat{\Psi}_o = k$, i.e.,

$$\frac{f'}{(f - \gamma)(f - \alpha)(f - \beta)} = \frac{k\mathcal{L}'}{(\mathcal{L} - \gamma)(\mathcal{L} - \alpha)(\mathcal{L} - \beta)}.$$

In that case clearly every 'γ' point of f will be (kp, p) fold point, where p is a positive integer.

Therefore

$$\frac{f'}{f - \gamma} - \frac{k\mathcal{L}'}{\mathcal{L} - \gamma} = R + \sum_{i=1}^{m} \frac{a_i}{z - b_i}, \tag{3.24}$$

where b_i $(i = 1, 2, \ldots, m)$ are poles of f and $\mathcal{L}$ with multiplicity a_i.

Now integrating both side of (3.24) we have

$$\frac{f - \gamma}{(\mathcal{L} - \gamma)^k} = c \prod_{i=1}^{m} (z - b_i)^{a_i} e^{\int R} = cQ.e^{\kappa} \tag{3.25}$$

It is easy to verify κ is a polynomial of degree ≤ 1. Then from (3.25) using the First Fundamental Theorem we have

$$kT(r, \mathcal{L}) \leq T(r, f) + O(r), \tag{3.26}$$

for $k \geq 2$ we will get a contradiction.

Again if we consider $\overline{N}_{(3,2)}(r, \gamma; f) \neq S(r, f)$, then we get $\hat{\Psi}_o = \frac{3}{2}$, i.e.,

$$\frac{2f'}{(f - \gamma)(f - \alpha)(f - \beta)} = \frac{3\mathcal{L}'}{(\mathcal{L} - \gamma)(\mathcal{L} - \alpha)(\mathcal{L} - \beta)}, \tag{3.27}$$

from (3.27) clearly every 'γ' point of f is $(3p, 2p)$ fold point.

Now let us consider the following function

$$G = \frac{(f-\gamma)^2.Q_0}{(\mathcal{L}-\gamma)^3},$$

where Q_0 is a rational function such that G has no zero and no pole.

Hence we can write it as

$$G = \frac{(f-\gamma)^2.Q_0}{(\mathcal{L}-\gamma)^3} = e^\tau,$$

where τ is a polynomial of degree ≤ 1.

Now using the Second Fundamental Theorem we have

$$\begin{aligned} T(r,\mathcal{L}) &\leq \overline{N}(r,\alpha;\mathcal{L}) + \overline{N}(r,\beta;\mathcal{L}) + \overline{N}(r,\infty;\mathcal{L}) + O(\log r) \\ &\leq \overline{N}\left(r, \frac{1}{\alpha-\gamma}; \frac{e^\tau}{Q_0}\right) + \overline{N}\left(r, \frac{1}{\beta-\gamma}; \frac{e^\tau}{Q_0}\right) + \overline{N}\left(r, \frac{(\beta-\gamma)^2}{(\alpha-\gamma)^3}; \frac{e^\tau}{Q_0}\right) \\ &\quad + \overline{N}\left(r, \frac{(\alpha-\gamma)^2}{(\beta-\gamma)^3}; \frac{e^\tau}{Q_0}\right) + O(\log r) \leq O\left(T(r, \frac{e^\tau}{Q_0})\right) + O(\log r). \end{aligned} \tag{3.28}$$

Next we will show none of the following holds.

(i) $\frac{(f-\gamma)^2}{(\mathcal{L}-\gamma)^3} \equiv \frac{1}{\alpha-\gamma}$,

(ii) $\frac{(f-\gamma)^2}{(\mathcal{L}-\gamma)^3} \equiv \frac{1}{\beta-\gamma}$,

(iii) $\frac{(f-\gamma)^2}{(\mathcal{L}-\gamma)^3} \equiv \frac{(\alpha-\gamma)^2}{(\beta-\gamma)^3}$,

(iv) $\frac{(f-\gamma)^2}{(\mathcal{L}-\gamma)^3} \equiv \frac{(\beta-\gamma)^2}{(\alpha-\gamma)^3}$.

