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Fractional Hermite-Hadamard Type Inequalities for Subadditive Functions

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Abstract. In this paper, we establish different variants of fractional Hermite-Hadamard inequalities for subadditive functions via Riemann-Liouville fractional integrals. Moreover, we offer some fractional integral inequalities for the product of two subadditive functions via Riemann-Liouville fractional integrals. It is also shown that the inequalities offered in this research are the generalization of the already given inequalities for convex functions and subadditive functions.

1. Introduction

The principal work on the general theory of subadditive functions is that of Hille and Phillips [9]. This reference also includes a part of the work of Rosenbaum [17] on subadditive functions of several variables. Additivity, subadditivity, and superadditivity are important concepts both in measure theory and in several fields of mathematics and mathematical inequalities. Especially, there are a lot of examples of additive, subadditive, and superadditive functions in various areas of mathematics such as norms, square roots, error function, growth rates, differential equations and integral means. Inequalities and especially subadditive function theory is one of the most extensively developed fields not only in theoretical and applied mathematics but also physics and other applied sciences. Here, we mention the results of [3, 9–13, 17] and the corresponding references cited therein.

Definition 1.1. *A function φ defined on a set H of real numbers and with the range contained in the set R^{+} of all positive real numbers, is subadditive on H if, for all elements $\varkappa$ and γ of H such that $\varkappa+\gamma$ is an element of H*

$$\varphi(\varkappa+\gamma)\leq\varphi(\varkappa)+\varphi(\gamma).$$

If the equality holds, φ is called additive; if the inequality is reversed, φ is superadditive. A function φ is convex on the (possibly infinite) interval D if, for all $\varkappa$ and γ in D and all τ which satisfy $0\leq\tau\leq1$,

$$\varphi(\tau\varkappa+(1-\tau)\gamma)\leq\tau\varphi(\varkappa)+(1-\tau)\varphi(\gamma).$$

If this inequality is reversed, φ is concave on D.

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Remark 1.2. *If φ is convex and subadditive on H and if $\varphi(0) = 0$, then φ is additive on H.*

Definition 1.3. *The function $\varphi : [0, v] \to R$, $v > 0$ is said to be starshaped if for every $\varkappa \in [0, v]$ and $\tau \in [0, 1]$ we have $\varphi(\tau\varkappa) \leq \tau\varphi(\varkappa)$.*

According to the above definitions, if a subadditive function $\varphi : A \subset [0, \infty) \to R$ is also starshaped, then φ is a convex function.

The following inequality is well known in the literature as the Hermite-Hadamard integral inequality (see, [6, 16]):

$$\varphi\left(\frac{u+v}{2}\right) \leq \frac{1}{v-u}\int_u^v \varphi(\varkappa)d\varkappa \leq \frac{\varphi(u)+\varphi(v)}{2} \tag{1}$$

where $\varphi : I \subset \mathbb{R} \to \mathbb{R}$ is a convex function on the interval I of real numbers and $u, v \in I$ with $u < v$.

The inequalities (1) have grown into a significant pillar in mathematical analysis and optimization, besides, by looking into a variety of settings, these inequalities are found to have a number of uses. What is more, for a specific choice of the function φ, many inequalities with special means are obtainable. Hermite Hadamard's inequality (1), for example, is significant in its rich geometry and hence there are many studies on it to demonstrate its new proofs, refinements, extensions and generalizations. You can check [1, 2, 4–7, 15, 18–28] and the references included there.

The most well-known inequalities related to the integral mean of a convex function are the Hermite Hadamard inequalities. Since then, some refinements of the Hermite-Hadamard inequality on convex functions have been extensively investigated by a number of authors.

Recently, Sarikaya and Ali [19] proved the following interesting integral inequalities of Hermite-Hadamard type for continuous subadditive functions.

Theorem 1.4. *If a continuous function $\varphi : I = [0, \infty) \to R$ is subadditive, then the following inequalities hold*

$$\frac{1}{2}\varphi(u+v) \leq \frac{1}{v-u}\int_u^v \varphi(\varkappa)d\varkappa \leq \frac{1}{u}\int_0^u \varphi(\varkappa)d\varkappa + \frac{1}{v}\int_0^v \varphi(\varkappa)d\varkappa. \tag{2}$$

Theorem 1.5. *Let φ, $\phi : I = [0, \infty) \to R$ be two continuous subadditive functions, then the following inequalities hold*

$$\begin{aligned}\frac{1}{2}\varphi(u+v)\phi(u+v) \leq\ & \frac{1}{v-u}\int_u^v \varphi(\varkappa)\phi(\varkappa)d\varkappa + \int_0^1 \left[\varphi(u\tau)\phi((1-\tau)u) + \varphi((1-\tau)v)\phi(\tau v)\right]d\tau \\ & + \int_0^1 \left[\varphi(u\tau)\phi(\tau v) + \varphi(\tau v)\phi(\tau u)\right]d\tau\end{aligned} \tag{3}$$

and

$$\begin{aligned}\frac{1}{v-u}\int_u^v \varphi(\varkappa)\phi(\varkappa)d\varkappa \leq\ & \frac{1}{u}\int_0^u \varphi(\varkappa)\phi(\varkappa)d\varkappa + \frac{1}{v}\int_0^v \varphi(\varkappa)\phi(\varkappa)d\varkappa \\ & + \int_0^1 \varphi(u\tau)\phi((1-\tau)v)d\tau + \int_0^1 \varphi(\tau v)\phi((1-\tau)u)d\tau.\end{aligned} \tag{4}$$

In the following, we give some necessary definitions and mathematical preliminaries of fractional calculus theory which will be used in the paper. For more details, one can consult [8, 14].

Definition 1.6. *Let $\varphi \in L_1[u, v]$. The left-sided Riemann-Liouville fractional integral $J_{u+}^{\alpha}\varphi$ and the right-sided Riemann-Liouville fractional integral $J_{v-}^{\alpha}\varphi$ of order $\alpha > 0$ with $u \geq 0$ are defined by*

$$J_{u+}^{\alpha}\varphi(\varkappa) = \frac{1}{\Gamma(\alpha)}\int_u^{\varkappa} (\varkappa-\tau)^{\alpha-1}\varphi(\tau)d\tau, \quad u < \varkappa$$

and

$$J_{v-}^{\alpha}\varphi(\varkappa)=\frac{1}{\Gamma(\alpha)}\int_{\varkappa}^{v}(\tau-\varkappa)^{\alpha-1}\varphi(\tau)d\tau,\quad \varkappa<v$$

respectively. Here $\Gamma(\alpha)$ *is Gamma function and* $J_{u+}^{0}\varphi(\varkappa)=J_{v-}^{0}\varphi(\varkappa)=\varphi(\varkappa)$.

In [23], Sarikaya et al. gave the following inequalities of Hermite-Hadamard type for convex functions involving Riemann-Liouville fractional integrals.

Theorem 1.7. *Let* $\varphi:[u,v]\to R$ *be a positive function with* $0\le u<v$ *and* $\varphi\in L_1[u,v]$. *If* f *is a convex function on* $[u,v]$, *then the following inequalities hold for the Riemann-Liouville fractional integrals:*

$$\varphi\left(\frac{u+v}{2}\right)\le\frac{\Gamma(\alpha+1)}{2(v-u)^{\alpha}}[J_{u+}^{\alpha}\varphi(v)+J_{v-}^{\alpha}\varphi(u)]\le\frac{\varphi(u)+\varphi(v)}{2},\tag{5}$$

where $\alpha>0$.

In [24], Sarikaya and Yildirim established the following inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals.