Let us assume (i) holds, i.e., $\frac{(f-\gamma)^2}{(\mathcal{L}-\gamma)^3} \equiv \frac{1}{\alpha-\gamma}$. Now it is easy to verify that $\overline{E}(\mathcal{L},\beta) \subseteq \overline{E}(f,\alpha)$ ($\overline{E}(\mathcal{L},\beta)$ is the collection of all β points of $\mathcal{L}$ counted exactly once, ignoring multiplicity) and from this we also have $(\beta-\gamma)^3 = (\alpha-\gamma)^3$. Now let $z_0 \in \overline{E}(\mathcal{L},\beta)$ and it is a (p,q) 'β' point of $\mathcal{L}$, i.e., $\mathcal{L}-\beta$ has zero of order p at z_0 and $f-\alpha$ has zero of order q at z_0. Then from (3.27) we get

$$\frac{2q}{3p} = -\frac{\alpha-\gamma}{\beta-\gamma}. \tag{3.29}$$

Since α, β are distinct and $2\gamma \neq \alpha+\beta$ also $\frac{2q}{3p} \in \mathbb{Q}$ then from $(\beta-\gamma)^3 = (\alpha-\gamma)^3$ and (3.27) we have a contradiction. Therefore (i) can not hold.

Proceeding similarly we can discard the case (ii).

Now if (iii) holds, i.e., $\frac{(f-\gamma)^2}{(\mathcal{L}-\gamma)^3} \equiv \frac{(\alpha-\gamma)^2}{(\beta-\gamma)^3}$, then it is easy to verify that $\overline{E}(\mathcal{L},\beta) \subseteq \overline{E}(f,\alpha)$. Now let at some point z_0, $f(z_0) = \alpha = \mathcal{L}(z_0)$, then from the given relation (iii) we get $(\beta-\gamma)^3 = (\alpha-\gamma)^3$. From $(\alpha-\gamma)^3 = (\beta-\gamma)^3$ we have the relation between α and β, i.e., $\frac{\beta-\gamma}{\alpha-\gamma} = 1$ *or* ω *or* ω^2. Again let us consider $z_1 \in \overline{E}(\mathcal{L},\beta) \subseteq \overline{E}(f,\alpha)$ such that $\mathcal{L}-\beta$ has zero of order p at z_1 and $f-\alpha$ has zero of order q at z_1. Then from (3.27) we get

$$\frac{2q}{3p} = -\frac{\alpha-\gamma}{\beta-\gamma}. \tag{3.30}$$

Since α, β are distinct and $\frac{2q}{3p}$ is a rational then from (3.30) we arrive at a contradiction.

Therefore $f(z_0) = \alpha \implies \mathcal{L}(z_0) = \beta \implies \overline{E}(f,\alpha) \subseteq \overline{E}(\mathcal{L},\beta)$ and hence $\overline{E}(f,\alpha) = \overline{E}(\mathcal{L},\beta)$. Immediately we have $\overline{E}(f,\beta) = \overline{E}(\mathcal{L},\alpha)$. Now let $z_1' \in \overline{E}(\mathcal{L},\alpha)$ such that $\mathcal{L}-\alpha$ has zero at z_1' of order p and $f-\beta$ has zero at z_1' of order q then from the relation (iii) we get $(\beta-\gamma)^5 = (\alpha-\gamma)^5$ and from (3.27) we get

$$\frac{2q}{3p} = -\frac{\beta-\gamma}{\alpha-\gamma}$$

since $\frac{2q}{3p} \in \mathbb{Q}$ and $\alpha,\ \beta$ are distinct also we get $(\beta-\gamma)^5 = (\alpha-\gamma)^5$ which gives a contradiction. Therefore $\overline{E}(\mathcal{L},\beta) = empty\ set$ again a contradiction.
Proceeding similarly we can discard the case (iv).

Then from (3.28) we get

$$T(r,\mathcal{L}) \leq O(r),$$

a contradiction.

Again if we consider $\overline{N}_{(l,2)}(r,\gamma;f) \neq S(r,f)$ (*or* $\overline{N}_{(2,l)}(r,\gamma;f) \neq S(r,f)$) for $l = 2,4,5,\ldots 15$, then proceeding same as done in above we will get contradiction.