Theorem 1.8. *Let* $\varphi:[u,v]\to R$ *be a positive function with* $0\le u<v$ *and* $\varphi\in L_1[u,v]$. *If* φ *is a convex function on* $[u,v]$, *then the following inequalities hold for Riemann-Liouville fractional integrals:*

$$\varphi\left(\frac{u+v}{2}\right)\le\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(v-u)^{\alpha}}[J_{\left(\frac{u+v}{2}\right)+}^{\alpha}\varphi(v)+J_{\left(\frac{u+v}{2}\right)-}^{\alpha}\varphi(u)]\le\frac{\varphi(u)+\varphi(v)}{2},\tag{6}$$

where $\alpha>0$.

In [1], Budak et al. gave the following inequalities of Hermite-Hadamard type for convex functions via Riemann-Liouville fractional integrals.

Theorem 1.9. *Let* $\varphi:[u,v]\to R$ *be a positive function with* $0\le u<v$ *and* $\varphi\in L_1[u,v]$. *If* φ *is a convex function on* $[u,v]$, *then the following inequalities hold for the Riemann-Liouville fractional integrals:*

$$\begin{aligned}\varphi\left(\frac{u+v}{2}\right)&\le\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(v-u)^{\alpha}}\left[J_{u+}^{\alpha}\varphi\left(\frac{u+v}{2}\right)+J_{v-}^{\alpha}\varphi\left(\frac{u+v}{2}\right)\right]\\&\le\frac{\varphi(u)+\varphi(v)}{2},\end{aligned}\tag{7}$$

where $\alpha>0$.

Involving to the products of convex functions, Chen gave two important new Hermite-Hadamard type inequalities involving Riemann-Liouville fractional integrals as follows in [2]:

Theorem 1.10. *Let* $\varphi,\phi:[u,v]\to R$ *be two positive functions with* $0\le u<v$ *and* $\varphi,\phi\in L_1[u,v]$. *If* φ,ϕ *are convex functions on* $[u,v]$, *then the following inequalities hold for the Riemann-Liouville fractional integrals:*

$$\frac{\Gamma(\alpha+1)}{2(v-u)^{\alpha}}[J_{u+}^{\alpha}\varphi(v)\phi(v)+J_{v-}^{\alpha}\varphi(u)\phi(u)]\le\left(\frac{\alpha}{\alpha+2}-\frac{\alpha}{\alpha+1}+\frac{1}{2}\right)M(u,v)+\frac{\alpha}{(\alpha+1)(\alpha+2)}N(u,v)\tag{8}$$

and

$$\begin{aligned}2\varphi\left(\frac{u+v}{2}\right)\phi\left(\frac{u+v}{2}\right)&\le\frac{\Gamma(\alpha+1)}{2(v-u)^{\alpha}}[J_{u+}^{\alpha}\varphi(v)\phi(v)+J_{v-}^{\alpha}\varphi(u)\phi(u)]\\&\quad+\frac{\alpha}{(\alpha+1)(\alpha+2)}M(u,v)+\left(\frac{\alpha}{\alpha+2}-\frac{\alpha}{\alpha+1}+\frac{1}{2}\right)N(u,v),\end{aligned}\tag{9}$$

where

$$\begin{aligned}M(u,v)&=\varphi(u)\phi(u)+\varphi(v)\phi(v),\\N(u,v)&=\varphi(u)\phi(v)+\varphi(v)\phi(u).\end{aligned}\tag{10}$$

In this article, we are interested to give the fractional version of Hermite-Hadamard inequalities for the subadditive functions by using the Riemann-Liouville fractional integrals. Moreover, we prove some fractional integral inequalities related to Hermite-Hadamard inequalities for subadditive functions via Riemann-Liouville fractional integrals.

2. Main Results

The fractional Hermite-Hadamard type inequalities for subadditive functions are given the following form:

Theorem 2.1. *If a continuous function* $\varphi : I = [0,\infty) \to R$ *is subadditive,* $u, v \in I^{\circ}$ *and* $u < v$*, then the following inequalities hold for the Riemann-Liouville fractional integrals:*

$$\begin{aligned} \frac{1}{2}\varphi(u+v) &\leq \frac{\Gamma(\alpha+1)}{2(v-u)^{\alpha}}\left[J_{u+}^{\alpha}\varphi(v)+J_{v-}^{\alpha}\varphi(u)\right] \\ &\leq \frac{\alpha}{2u^{\alpha}}\int_{0}^{u}\left[\varkappa^{\alpha-1}+(u-\varkappa)^{\alpha-1}\right]\varphi(\varkappa)\,d\varkappa+\frac{\alpha}{2v^{\alpha}}\int_{0}^{v}\left[\varkappa^{\alpha-1}+(v-\varkappa)^{\alpha-1}\right]\varphi(\varkappa)\,d\varkappa \end{aligned} \tag{11}$$

with $\alpha > 0$.

Proof. Since φ is a subadditive function on I, we have

$$\varphi(u+v) \leq \varphi(u\tau+(1-\tau)v)+\varphi((1-\tau)u+\tau v). \tag{12}$$

Multiplying both sides of (12) by $\tau^{\alpha-1}$ and integrating the resultant inequality with respect to τ over $[0,1]$, we have

$$\begin{aligned} \frac{1}{\alpha}\varphi(u+v) &\leq \int_{0}^{1}\tau^{\alpha-1}\varphi(u\tau+(1-\tau)v)\,d\tau+\int_{0}^{1}\tau^{\alpha-1}\varphi((1-\tau)u+\tau v)\,d\tau \\ &= \int_{u}^{v}\left(\frac{v-u}{v-u}\right)^{\alpha-1}\varphi(u)\frac{du}{v-u}+\int_{u}^{v}\left(\frac{v-u}{v-u}\right)^{\alpha-1}\varphi(v)\frac{dv}{v-u} \\ &= \frac{\Gamma(\alpha)}{(v-u)^{\alpha}}\left[J_{u+}^{\alpha}\varphi(v)+J_{v-}^{\alpha}\varphi(u)\right] \end{aligned}$$

i.e.

$$\frac{1}{2}\varphi(u+v) \leq \frac{\Gamma(\alpha+1)}{2(v-u)^{\alpha}}\left[J_{u+}^{\alpha}\varphi(v)+J_{v-}^{\alpha}\varphi(u)\right]$$

and the first inequality in (11) is proved.

For the proof of the second inequality in (11), we first note that φ is a subadditive function on I, then for $\tau \in [0,1]$, it yields

$$\varphi(\tau u+(1-\tau)v) \leq \varphi(\tau u)+\varphi((1-\tau)v) \tag{13}$$

and

$$\varphi(\tau v+(1-\tau)u) \leq \varphi(\tau v)+\varphi((1-\tau)u). \tag{14}$$

By adding the inequalities (13) and (14), we have

$$\varphi(\tau u+(1-\tau)v)+\varphi(\tau v+(1-\tau)u) \leq \varphi(\tau u)+\varphi((1-\tau)v)+\varphi(\tau v)+\varphi((1-\tau)u). \tag{15}$$

Multiplying both sides of (15) by $\tau^{\alpha-1}$ and integrating the resultant one with respect to τ over $[0,1]$ and using the change of variables, we have the second inequality in (11). Hence, the proof is finished. □

Remark 2.2. *Under the assumptions of Theorem 2.1, if we take $\alpha = 1$, then we have the inequality (2).*

Corollary 2.3. *Under the conditions of Theorem 2.1, if we take $\varphi(\tau\varkappa) \leq \tau\varphi(\varkappa)$, then we get*

$$\begin{aligned} \varphi\left(\frac{u+v}{2}\right) &\leq \frac{1}{2}\varphi(u+v) \\ &\leq \frac{\Gamma(\alpha+1)}{2(v-u)^{\alpha}}\left[J_{u+}^{\alpha}\varphi(v)+J_{v-}^{\alpha}\varphi(u)\right] \leq \frac{\varphi(u)+\varphi(v)}{2} \end{aligned} \tag{16}$$

which coincides with the inequalities for convex functions given in (5).