Now let us consider (3.21) holds and $\overline{N}_{(k,1)}(r,\alpha;f) \neq S(r,f)$. Now let $\overline{N}_{(1,1)}(r,\alpha;f) \neq S(r,f)$ then two cases can occur $(i_a)\overline{N}(r,\alpha;f \mid \mathcal{L}=\alpha) \neq S(r,f)$, $(i_b)\overline{N}(r,\alpha;f \mid \mathcal{L}=\beta) \neq S(r,f)$.

If (i_a) holds then we get $\hat{\Psi}_o \equiv 1$ and if (i_b) holds then we get $\hat{\Psi}_o \equiv -\frac{(\beta-\gamma)}{(\alpha-\gamma)}$. Clearly both (i_a) and (i_b) can not hold together.

Now if $\hat{\Psi}_o \equiv 1$ then we get f and $\mathcal{L}$ share $\{\gamma\}$ CM and $\{\alpha,\ \beta\}$ then from *Theorem 1.10* we will have $f \equiv \mathcal{L}$.

Again if (i_b) holds then we will get

$$\frac{f'(\mathcal{L}-\gamma)(\mathcal{L}-\alpha)(\mathcal{L}-\beta)}{\mathcal{L}'(f-\gamma)(f-\alpha)(f-\beta)} \equiv -\frac{(\beta-\gamma)}{(\alpha-\gamma)}. \tag{3.31}$$

Now let z_0 be a (p,q) 'γ' point of f. Then from (3.31) we get $\frac{p}{q} = -\frac{(\beta-\gamma)}{(\alpha-\gamma)}$. Now if $-\frac{(\beta-\gamma)}{(\alpha-\gamma)}$ is negetive or not rational then clearly we get a contradiction. Now if $-\frac{(\beta-\gamma)}{(\alpha-\gamma)}$ is a non-negetive rational then let $\frac{p}{q} = -\frac{(\beta-\gamma)}{(\alpha-\gamma)} = \frac{n}{m}$, where $sn = -(\beta-\gamma)$, $sm = (\alpha-\gamma)$, $gcd(n,m) = 1$ and s is a positive integer. Clearly here $\frac{n}{m} \neq 1$. So according to our constructions z_0 is a 'γ' point of f of (ns,ms) type.

Next let us consider the following function

$$G_1 = \frac{Q_1.(f-\gamma)^m}{(\mathcal{L}-\gamma)^n} = e^{\tau_1}, \tag{3.32}$$

where τ_1 is a polynomial of degree ≤ 1 and Q_1 is a rational function. Then from (3.32) using the First Fundamental theorem we have when $\frac{n}{m} > \frac{3}{2}$ or $\frac{n}{m} < \frac{2}{3}$, we get a contradiction.

Now let us consider the case $2/3 \leq n/m \leq 3/2$. Clearly $n \neq 1$ and $m \neq 1$. Again using the Second Fundamental Theorem we get from (3.32)

$$\begin{aligned}
& T(r,\mathcal{L}) \\
\leq\ & \overline{N}(r,\alpha;\mathcal{L}) + \overline{N}(r,\beta;\mathcal{L}) + \overline{N}(r,\infty;\mathcal{L}) + O(\log r) \\
\leq\ & \overline{N}\left(r,0;\frac{e^{\tau_1}}{Q_1} - \frac{1}{(\alpha-\gamma)^{n-m}}\right) + \overline{N}\left(r,0;\frac{e^{\tau_1}}{Q_1} - \frac{1}{(\beta-\gamma)^{n-m}}\right) + \overline{N}\left(r,0;\frac{e^{\tau_1}}{Q_1} - \frac{(\beta-\gamma)^m}{(\alpha-\gamma)^n}\right) \\
& + \overline{N}\left(r,0;\frac{e^{\tau_1}}{Q_1} - \frac{(\alpha-\gamma)^m}{(\beta-\gamma)^n}\right) + O(\log r).
\end{aligned} \tag{3.33}$$

Now we will show that none of the following holds:

$(i_{b_1})\frac{(f-\gamma)^m}{(\mathcal{L}-\gamma)^n} \equiv \frac{1}{(\alpha-\gamma)^{n-m}}$,
$(i_{b_2})\frac{(f-\gamma)^m}{(\mathcal{L}-\gamma)^n} \equiv \frac{1}{(\beta-\gamma)^{n-m}}$,
$(i_{b_3})\frac{(f-\gamma)^m}{(\mathcal{L}-\gamma)^n} \equiv \frac{(\beta-\gamma)^m}{(\alpha-\gamma)^n}$,
$(i_{b_4})\frac{(f-\gamma)^m}{(\mathcal{L}-\gamma)^n} \equiv \frac{(\alpha-\gamma)^m}{(\beta-\gamma)^n}$.