Theorem 2.4. *If $\varphi, \phi : I = [0,\infty) \to R$ are two continuous subadditive functions, $u, v \in I^{\circ}$, $u < v$, then the following inequalities hold for the Riemann-Liouville fractional integrals:*

$$\begin{aligned} \frac{1}{2}\varphi(u+v)\phi(u+v) \leq\ & \frac{\Gamma(\alpha+1)}{2(v-u)^{\alpha}}\left[J_{u+}^{\alpha}\varphi(v)\phi(v)+J_{v-}^{\alpha}\varphi(u)\phi(u)\right] \\ &+\frac{\alpha}{2}\left[\frac{1}{u^{\alpha}}\int_0^u\left[\varkappa^{\alpha-1}+(u-\varkappa)^{\alpha-1}\right]\varphi(\varkappa)\phi(u-\varkappa)\,d\varkappa\right. \\ &+\frac{1}{v^{\alpha}}\int_0^v\left[\varkappa^{\alpha-1}+(v-\varkappa)^{\alpha-1}\right]\varphi(\varkappa)\phi(v-\varkappa)\,d\varkappa \\ &+\int_0^1\left[\tau^{\alpha-1}+(1-\tau)^{\alpha-1}\right]\varphi(u\tau)\phi(\tau v)\,d\tau \\ &\left.+\int_0^1\left[\tau^{\alpha-1}+(1-\tau)^{\alpha-1}\right]\varphi(\tau v)\phi(\tau u)\,d\tau\right] \end{aligned} \tag{17}$$

and

$$\begin{aligned} \frac{\Gamma(\alpha+1)}{2(v-u)^{\alpha}}\left[J_{u+}^{\alpha}\varphi(v)\phi(v)+J_{v-}^{\alpha}\varphi(u)\phi(u)\right] \leq\ & \frac{\alpha}{2}\left[\frac{1}{u^{\alpha}}\int_0^u\left[\varkappa^{\alpha-1}+(u-\varkappa)^{\alpha-1}\right]\varphi(\varkappa)\phi(\varkappa)d\varkappa\right. \\ &+\frac{1}{v^{\alpha}}\int_0^v\left[\varkappa^{\alpha-1}+(v-\varkappa)^{\alpha-1}\right]\varphi(\varkappa)\phi(\varkappa)d\varkappa \\ &+\int_0^1\left[\tau^{\alpha-1}+(1-\tau)^{\alpha-1}\right]\varphi(\tau u)\phi((1-\tau)v)d\tau \\ &\left.+\int_0^1\left[\tau^{\alpha-1}+(1-\tau)^{\alpha-1}\right]\varphi(\tau v)\phi((1-\tau)u)d\tau\right] \end{aligned} \tag{18}$$

with $\alpha > 0$.

Proof. Since φ and ϕ are subadditive functions on I, by using the subadditivity of φ and ϕ, we have

$$\begin{aligned} \varphi(u+v) &= \varphi(\tau u+(1-\tau)v+\tau v+(1-\tau)u) \\ &\leq \varphi(\tau u+(1-\tau)v)+\varphi(\tau v+(1-\tau)u), \end{aligned} \tag{19}$$

and

$$\begin{aligned} \phi(u+v) &= \phi(\tau u+(1-\tau)v+\tau v+(1-\tau)u) \\ &\leq \phi(\tau u+(1-\tau)v)+\phi(\tau v+(1-\tau)u). \end{aligned} \tag{20}$$

From the inequalities (19) and (20), we have

$$\begin{aligned}
& \varphi(u+v)\phi(u+v) \quad (21)\\
\leq\ & \Big[\varphi(u\tau+(1-\tau)v)\phi(u\tau+(1-\tau)v)+\varphi((1-\tau)u+\tau v)\phi((1-\tau)u+\tau v)\Big]\\
& +\Big[\varphi(u\tau+(1-\tau)v)\phi((1-\tau)u+\tau v)+\varphi((1-\tau)u+\tau v)\phi(u\tau+(1-\tau)v)\Big]\\
\leq\ & \Big[\varphi(u\tau+(1-\tau)v)\phi(u\tau+(1-\tau)v)+\varphi((1-\tau)u+\tau v)\phi((1-\tau)u+\tau v)\Big]\\
& +[\varphi(u\tau)+\varphi((1-\tau)v)]\Big[\phi((1-\tau)u)+\phi(\tau v)\Big]\\
& +[\varphi((1-\tau)u)+\varphi(\tau v)]\Big[\phi(u\tau)+\phi((1-\tau)v)\Big]\\
=\ & \Big[\varphi(u\tau+(1-\tau)v)\phi(u\tau+(1-\tau)v)+\varphi((1-\tau)u+\tau v)\phi((1-\tau)u+\tau v)\Big]\\
& +\Big[\varphi(u\tau)\phi((1-\tau)u)+\varphi(u\tau)\phi(\tau v)+\varphi((1-\tau)v)\phi((1-\tau)u)+\varphi((1-\tau)v)\phi(\tau v)\Big]\\
& +\Big[\varphi((1-\tau)u)\phi(\tau u)+\varphi((1-\tau)u)\phi((1-\tau)v)+\varphi(\tau v)\phi(\tau u)+\varphi(\tau v)\phi((1-\tau)v)\Big].
\end{aligned}$$

Multiplying both sides of the inequality (21) by $\tau^{\alpha-1}$ and integrating the resultant inequality with respect to τ over $[0,1]$, we obtain

$$\begin{aligned}
\frac{1}{\alpha}\varphi(u+v)\phi(u+v) \leq\ & \int_0^1 \tau^{\alpha-1}\Big[\varphi(u\tau+(1-\tau)v)\phi(u\tau+(1-\tau)v)+\varphi((1-\tau)u+\tau v)\phi((1-\tau)u+\tau v)\Big]d\tau\\
& +\int_0^1 \tau^{\alpha-1}\varphi(u\tau)\phi((1-\tau)u)d\tau+\int_0^1 \tau^{\alpha-1}\varphi((1-\tau)u)\phi(\tau u)d\tau\\
& +\int_0^1 \tau^{\alpha-1}\varphi(\tau v)\phi(v(1-\tau))d\tau+\int_0^1 \tau^{\alpha-1}\varphi((1-\tau)v)\phi(\tau v)d\tau\\
& +\int_0^1 \tau^{\alpha-1}\varphi(\tau u)\phi(\tau v)d\tau+\int_0^1 \tau^{\alpha-1}\varphi((1-\tau)u)\phi((1-\tau)v)d\tau\\
& +\int_0^1 \tau^{\alpha-1}\varphi(\tau v)\phi(\tau u)d\tau+\int_0^1 \tau^{\alpha-1}\varphi((1-\tau)v)\phi((1-\tau)u)d\tau
\end{aligned}$$

and changing the variables of integration, we obtain the inequality (17).

Since φ and ϕ are subadditive functions on I, then for $\tau\in[0,1]$, we have

$$\varphi(\tau u+(1-\tau)v)\leq\varphi(\tau u)+\varphi((1-\tau)v) \tag{22}$$

and

$$\phi(\tau u+(1-\tau)v)\leq\phi(\tau u)+\phi((1-\tau)v). \tag{23}$$

From inequalities (22) and (23), we have

$$\begin{aligned}
\varphi(\tau u+(1-\tau)v)\phi(\tau u+(1-\tau)v) \leq\ & \varphi(\tau u)\phi(\tau u)+\varphi(\tau u)\phi((1-\tau)v)\\
& +\varphi((1-\tau)v)\phi(\tau u)+\varphi((1-\tau)v)\phi((1-\tau)v).
\end{aligned} \tag{24}$$

Similarly, we have

$$\begin{aligned}
\varphi((1-\tau)u+\tau v)\phi((1-\tau)u+\tau v) \leq\ & \varphi((1-\tau)u)\phi((1-\tau)u)+\varphi((1-\tau)u)\phi(\tau v)\\
& +\varphi(\tau v)\phi((1-\tau)u)+\varphi(\tau v)\phi(\tau v).
\end{aligned} \tag{25}$$

Adding the inequalities (24) and (25), we get

$$\begin{aligned}
& \varphi(\tau u+(1-\tau)v)\phi(\tau u+(1-\tau)v)+\varphi((1-\tau)u+\tau v)\phi((1-\tau)u+\tau v)\\
\leq\ & \varphi(\tau u)\phi(\tau u)+\varphi((1-\tau)u)\phi((1-\tau)u)+\varphi(\tau u)\phi((1-\tau)v)+\varphi((1-\tau)u)\phi(\tau v)\\
& +\varphi((1-\tau)v)\phi(\tau u)+\varphi(\tau v)\phi((1-\tau)u)+\varphi((1-\tau)v)\phi((1-\tau)v)+\varphi(\tau v)\phi(\tau v).
\end{aligned}$$

Multiplying both sides of the above inequality by $\tau^{\alpha-1}$ and integrating the resultant inequality with respect to τ over $[0,1]$, we have

$$\begin{aligned}
&\int_0^1 \tau^{\alpha-1}\left[\varphi(\tau u+(1-\tau)v)\phi(\tau u+(1-\tau)v)+\varphi((1-\tau)u+\tau v)\phi((1-\tau)u+\tau v)\right]d\tau\\
\leq\ & \int_0^1 \tau^{\alpha-1}\varphi(\tau u)\phi(\tau u)d\tau+\int_0^1 \tau^{\alpha-1}\varphi((1-\tau)u)\phi((1-\tau)u)d\tau\\
&+\int_0^1 \tau^{\alpha-1}\varphi(\tau v)\phi(\tau v)d\tau+\int_0^1 \tau^{\alpha-1}\varphi((1-\tau)v)\phi((1-\tau)v)d\tau\\
&+\int_0^1 \tau^{\alpha-1}\varphi(\tau u)\phi((1-\tau)v)d\tau+\int_0^1 \tau^{\alpha-1}\varphi\left((1-\tau)u\right)\phi\left(\tau v\right)d\tau\\
&+\int_0^1 \tau^{\alpha-1}\varphi(\tau v)\phi((1-\tau)u)d\tau+\int_0^1 \tau^{\alpha-1}\varphi\left((1-\tau)v\right)\phi\left(\tau u\right)d\tau
\end{aligned}$$

and by changing the variables of integration, we obtain the inequality (18). □

Remark 2.5. *Under the assumptions of Theorem 2.4, if we take $\alpha = 1$, then we have inequalities (3) and (4).*

Corollary 2.6. *Under the assumptions of Theorem 2.4, if we take $\varphi(\tau\varkappa) \leq \tau\varphi(\varkappa)$, then we get*

$$\begin{aligned}
2\varphi\left(\frac{u+v}{2}\right)\phi\left(\frac{u+v}{2}\right) &\leq \frac{1}{2}\varphi(u+v)\phi(u+v)\\
&\leq \frac{\Gamma(\alpha+1)}{2(v-u)^{\alpha}}\left[J_{u+}^{\alpha}\varphi(v)+J_{v-}^{\alpha}\varphi(u)\right]+M(u,v)\frac{\alpha}{(\alpha+1)(\alpha+2)}+N(u,v)\left(\frac{\alpha}{\alpha+2}-\frac{\alpha}{\alpha+1}+\frac{1}{2}\right)
\end{aligned}$$

and

$$\frac{\Gamma(\alpha+1)}{2(v-u)^{\alpha}}\left[J_{u+}^{\alpha}\varphi(v)+J_{v-}^{\alpha}\varphi(u)\right] \leq M(u,v)\left(\frac{\alpha}{\alpha+2}-\frac{\alpha}{\alpha+1}+\frac{1}{2}\right)+N(u,v)\frac{\alpha}{(\alpha+1)(\alpha+2)},$$

where $M(u,v)$ and $N(u,v)$ are defined by (10). The above inequalities coincide with the inequalities (8) and (9) for convex functions.

Theorem 2.7. *If a continuous function $\varphi : I=[0,\infty)\to R$ is subadditive, $u,v\in I^{\circ}$ and $u<v$, then the following inequalities hold for the Riemann-Liouville fractional integrals:*

$$\begin{aligned}
\frac{1}{2}\varphi\left(u+v\right) &\leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(v-u)^{\alpha}}\left[J_{\left(\frac{u+v}{2}\right)+}^{\alpha}\varphi(v)+J_{\left(\frac{u+v}{2}\right)-}^{\alpha}\varphi(u)\right] \qquad (26)\\
&\leq \frac{2^{\alpha-1}\alpha}{u^{\alpha}}\int_0^{\frac{u}{2}}\varkappa^{\alpha-1}\varphi\left(\varkappa\right)d\varkappa+\frac{2^{\alpha-1}\alpha}{u^{\alpha}}\int_{\frac{u}{2}}^{u}(u-\varkappa)^{\alpha-1}\varphi\left(\varkappa\right)d\varkappa\\
&\quad+\frac{2^{\alpha-1}\alpha}{v^{\alpha}}\int_0^{\frac{v}{2}}\varkappa^{\alpha-1}\varphi\left(\varkappa\right)d\varkappa+\frac{2^{\alpha-1}\alpha}{v^{\alpha}}\int_{\frac{v}{2}}^{v}(v-\varkappa)^{\alpha-1}\varphi\left(\varkappa\right)d\varkappa
\end{aligned}$$

with $\alpha > 0$.

Proof. By using the subadditivity of functions φ and ϕ, we have

$$\begin{aligned}
\varphi(u+v) &= \varphi\left(\frac{\tau}{2}u+\frac{2-\tau}{2}v+\frac{2-\tau}{2}u+\frac{\tau}{2}v\right) \qquad (27)\\
&\leq \varphi\left(\frac{\tau}{2}u+\frac{2-\tau}{2}v\right)+\varphi\left(\frac{2-\tau}{2}u+\frac{\tau}{2}v\right).
\end{aligned}$$

Multiplying both sides of the inequality (27) by $\tau^{\alpha-1}$ and integrating the resultant inequality with respect to τ over $[0,1]$, we obtain

$$\begin{aligned}\frac{1}{\alpha}\varphi(u+v) &\leq \int_0^1 \tau^{\alpha-1}\varphi\left(\frac{\tau}{2}u+\frac{2-\tau}{2}v\right)d\tau+\int_0^1 \tau^{\alpha-1}\varphi\left(\frac{2-\tau}{2}u+\frac{\tau}{2}v\right)d\tau \\ &= \frac{2^{\alpha}\Gamma(\alpha)}{(v-u)^{\alpha}}\left[J^{\alpha}_{\left(\frac{u+v}{2}\right)+}\varphi(v)+J^{\alpha}_{\left(\frac{u+v}{2}\right)-}\varphi(u)\right]\end{aligned}$$

and the proof of the first inequality in (26) is completed.