Let us assume (i_{b_1}) holds, i.e., $\frac{(f-\gamma)^m}{(\mathcal{L}-\gamma)^n} \equiv \frac{1}{(\alpha-\gamma)^{n-m}}$. Let for some z_0, $\mathcal{L}(z_0) = \alpha$ and $f(z_0) = \beta$, then we get from (i_{b_1}), $(\beta-\gamma)^m = (\alpha-\gamma)^m \implies (-n)^m = m^m \implies m \mid n$, a contradiction. Hence in this case $\mathcal{L}(z_0) = \alpha \implies f(z_0) = \alpha$. Therefore $\overline{E}(\mathcal{L},\alpha) \subseteq \overline{E}(f,\alpha)$.

Also it is easy to verify that $\overline{E}(\mathcal{L},\alpha) \supseteq \overline{E}(f,\alpha)$ and hence $\overline{E}(\mathcal{L},\alpha) = \overline{E}(f,\alpha)$. Since $f, \mathcal{L}$ share $\{\alpha,\beta\}$ IM, then immediately we have $\overline{E}(\mathcal{L},\beta) = \overline{E}(f,\beta)$ and since $\mathcal{L}$ has no generelized exceptional value and here $\frac{\beta-\gamma}{\alpha-\gamma}$ is always rational, then we get $(\beta-\gamma)^{n-m} = (\alpha-\gamma)^{n-m} \implies (\frac{\beta-\gamma}{\alpha-\gamma})^{n-m} = 1 \implies \beta-\gamma = \alpha-\gamma$ (or $-(\alpha-\gamma)$), again a contradiction. Hence (i_{b_1}) does not hold.

Proceeding similarly we can discard the option (i_{b_2}).

Again if we consider (i_{b_3}) holds, i.e., $\frac{(f-\gamma)^m}{(\mathcal{L}-\gamma)^n} \equiv \frac{(\beta-\gamma)^m}{(\alpha-\gamma)^n}$. It is easy to verify that $\overline{E}(\mathcal{L},\alpha) \subseteq \overline{E}(f,\beta)$ and $\overline{E}(\mathcal{L},\alpha) \supseteq \overline{E}(f,\beta)$ and this implies $\overline{E}(\mathcal{L},\alpha) = \overline{E}(f,\beta)$. Hence immediately we get $\overline{E}(\mathcal{L},\beta) = \overline{E}(f,\alpha)$ and which implies $(\beta-\gamma)^{n+m} = (\alpha-\gamma)^{n+m}$, a contradiction.

Proceeding similarly we can discard (i_{b_4}).

Now from (3.33) we get

$$T(r,\mathcal{L}) \leq O(r),$$

a contradiction.

Next let us consider $\overline{N}_{(k,1)}(r,\alpha;f) \neq S(r,f)$ $(2 \leq k \leq 15)$ then two cases can occur which are $(ii_a)\overline{N}_{(k,1)}(r,\alpha;f \mid \mathcal{L}=\alpha) \neq S(r,f)$, $(ii_b)\overline{N}_{(k,1)}(r,\alpha;f \mid \mathcal{L}=\beta) \neq S(r,f)$.

Now if (ii_a) hold then we get $\hat{\Psi}_o = k$ and if (ii_b) holds then we will get $\hat{\Psi}_o = -\frac{k(\beta-\gamma)}{(\alpha-\gamma)}$. Hence clearly both (ii_a) and (ii_b) can not hold together.

Now if (ii_a) hold then dealing same as done in (3.24)-(3.26) we will get a contradiction.

Again if (ii_b) holds then $\hat{\Psi}_o = -\frac{k(\beta-\gamma)}{(\alpha-\gamma)}$. Considering any arbitrary (p,q) fold 'γ' point we will get $\frac{p}{q} = -\frac{k(\beta-\gamma)}{(\alpha-\gamma)}$. If $\frac{-(\beta-\gamma)}{(\alpha-\gamma)}$ is negetive or irrational then we get a contradiction and if it is non-negetive rational then taking $\frac{p}{q} = -\frac{k(\beta-\gamma)}{(\alpha-\gamma)} = \frac{n}{m}$ where $k(\beta-\gamma) = tn$, $(\alpha-\gamma) = tm$, $gsd(n,m) = 1$ and proceeding in the same manner as in (3.32), (3.33) we will get a contradiction again.