For the proof of the second inequality in (26) first we note that φ is a subadditive function on I, then for $\tau\in[0,1]$ it yields

$$\varphi\left(\frac{\tau}{2}u+\frac{2-\tau}{2}v\right)+\varphi\left(\frac{2-\tau}{2}u+\frac{\tau}{2}v\right) \leq \varphi\left(\frac{\tau}{2}u\right)+\varphi\left(\frac{2-\tau}{2}v\right)+\varphi\left(\frac{2-\tau}{2}u\right)+\varphi\left(\frac{\tau}{2}v\right). \tag{28}$$

Multiplying both sides of the inequality (28) by $\tau^{\alpha-1}$ and integrating the resultant inequality with respect to τ over $[0,1]$, we obtain

$$\begin{aligned}&\int_0^1 \tau^{\alpha-1}\varphi\left(\frac{\tau}{2}u+\frac{2-\tau}{2}v\right)d\tau+\int_0^1 \tau^{\alpha-1}\varphi\left(\frac{2-\tau}{2}u+\frac{\tau}{2}v\right)d\tau \\ \leq\ &\int_0^1 \tau^{\alpha-1}\varphi\left(\frac{\tau}{2}u\right)d\tau+\int_0^1 \tau^{\alpha-1}\varphi\left(\frac{\tau}{2}v\right)d\tau+\int_0^1 \tau^{\alpha-1}\varphi\left(\frac{2-\tau}{2}v\right)d\tau+\int_0^1 \tau^{\alpha-1}\varphi\left(\frac{2-\tau}{2}u\right)d\tau\end{aligned}$$

and by changing the variables of integration, we obtain the second inequality of (26). □

Remark 2.8. *Under the assumptions of Theorem 2.7, if we take $\alpha=1$, then we have the inequality (2).*

Corollary 2.9. *Under the conditions of Theorem 2.7, if we take $\varphi(\tau\varkappa)\leq\tau\varphi(\varkappa)$, then we get*

$$\begin{aligned}\varphi\left(\frac{u+v}{2}\right) &\leq \frac{1}{2}\varphi(u+v) \\ &\leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(v-u)^{\alpha}}\left[J^{\alpha}_{\left(\frac{u+v}{2}\right)+}\varphi(v)+J^{\alpha}_{\left(\frac{u+v}{2}\right)-}\varphi(u)\right] \\ &\leq \frac{\varphi(u)+\varphi(v)}{2}.\end{aligned}$$

which coincides with the inequalities for convex functions given in (6).

Theorem 2.10. *If $\varphi,\phi:[u,v]\subset[0,\infty)\to R$ are two continuous subadditive functions, $u,v\in I^{\circ}$ and $u<v$, then the following inequalities hold for Riemann-Liouville fractional integrals:*

$$\varphi(u+v)\phi(u+v) \leq \frac{2^{\alpha}\Gamma(\alpha+1)}{(v-u)^{\alpha}}\left[J^{\alpha}_{\left(\frac{u+v}{2}\right)+}\left[\varphi(v)\phi(v)\right]+J^{\alpha}_{\left(\frac{u+v}{2}\right)-}\left[\varphi(u)\phi(u)\right]\right]+I_1+I_2 \tag{29}$$

and

$$\begin{aligned}&\frac{2^{\alpha}\Gamma(\alpha+1)}{(v-u)^{\alpha}}\left[J^{\alpha}_{\left(\frac{u+v}{2}\right)+}\left[\varphi(v)\phi(v)\right]+J^{\alpha}_{\left(\frac{u+v}{2}\right)-}\left[\varphi(u)\phi(u)\right]\right] \\ \leq\ &\frac{2^{\alpha}\alpha}{u^{\alpha}}\int_0^{\frac{u}{2}}\varkappa^{\alpha-1}\varphi(\varkappa)\phi(\varkappa)d\varkappa+\frac{2^{\alpha}\alpha}{u^{\alpha}}\int_{\frac{u}{2}}^{u}(u-\varkappa)^{\alpha-1}\varphi(\varkappa)\phi(\varkappa)d\varkappa \\ &+\frac{2^{\alpha}\alpha}{v^{\alpha}}\int_0^{\frac{v}{2}}\varkappa^{\alpha-1}\varphi(\varkappa)\phi(\varkappa)d\varkappa+\frac{2^{\alpha}\alpha}{v^{\alpha}}\int_{\frac{v}{2}}^{v}(v-\varkappa)^{\alpha-1}\varphi(\varkappa)\phi(\varkappa)d\varkappa+I_1,\end{aligned} \tag{30}$$

where

$$\begin{aligned}I_1 &= \alpha\int_0^1 \tau^{\alpha-1}\left[\varphi\left(\frac{\tau}{2}u\right)\phi\left(\frac{2-\tau}{2}u\right)+\varphi\left(\frac{2-\tau}{2}u\right)\phi\left(\frac{\tau}{2}u\right)+\varphi\left(\frac{\tau}{2}v\right)\phi\left(\frac{2-\tau}{2}v\right)+\varphi\left(\frac{2-\tau}{2}v\right)\phi\left(\frac{\tau}{2}v\right)\right]d\tau,\\ I_2 &= \alpha\int_0^1 \tau^{\alpha-1}\left[\varphi\left(\frac{\tau}{2}u\right)\phi\left(\frac{\tau}{2}v\right)+\varphi\left(\frac{2-\tau}{2}u\right)\phi\left(\frac{2-\tau}{2}v\right)+\varphi\left(\frac{2-\tau}{2}v\right)\phi\left(\frac{2-\tau}{2}u\right)+\varphi\left(\frac{\tau}{2}v\right)\phi\left(\frac{\tau}{2}u\right)\right].\end{aligned}$$

Proof. Since φ and ϕ are subadditive functions, by using the subadditivity of φ and ϕ, we get

$$\begin{aligned}\varphi(u+v) &= \varphi\left(\frac{\tau}{2}u+\frac{2-\tau}{2}v+\frac{2-\tau}{2}u+\frac{\tau}{2}v\right)\\ &\le \varphi\left(\frac{\tau}{2}u+\frac{2-\tau}{2}v\right)+\varphi\left(\frac{2-\tau}{2}u+\frac{\tau}{2}v\right)\end{aligned}\tag{31}$$

and

$$\begin{aligned}\phi(u+v) &= \phi\left(\frac{\tau}{2}u+\frac{2-\tau}{2}v+\frac{2-\tau}{2}u+\frac{\tau}{2}v\right)\\ &\le \phi\left(\frac{\tau}{2}u+\frac{2-\tau}{2}v\right)+\phi\left(\frac{2-\tau}{2}u+\frac{\tau}{2}v\right).\end{aligned}\tag{32}$$

From inequalities (31) and (32), we have

$$\begin{aligned}\varphi(u+v)\phi(u+v) &\le \varphi\left(\frac{\tau}{2}u+\frac{2-\tau}{2}v\right)\phi\left(\frac{\tau}{2}u+\frac{2-\tau}{2}v\right)+\varphi\left(\frac{2-\tau}{2}u+\frac{\tau}{2}v\right)\phi\left(\frac{2-\tau}{2}u+\frac{\tau}{2}v\right)\\ &\quad+\varphi\left(\frac{\tau}{2}u+\frac{2-\tau}{2}v\right)\phi\left(\frac{2-\tau}{2}u+\frac{\tau}{2}v\right)+\varphi\left(\frac{2-\tau}{2}u+\frac{\tau}{2}v\right)\phi\left(\frac{\tau}{2}u+\frac{2-\tau}{2}v\right)\\ &\le \left[\varphi\left(\frac{\tau}{2}u+\frac{2-\tau}{2}v\right)\phi\left(\frac{\tau}{2}u+\frac{2-\tau}{2}v\right)+\varphi\left(\frac{2-\tau}{2}u+\frac{\tau}{2}v\right)\phi\left(\frac{2-\tau}{2}u+\frac{\tau}{2}v\right)\right]\\ &\quad+\left[\varphi\left(\frac{\tau}{2}u\right)\phi\left(\frac{2-\tau}{2}u\right)+\varphi\left(\frac{2-\tau}{2}u\right)\phi\left(\frac{\tau}{2}u\right)+\varphi\left(\frac{\tau}{2}v\right)\phi\left(\frac{2-\tau}{2}v\right)+\varphi\left(\frac{2-\tau}{2}v\right)\phi\left(\frac{\tau}{2}v\right)\right]\\ &\quad+\left[\varphi\left(\frac{\tau}{2}u\right)\phi\left(\frac{\tau}{2}v\right)+\varphi\left(\frac{2-\tau}{2}u\right)\phi\left(\frac{2-\tau}{2}v\right)+\varphi\left(\frac{\tau}{2}v\right)\phi\left(\frac{\tau}{2}u\right)+\varphi\left(\frac{2-\tau}{2}v\right)\phi\left(\frac{2-\tau}{2}u\right)\right].\end{aligned}\tag{33}$$

Multiplying both sides of the inequality (33) by $\tau^{\alpha-1}$ and integrating the resultant one with respect to τ over $[0,1]$, we obtain

$$\begin{aligned}\frac{1}{\alpha}\varphi(u+v)\phi(u+v) &\le \int_0^1\tau^{\alpha-1}\left[\varphi\left(\frac{\tau}{2}u+\frac{2-\tau}{2}v\right)\phi\left(\frac{\tau}{2}u+\frac{2-\tau}{2}v\right)\right.\\ &\quad\left.+\varphi\left(\frac{2-\tau}{2}u+\frac{\tau}{2}v\right)\phi\left(\frac{2-\tau}{2}u+\frac{\tau}{2}v\right)\right]d\tau\\ &\quad+\int_0^1\tau^{\alpha-1}\left[\varphi\left(\frac{\tau}{2}u\right)\phi\left(\frac{2-\tau}{2}u\right)+\varphi\left(\frac{2-\tau}{2}u\right)\phi\left(\frac{\tau}{2}u\right)\right.\\ &\quad\left.+\varphi\left(\frac{\tau}{2}v\right)\phi\left(\frac{2-\tau}{2}v\right)+\varphi\left(\frac{2-\tau}{2}v\right)\phi\left(\frac{\tau}{2}v\right)\right]d\tau\\ &\quad+\int_0^1\tau^{\alpha-1}\left[\varphi\left(\frac{\tau}{2}u\right)\phi\left(\frac{\tau}{2}v\right)+\varphi\left(\frac{2-\tau}{2}u\right)\phi\left(\frac{2-\tau}{2}v\right)\right.\\ &\quad\left.+\varphi\left(\frac{\tau}{2}v\right)\phi\left(\frac{\tau}{2}u\right)+\varphi\left(\frac{2-\tau}{2}v\right)\phi\left(\frac{2-\tau}{2}u\right)\right]d\tau\end{aligned}$$

and by changing the variables of integration, we have the inequality (29).

Since φ and ϕ are subadditive functions on I, then for $\tau\in[0,1]$, we have

$$\varphi\left(\frac{\tau}{2}u+\frac{2-\tau}{2}v\right)\le\varphi\left(\frac{\tau}{2}u\right)+\varphi\left(\frac{2-\tau}{2}v\right),\tag{34}$$

and

$$\phi\left(\frac{\tau}{2}u+\frac{2-\tau}{2}v\right)\leq\phi\left(\frac{\tau}{2}u\right)+\phi\left(\frac{2-\tau}{2}v\right). \tag{35}$$

From inequalities (34) and (35), we have

$$\begin{aligned}\varphi\left(\frac{\tau}{2}u+\frac{2-\tau}{2}v\right)\phi\left(\frac{\tau}{2}u+\frac{2-\tau}{2}v\right) \leq\ & \varphi\left(\frac{\tau}{2}u\right)\phi\left(\frac{\tau}{2}u\right)+\varphi\left(\frac{\tau}{2}u\right)\phi\left(\frac{2-\tau}{2}v\right)\\ &+\varphi\left(\frac{2-\tau}{2}v\right)\phi\left(\frac{\tau}{2}u\right)+\varphi\left(\frac{2-\tau}{2}v\right)\phi\left(\frac{2-\tau}{2}v\right).\end{aligned} \tag{36}$$

Similarly, we have

$$\begin{aligned}\varphi\left(\frac{2-\tau}{2}u+\frac{\tau}{2}v\right)\phi\left(\frac{2-\tau}{2}u+\frac{\tau}{2}v\right) \leq\ & \varphi\left(\frac{2-\tau}{2}u\right)\phi\left(\frac{2-\tau}{2}u\right)+\varphi\left(\frac{2-\tau}{2}u\right)\phi\left(\frac{\tau}{2}v\right)\\ &+\varphi\left(\frac{\tau}{2}v\right)\phi\left(\frac{2-\tau}{2}u\right)+\varphi\left(\frac{\tau}{2}v\right)\phi\left(\frac{\tau}{2}v\right).\end{aligned} \tag{37}$$

Adding the inequalities (36) and (37), we have

$$\begin{aligned}&\varphi\left(\frac{\tau}{2}u+\frac{2-\tau}{2}v\right)\phi\left(\frac{\tau}{2}u+\frac{2-\tau}{2}v\right)+\varphi\left(\frac{2-\tau}{2}u+\frac{\tau}{2}v\right)\phi\left(\frac{2-\tau}{2}u+\frac{\tau}{2}v\right)\\ \leq\ &\left[\varphi\left(\frac{\tau}{2}u\right)\phi\left(\frac{\tau}{2}u\right)+\varphi\left(\frac{2-\tau}{2}u\right)\phi\left(\frac{2-\tau}{2}u\right)+\varphi\left(\frac{2-\tau}{2}v\right)\phi\left(\frac{2-\tau}{2}v\right)+\varphi\left(\frac{\tau}{2}v\right)\phi\left(\frac{\tau}{2}v\right)\right]\\ &+\left[\varphi\left(\frac{\tau}{2}u\right)\phi\left(\frac{2-\tau}{2}v\right)+\varphi\left(\frac{2-\tau}{2}u\right)\phi\left(\frac{\tau}{2}v\right)+\varphi\left(\frac{2-\tau}{2}v\right)\phi\left(\frac{\tau}{2}u\right)+\varphi\left(\frac{\tau}{2}v\right)\phi\left(\frac{2-\tau}{2}u\right)\right].\end{aligned} \tag{38}$$

Multiplying both sides of the above inequality by $\tau^{\alpha-1}$ and integrating the resultant inequality with respect to τ over $[0,1]$, we have

$$\begin{aligned}&\int_0^1\tau^{\alpha-1}\left[\varphi\left(\frac{\tau}{2}u+\frac{2-\tau}{2}v\right)\phi\left(\frac{\tau}{2}u+\frac{2-\tau}{2}v\right)+\varphi\left(\frac{2-\tau}{2}u+\frac{\tau}{2}v\right)\phi\left(\frac{2-\tau}{2}u+\frac{\tau}{2}v\right)\right]d\tau\\ \leq\ &\int_0^1\tau^{\alpha-1}\left[\varphi\left(\frac{\tau}{2}u\right)\phi\left(\frac{\tau}{2}u\right)+\varphi\left(\frac{2-\tau}{2}u\right)\phi\left(\frac{2-\tau}{2}u\right)+\varphi\left(\frac{2-\tau}{2}v\right)\phi\left(\frac{2-\tau}{2}v\right)+\varphi\left(\frac{\tau}{2}v\right)\phi\left(\frac{\tau}{2}v\right)\right]d\tau\\ &+\int_0^1\tau^{\alpha-1}\left[\varphi\left(\frac{\tau}{2}u\right)\phi\left(\frac{2-\tau}{2}v\right)+\varphi\left(\frac{2-\tau}{2}u\right)\phi\left(\frac{\tau}{2}v\right)+\varphi\left(\frac{2-\tau}{2}v\right)\phi\left(\frac{\tau}{2}u\right)+\varphi\left(\frac{\tau}{2}v\right)\phi\left(\frac{2-\tau}{2}u\right)\right]\end{aligned}$$

and by changing the variables of integration, we get the inequality (30). □

Corollary 2.11. *Under the assumptions of Theorem 2.10, if we take $\varphi(\tau\varkappa)\leq\tau\varphi(\varkappa)$, then we get*

$$\begin{aligned}2\varphi\left(\frac{u+v}{2}\right)\phi\left(\frac{u+v}{2}\right) \leq\ & \frac{1}{2}\varphi(u+v)\phi(u+v)\\ \leq\ & \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(v-u)^{\alpha}}\left[J^{\alpha}_{\left(\frac{u+v}{2}\right)+}\varphi(v)\phi(v)+J^{\alpha}_{\left(\frac{u+v}{2}\right)-}\varphi(u)\phi(u)\right]\\ &+M(u,v)\frac{\alpha}{(\alpha+1)(\alpha+2)}+N(u,v)\left(\frac{\alpha}{\alpha+2}-\frac{\alpha}{\alpha+1}+\frac{1}{2}\right)\end{aligned}$$

and

$$\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(v-u)^{\alpha}}\left[J^{\alpha}_{\left(\frac{u+v}{2}\right)+}\varphi(v)\phi(v)+J^{\alpha}_{\left(\frac{u+v}{2}\right)-}\varphi(u)\phi(u)\right] \leq M(u,v)\left(\frac{\alpha}{\alpha+2}-\frac{\alpha}{\alpha+1}+\frac{1}{2}\right)+N(u,v)\frac{\alpha}{(\alpha+1)(\alpha+2)},$$

where $M(u,v)$ and $N(u,v)$ are defined by (10).