Next we consider $\overline{N}_{(l,2)}(r,\alpha;f) \neq S(r,f)$ then for $l = 2,3,4,\ldots 15$ adopiting the process same as done in above we will get contradiction. And if $\frac{n}{m} = 1$ then we will have f and $\mathcal{L}$ share γ CM then from *Theorem 1.10* we will get $f \equiv \mathcal{L}$.

Proceeding similarly the other cases can be disposed off and in each case we can get a contradiction. Therefore, one must have $f \equiv \mathcal{L}$. □

Proof. [Proof of Theorem 1.19] Suppose that $f \not\equiv \mathcal{L}$. It is given that f and $\mathcal{L}$ share $\{\alpha\}$, $\{\beta\}$ IM, it is easy to verify that $\rho(f) = \rho(\mathcal{L})$ $(= 1)$ and therefore we have $S(r,f) = S(r,\mathcal{L}) = O(\log r)$.

Now let us consider the following function

$$\Psi_1 = \frac{f''}{f'} - \frac{f'}{f-\alpha} - \frac{f'}{f-\beta} - \left(\frac{\mathcal{L}''}{\mathcal{L}'} - \frac{\mathcal{L}'}{\mathcal{L}-\alpha} - \frac{\mathcal{L}'}{\mathcal{L}-\beta}\right). \tag{3.34}$$

Since f and $\mathcal{L}$ share $(\alpha,0)$, $(\beta,0)$ then clearly the only poles of Ψ_1 come from the infinity points of f and $\mathcal{L}$ and from the zeros of f', $\mathcal{L}'$ where $f \neq \alpha,\beta$ $(\mathcal{L} \neq \alpha,\beta)$.

It is easy to verify that

$$m(r,\Psi_1) = O(\log r)$$

whereas from the given condition we have

$$N(r,\infty;\Psi_1) = \overline{N}(r,\infty;\Psi_1) = O(\log r), \tag{3.35}$$

hence $T(r,\psi_1) = O(\log r)$.

Now proceeding same as in (3.16)-(3.17) we can get

$$\hat{\Psi}_1 = \frac{f'(\mathcal{L}-\alpha)(\mathcal{L}-\beta)}{\mathcal{L}'(f-\alpha)(f-\beta)} = Q.e^{\eta_1}, \tag{3.36}$$

for some entire function η_1.

Now using *Lemma 2.5* we have

$$\rho(e^{\eta_1}) \leq \rho(f) = \rho(\mathcal{L}) = 1, \tag{3.37}$$

since $\rho(Q) = 0$. Therefore degree of η_1 is atmost one.

Next letting z_0 be a (r,s) fold 'α' point of f and $\mathcal{L}$, from above we get

$$\hat{\Psi}_1(z_0) = \frac{r}{s}. \tag{3.38}$$

Again if $\hat{\Psi}_1$ is non-constant then in view of *Lemma 2.3* we will get

$$T(r,\mathcal{L}) \leq O(T(r,\hat{\Psi}_1)) \leq O(r), \tag{3.39}$$

which implies the degree of $\mathcal{L}$ is zero and the same becomes a constant, a contradiction. Thus $\hat{\Psi}_1$ is a constant.

Next from *Lemma 2.3* at least one of

$$\sum_{k=1}^{18} \overline{N}_{(1,k)}(r,\alpha;f) + \sum_{k=2}^{18} \overline{N}_{(k,1)}(r,\alpha;f) + \sum_{l=2}^{18} \overline{N}_{(2,l)}(r,\alpha;f) + \sum_{l=3}^{18} \overline{N}_{(l,2)}(r,\alpha;f) \neq S(r,f), \tag{3.40}$$

$$\sum_{k=1}^{18} \overline{N}_{(1,k)}(r,\beta;f) + \sum_{k=2}^{18} \overline{N}_{(k,1)}(r,\beta;f) + \sum_{l=2}^{18} \overline{N}_{(2,l)}(r,\beta;f) + \sum_{l=3}^{18} \overline{N}_{(l,2)}(r,\beta;f) \neq S(r,f), \tag{3.41}$$

holds. Now without loss of generelity let us assume (3.40) holds.