Theorem 2.12. *If a continuous function $\varphi : I=[0,\infty)\to R$ is subadditive, $u,v\in I^{\circ}$ and $u<v$, then the following inequalities hold for the Riemann-Liouville fractional integrals:*

$$\begin{aligned}
&\frac{1}{2}\varphi(u+v)\leq\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(v-u)^{\alpha}}\left[J_{u+}^{\alpha}\varphi\left(\frac{u+v}{2}\right)+J_{v-}^{\alpha}\varphi\left(\frac{u+v}{2}\right)\right] \qquad (39)\\
\leq\;&\frac{2^{\alpha-1}\alpha}{u^{\alpha}}\int_{0}^{\frac{u}{2}}\left(\frac{u}{2}-\varkappa\right)^{\alpha-1}\varphi(\varkappa)d\varkappa+\frac{2^{\alpha-1}\alpha}{u^{\alpha}}\int_{\frac{u}{2}}^{u}\left(\varkappa-\frac{u}{2}\right)^{\alpha-1}\varphi(\varkappa)d\varkappa\\
&+\frac{2^{\alpha-1}\alpha}{v^{\alpha}}\int_{0}^{\frac{v}{2}}\left(\frac{v}{2}-\varkappa\right)^{\alpha-1}\varphi(\varkappa)d\varkappa+\frac{2^{\alpha-1}\alpha}{v^{\alpha}}\int_{\frac{v}{2}}^{v}\left(\varkappa-\frac{v}{2}\right)^{\alpha-1}\varphi(\varkappa)d\varkappa
\end{aligned}$$

with $\alpha>0$.

Proof. By using the subadditivity of the function φ, we have

$$\begin{aligned}
\varphi(u+v) &= \varphi\left(\frac{1-\tau}{2}u+\frac{1+\tau}{2}v+\frac{1+\tau}{2}u+\frac{1-\tau}{2}v\right) \qquad (40)\\
&\leq \varphi\left(\frac{1-\tau}{2}u+\frac{1+\tau}{2}v\right)+\varphi\left(\frac{1+\tau}{2}u+\frac{1-\tau}{2}v\right).
\end{aligned}$$

Multiplying both sides of the above inequality by $\tau^{\alpha-1}$ and integrating the resultant inequality with respect to τ over $[0,1]$, we obtain

$$\begin{aligned}
\frac{1}{\alpha}\varphi(u+v) &\leq \int_{0}^{1}\tau^{\alpha-1}\left[\varphi\left(\frac{1-\tau}{2}u+\frac{1+\tau}{2}v\right)+\varphi\left(\frac{1+\tau}{2}u+\frac{1-\tau}{2}v\right)\right]d\tau\\
&= \frac{2^{\alpha}\Gamma(\alpha)}{(v-u)^{\alpha}}\left[J_{u+}^{\alpha}\varphi\left(\frac{u+v}{2}\right)+J_{v-}^{\alpha}\varphi\left(\frac{u+v}{2}\right)\right]
\end{aligned}$$

and the proof of the first inequality in (39) is completed.

For the proof of the second inequality in (39), first we note that φ is a subadditive function on I, then for $\tau\in[0,1]$ it yields

$$\varphi\left(\frac{1-\tau}{2}u+\frac{1+\tau}{2}v\right)\leq\varphi\left(\frac{1-\tau}{2}u\right)+\varphi\left(\frac{1+\tau}{2}v\right) \qquad (41)$$

and

$$\varphi\left(\frac{1+\tau}{2}u+\frac{1-\tau}{2}v\right)\leq\varphi\left(\frac{1+\tau}{2}u\right)+\varphi\left(\frac{1-\tau}{2}v\right). \qquad (42)$$

Adding the inequalities (41) and (42), we get

$$\begin{aligned}
&\varphi\left(\frac{1-\tau}{2}u+\frac{1+\tau}{2}v\right)+\varphi\left(\frac{1+\tau}{2}u+\frac{1-\tau}{2}v\right) \qquad (43)\\
\leq\;&\varphi\left(\frac{1-\tau}{2}u\right)+\varphi\left(\frac{1+\tau}{2}v\right)+\varphi\left(\frac{1+\tau}{2}u\right)+\varphi\left(\frac{1-\tau}{2}v\right).
\end{aligned}$$

Multiplying both sides of the above inequality by $\tau^{\alpha-1}$ and integrating the resultant inequality with respect to τ over $[0,1]$, we obtain

$$\begin{aligned}
&\int_{0}^{1}\tau^{\alpha-1}\left[\varphi\left(\frac{1-\tau}{2}u+\frac{1+\tau}{2}v\right)+\varphi\left(\frac{1+\tau}{2}u+\frac{1-\tau}{2}v\right)\right]d\tau\\
\leq\;&\int_{0}^{1}\tau^{\alpha-1}\left[\varphi\left(\frac{1-\tau}{2}u\right)+\varphi\left(\frac{1+\tau}{2}v\right)+\varphi\left(\frac{1+\tau}{2}u\right)+\varphi\left(\frac{1-\tau}{2}v\right)\right]d\tau
\end{aligned}$$

and changing the variables of integration, we have the second inequality of (39). □

Remark 2.13. *Under the assumptions of Theorem 2.12, if we take $\alpha = 1$, then we get the inequality (2).*

Corollary 2.14. *Under the conditions of Theorem 2.12, if we take $\varphi(\tau\varkappa) \leq \tau\varphi(\varkappa)$, then we get*

$$\begin{aligned}\varphi\left(\frac{u+v}{2}\right) &\leq \frac{1}{2}\varphi(u+v) \leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(v-u)^{\alpha}}\left[J_{u+}^{\alpha}\varphi\left(\frac{u+v}{2}\right)+J_{v-}^{\alpha}\varphi\left(\frac{u+v}{2}\right)\right]\\ &\leq \frac{\varphi(u)+\varphi(v)}{2}\end{aligned}$$

which coincides with the inequalities for convex functions given in (7).