Since $\hat{\Psi}_1$ is a constant in view of (3.38), exactly one of the following holds:

$$\overline{N}_{(s,s)}(r,\alpha;f) \neq S(r,f) \text{ at least for some } s = 1,2;$$

or

$$\overline{N}_{(1,k)}(r,\alpha;f) \neq S(r,f);$$

or

$$\overline{N}_{(k,1)}(r,\alpha;f) \neq S(r,f);$$

or

$$\overline{N}_{(l,2)}(r,\alpha;f) \neq S(r,f);$$

or

$$\overline{N}_{(2,l)}(r,\alpha;f) \neq S(r,f);$$

for some $l \neq 2k$, $2 \leq k \leq 18$ and $3 \leq l \leq 18$.

For the first case we have, $\hat{\Psi}_1 = 1$ and finally we get

$$\frac{f'}{(f-\alpha)(f-\beta)} = \frac{\mathcal{L}'}{(\mathcal{L}-\alpha)(\mathcal{L}-\beta)}, \tag{3.42}$$

which implies f and $\mathcal{L}$ share (α,∞), (β,∞) and hence by *Theorem 1.7* we get $f \equiv \mathcal{L}$.

Next for the sake of convenience let us consider the case $\overline{N}_{(k,1)}(r,\alpha;f) \neq S(r,f)$ for some $2 \leq k \leq 18$. Then we have $\hat{\Psi}_1 = k$, a constant.

Clearly $\overline{N}_{(1,k)}(r,\alpha;f) = \overline{N}_{(1,k)}(r,\beta;f) = \overline{N}_{(2,l)}(r,\alpha;f) = \overline{N}_{(2,l)}(r,\beta;f) = S(r,f)$, where $2 \leq l \leq 18$. Also clearly for $l \neq k$, $\overline{N}_{(l,1)}(r,\alpha;f) = \overline{N}_{(l,1)}(r,\beta;f) = S(r,f)$. On the other hand for $l \neq 2k$, $\overline{N}_{(l,2)}(r,\alpha;f) = \overline{N}_{(l,2)}(r,\beta;f) = S(r,f)$.

Hence every α, β point of f and $\mathcal{L}$ is a (kp,p) fold for some positive integer p.

Then from $\hat{\Psi}_1 = k$ we have

$$\frac{f'}{(f-\alpha)(f-\beta)} = \frac{k\mathcal{L}'}{(\mathcal{L}-\alpha)(\mathcal{L}-\beta)}. \tag{3.43}$$

Integrating both side of (3.43) we have

$$\left(\frac{\mathcal{L}-\beta}{\mathcal{L}-\alpha}\right)^k = c_1\left(\frac{f-\beta}{f-\alpha}\right), \tag{3.44}$$

where c_1 is a constant and using the First Fundamenntal theorem we get from (3.44)

$$kT(r,\mathcal{L}) \leq T(r,f) + T(r,c_1) \leq 2T(r,\mathcal{L}) + O(\log r). \tag{3.45}$$

Clearly for $k \geq 3$, (3.45) leads to a contradiction.

Again if $k = 2$ then we have each α, β points of f and $\mathcal{L}$ are $(2p, p)$ fold point.

Now let us consider the function

$$\hat{\mathcal{H}}_\alpha = \frac{(f-\alpha)Q_0}{(\mathcal{L}-\alpha)^2},$$

where Q_0 is a rational s.t $\hat{\mathcal{H}}_\alpha$ is a zero free entire function. Since every α point is a $(2p, p)$ fold point and f, $\mathcal{L}$ have finite number of poles, so it is possible to construct such rational Q_0. Therefore we can write $\hat{\mathcal{H}}_\alpha$ as

$$\hat{\mathcal{H}}_\alpha = \frac{(f-\alpha)Q_0}{(\mathcal{L}-\alpha)^2} = e^\mu, \tag{3.46}$$

for some entire function μ with $\rho(e^\mu) \leq 1$. So we can write $\mu = sz + t$, for some complex number s, t.

By the similar arguments we can get a function

$$\hat{\mathcal{H}}_\beta = \frac{(f-\beta)Q_0'}{(\mathcal{L}-\beta)^2} = e^\delta, \tag{3.47}$$

for some $\delta = az + b$, where a, b are finite complex numbers.

Now using the Second Fundamental Theorem and from (3.46), (3.47) we have

$$\begin{aligned} T(r,\mathcal{L}) &\leq \overline{N}(r,\alpha;\mathcal{L}) + \overline{N}(r,\beta;\mathcal{L}) + \overline{N}(r,\infty;\mathcal{L}) + O(\log r) \\ &\leq \overline{N}\left(r, \frac{1}{\alpha-\beta}; \frac{e^\delta}{Q_0'}\right) + \overline{N}\left(r, \frac{1}{\beta-\alpha}; \frac{e^\mu}{Q_0}\right) + O(\log r). \end{aligned} \tag{3.48}$$

Now we will show none of the following conditions (i), (ii) holds.

$$(i)\frac{f-\alpha}{(\mathcal{L}-\alpha)^2} \equiv \frac{1}{\beta-\alpha}$$

$$(ii)\frac{f-\beta}{(\mathcal{L}-\beta)^2} \equiv \frac{1}{\alpha-\beta}.$$

First we claim both (i), (ii) can not hold together. If not let us assume both (i), (ii) hold. Then we will get

$$(\mathcal{L}-\alpha)^2 + (\mathcal{L}-\beta)^2 = (\beta-\alpha)^2,$$

implies $\mathcal{L}$ is a constant, a contradiction.

Therefore either (i) or (ii) holds.

Again with out loss of generility let us consider $\frac{f-\alpha}{(\mathcal{L}-\alpha)^2} \equiv \frac{1}{\beta-\alpha}$. Then for $k = 2$ using this from (3.44) we get $\frac{f-\beta}{(\mathcal{L}-\beta)^2} \equiv \frac{1}{\alpha-\beta}$. Hence (i) holds implies (ii) holds and vice versa, i.e., (i) and (ii) always stand together.

Since both (i) and (ii) can not hold together, therefore none of the (i) and (ii) holds.

Then from (3.48) we get

$$T(r,\mathcal{L}) \leq O(r),$$

a contradiction.

Again if we consider $\overline{N}_{(l,2)}(r,\alpha;f) \neq S(r,f)$ then for $l = 2, 3, 4, 5, \ldots 18$ proceeding same as done in above we will get contradiction.

Proceeding similarly the other cases can be disposed off and in each case we will get a contradiction. Therefore, one must have $f \equiv \mathcal{L}$. □

Proof. [Proof of Theorem 1.22] Let us assume $f \not\equiv \mathcal{L}$. Without loss of generility assume that $S_1 = \{\alpha_1, \alpha_2\}$ and $S_2 = \{\beta_1, \beta_2\}$. It is given that f and g share the sets S_1 and S_2 with weight m_1 and m_2 respectively. Proceeding in the same way as done in (3.12)-(3.13), we can get $\rho(f) = \rho(\mathcal{L}) = 1$. Let's consider the following functions

$$\chi_0 = \frac{F'}{F} - \frac{G'}{G}, \tag{3.49}$$

where $F = (f-\alpha_1)(f-\alpha_2)$ and $G = (\mathcal{L}-\alpha_1)(\mathcal{L}-\alpha_2)$. Clearly F, G share $(0, m_1)$.

Set

$$\chi_1 = \frac{F_o'}{F_o} - \frac{G_o'}{G_o}, \tag{3.50}$$

where $F_o = (f-\beta_1)(f-\beta_2)$ and $G_o = (\mathcal{L}-\beta_1)(\mathcal{L}-\beta_2)$. Clearly F, G share $(0, m_2)$.

Next suppose that $\chi_0 \not\equiv 0$ and $\chi_1 \not\equiv 0$. Now,

$$\begin{aligned} &\overline{N}(r,\beta_1; f \mid\geq m_2+1) + \overline{N}(r,\beta_2; f \mid\geq m_2+1) \leq \frac{1}{m_2} T(r,\chi_0) + O(1) \\ \leq\ & \frac{1}{m_2}\Big(\overline{N}(r,\infty;G) + \overline{N}(r,\infty;F) + \overline{N}(r,0;F \mid\geq m_1+1)\Big) + O(\log r) \\ \leq\ & \frac{1}{m_2}\Big(\overline{N}(r,\alpha_1; f \mid\geq m_1+1) + \overline{N}(r,\alpha_2; f \mid\geq m_1+1)\Big) + O(\log r). \end{aligned} \tag{3.51}$$

Similarly,

$$\begin{aligned} &\overline{N}(r,\alpha_1; f \mid\geq m_1+1) + \overline{N}(r,\alpha_2; f \mid\geq m_1+1) \leq \frac{1}{m_1} T(r,\chi_1) + O(1) \\ \leq\ & \frac{1}{m_1} N(r,\infty;\chi_1) + O(\log r) \leq \frac{1}{m_1}\Big(\overline{N}(r,\beta_1; f \mid\geq m_2+1) + \overline{N}(r,\beta_2; f \mid\geq m_2+1)\Big) + O(\log r). \end{aligned} \tag{3.52}$$

From (3.51) and (3.52) we have for $m_1.m_2 > 1$,

$$\overline{N}(r,\alpha_1; f \mid\geq m_1+1) + \overline{N}(r,\alpha_2; f \mid\geq m_1+1) \leq O(\log r),$$

$$\overline{N}(r,\beta_1; f \mid\geq m_2+1) + \overline{N}(r,\beta_2; f \mid\geq m_2+1) \leq O(\log r).$$

Let's consider the following function

$$\Delta_\alpha = \frac{(f-\alpha_1)(f-\alpha_2)}{(\mathcal{L}-\alpha_1)(\mathcal{L}-\alpha_2)}.$$

Clearly,

$$\overline{N}(r,0;\Delta_\alpha) \leq \sum_{i=1}^{2} \overline{N}(r,\alpha_i; f \mid\geq m_1+1) + \overline{N}(r,\infty;\mathcal{L}) \leq O(\log r) \tag{3.53}$$

and

$$\overline{N}(r,\infty;\Delta_\alpha) \leq \sum_{i=1}^{2} \overline{N}(r,\alpha_i; \mathcal{L} \mid\geq m_1+1) + \overline{N}(r,\infty;f) \leq O(\log r). \tag{3.54}$$

Now from (3.53)-(3.54) we can get a rational function H such that

$$\Delta_\alpha = \frac{H(f-\alpha_1)(f-\alpha_2)}{(\mathcal{L}-\alpha_1)(\mathcal{L}-\alpha_2)},$$

is a zero free entire function. As usual We can write $\Delta_\alpha = \frac{H(f-\alpha_1)(f-\alpha_2)}{(\mathcal{L}-\alpha_1)(\mathcal{L}-\alpha_2)} = e^\nu$, where ν is an entire function s.t $\rho(e^\nu) \leq 1$.

Now Let

$$\Sigma = \left(\frac{e^\nu}{H} - 1\right)\left(\frac{e^\nu}{H} - \frac{(\beta_1-\alpha_1)(\beta_1-\alpha_2)}{(\beta_2-\alpha_1)(\beta_2-\alpha_2)}\right)\left(\frac{e^\nu}{H} - \frac{(\beta_2-\alpha_1)(\beta_2-\alpha_2)}{(\beta_1-\alpha_1)(\beta_1-\alpha_2)}\right).$$

If $\Sigma \neq 0$ then we have,

$$T(r,\mathcal{L}) \leq \sum_{i=1}^{2} \overline{N}(r,\beta_i;\mathcal{L}) + \overline{N}(r,\infty;\mathcal{L}) \quad \leq \quad \overline{N}(r,0;\Sigma) \leq O(r),$$

a contradiction. Therefore $\Sigma \equiv 0$.

Now proceeding same as done in the last part Proposition 2.6 in [11] we will get $f \equiv \mathcal{L}$. Since our assumption was $f \not\equiv \mathcal{L}$ then one must have at least one of χ_0, χ_1 will equal to zero, i.e., f and $\mathcal{L}$ share S_1 or S_2 with weight ∞. Then again from the same in [11] we can get the result.

Similarly if S_1 contains one and S_2 contains two (one) elements then proceeding same as done in above and the help of *Theorem 1.7* in [11] we will get $f \equiv \mathcal{L}$. □

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