Theorem 2.15. *If $\varphi, \phi : I = [0,\infty) \to R$ are two continuous subadditive functions, $u, v \in I^{\circ}$ and $u < v$, then the following inequalities hold for Riemann-Liouville fractional integrals:*

$$\begin{aligned}&\varphi(u+v)\phi(u+v)\\ \leq\ & \frac{2^{\alpha}\Gamma(\alpha+1)}{(v-u)^{\alpha}}\left[J_{u+}^{\alpha}\varphi\left(\frac{u+v}{2}\right)\phi\left(\frac{u+v}{2}\right)+J_{v-}^{\alpha}\varphi\left(\frac{u+v}{2}\right)\phi\left(\frac{u+v}{2}\right)\right]+\Im_1+\Im_2\end{aligned} \tag{44}$$

and

$$\begin{aligned}&\frac{2^{\alpha}\Gamma(\alpha+1)}{(v-u)^{\alpha}}\left[J_{u+}^{\alpha}\varphi\left(\frac{u+v}{2}\right)\phi\left(\frac{u+v}{2}\right)+J_{v-}^{\alpha}\varphi\left(\frac{u+v}{2}\right)\phi\left(\frac{u+v}{2}\right)\right]\\ \leq\ & \frac{2^{\alpha}\alpha}{u^{\alpha}}\int_0^{\frac{u}{2}}\left(\frac{u}{2}-\varkappa\right)^{\alpha-1}\varphi(\varkappa)\phi(\varkappa)d\varkappa+\frac{2^{\alpha}\alpha}{u^{\alpha}}\int_{\frac{u}{2}}^{u}\left(\varkappa-\frac{u}{2}\right)^{\alpha-1}\varphi(\varkappa)\phi(\varkappa)d\varkappa\\ &+\frac{2^{\alpha}\alpha}{v^{\alpha}}\int_0^{\frac{v}{2}}\left(\frac{v}{2}-\varkappa\right)^{\alpha-1}\varphi(\varkappa)\phi(\varkappa)d\varkappa+\frac{2^{\alpha}\alpha}{v^{\alpha}}\int_{\frac{v}{2}}^{v}\left(\varkappa-\frac{v}{2}\right)^{\alpha-1}\varphi(\varkappa)\phi(\varkappa)d\varkappa+\Im_1,\end{aligned} \tag{45}$$

where

$$\begin{aligned}\Im_1 &= \alpha\int_0^1\tau^{\alpha-1}\left[\varphi\left(\frac{1-\tau}{2}u\right)\phi\left(\frac{1+\tau}{2}u\right)+\varphi\left(\frac{1+\tau}{2}u\right)\phi\left(\frac{1-\tau}{2}u\right)\right.\\ &\quad\left.+\varphi\left(\frac{1+\tau}{2}v\right)\phi\left(\frac{1-\tau}{2}v\right)+\varphi\left(\frac{1-\tau}{2}v\right)\phi\left(\frac{1+\tau}{2}v\right)\right]d\tau,\\ \Im_2 &= \alpha\int_0^1\tau^{\alpha-1}\left[\varphi\left(\frac{1-\tau}{2}u\right)\phi\left(\frac{1-\tau}{2}v\right)+\varphi\left(\frac{1+\tau}{2}u\right)\phi\left(\frac{1+\tau}{2}v\right)\right.\\ &\quad\left.+\varphi\left(\frac{1-\tau}{2}v\right)\phi\left(\frac{1-\tau}{2}u\right)+\varphi\left(\frac{1+\tau}{2}v\right)\phi\left(\frac{1+\tau}{2}u\right)\right].\end{aligned}$$

Proof. Since φ and ϕ are subadditive functions, by using the subadditivity of φ and ϕ, we get

$$\begin{aligned}\varphi(u+v) &= \varphi\left(\frac{1-\tau}{2}u+\frac{1+\tau}{2}v+\frac{1+\tau}{2}u+\frac{1-\tau}{2}v\right)\\ &\leq \varphi\left(\frac{1-\tau}{2}u+\frac{1+\tau}{2}v\right)+\varphi\left(\frac{1+\tau}{2}u+\frac{1-\tau}{2}v\right)\end{aligned} \tag{46}$$

and

$$\begin{aligned}\phi(u+v) &= \phi\left(\frac{1-\tau}{2}u+\frac{1+\tau}{2}v+\frac{1+\tau}{2}u+\frac{1-\tau}{2}v\right)\\ &\leq \phi\left(\frac{1-\tau}{2}u+\frac{1+\tau}{2}v\right)+\phi\left(\frac{1+\tau}{2}u+\frac{1-\tau}{2}v\right).\end{aligned} \tag{47}$$

From the inequalities (46) and (47), we have

$$\begin{aligned}
\varphi(u+v)\phi(u+v) \leq\; & \varphi\left(\frac{1-\tau}{2}u+\frac{1+\tau}{2}v\right)\phi\left(\frac{1-\tau}{2}u+\frac{1+\tau}{2}v\right) \\
&+\varphi\left(\frac{1+\tau}{2}u+\frac{1-\tau}{2}v\right)\phi\left(\frac{1+\tau}{2}u+\frac{1-\tau}{2}v\right) \\
&+\varphi\left(\frac{1-\tau}{2}u+\frac{1+\tau}{2}v\right)\phi\left(\frac{1+\tau}{2}u+\frac{1-\tau}{2}v\right) \\
&+\varphi\left(\frac{1+\tau}{2}u+\frac{1-\tau}{2}v\right)\phi\left(\frac{1-\tau}{2}u+\frac{1+\tau}{2}v\right) \\
\leq\; & \left[\varphi\left(\frac{1-\tau}{2}u+\frac{1+\tau}{2}v\right)\phi\left(\frac{1-\tau}{2}u+\frac{1+\tau}{2}v\right)\right. \\
&\left.+\varphi\left(\frac{1+\tau}{2}u+\frac{1-\tau}{2}v\right)\phi\left(\frac{1+\tau}{2}u+\frac{1-\tau}{2}v\right)\right] \\
&+\left[\varphi\left(\frac{1-\tau}{2}u\right)\phi\left(\frac{1+\tau}{2}u\right)+\varphi\left(\frac{1+\tau}{2}u\right)\phi\left(\frac{1-\tau}{2}u\right)\right. \\
&\left.+\varphi\left(\frac{1-\tau}{2}v\right)\phi\left(\frac{1+\tau}{2}v\right)+\varphi\left(\frac{1+\tau}{2}v\right)\phi\left(\frac{1-\tau}{2}v\right)\right] \\
&+\left[\varphi\left(\frac{1-\tau}{2}u\right)\phi\left(\frac{1-\tau}{2}v\right)+\varphi\left(\frac{1+\tau}{2}u\right)\phi\left(\frac{1+\tau}{2}v\right)\right. \\
&\left.+\varphi\left(\frac{1-\tau}{2}v\right)\phi\left(\frac{1-\tau}{2}u\right)+\varphi\left(\frac{1+\tau}{2}v\right)\phi\left(\frac{1+\tau}{2}u\right)\right].
\end{aligned} \tag{48}$$

Multiplying both sides of the inequality (48) by $\tau^{\alpha-1}$ and integrating the resultant one with respect to τ over $[0,1]$, we obtain

$$\begin{aligned}
\frac{1}{\alpha}\varphi(u+v)\phi(u+v) \leq\; & \int_0^1 \tau^{\alpha-1}\left[\varphi\left(\frac{1-\tau}{2}u+\frac{1+\tau}{2}v\right)\phi\left(\frac{1-\tau}{2}u+\frac{1+\tau}{2}v\right)\right. \\
&\left.+\varphi\left(\frac{1+\tau}{2}u+\frac{1-\tau}{2}v\right)\phi\left(\frac{1+\tau}{2}u+\frac{1-\tau}{2}v\right)\right]d\tau \\
&+\int_0^1 \tau^{\alpha-1}\left[\varphi\left(\frac{1-\tau}{2}u\right)\phi\left(\frac{1+\tau}{2}u\right)+\varphi\left(\frac{1+\tau}{2}u\right)\phi\left(\frac{1-\tau}{2}u\right)\right. \\
&\left.+\varphi\left(\frac{1-\tau}{2}v\right)\phi\left(\frac{1+\tau}{2}v\right)+\varphi\left(\frac{1+\tau}{2}v\right)\phi\left(\frac{1-\tau}{2}v\right)\right]d\tau \\
&+\int_0^1 \tau^{\alpha-1}\left[\varphi\left(\frac{1-\tau}{2}u\right)\phi\left(\frac{1-\tau}{2}v\right)+\varphi\left(\frac{1+\tau}{2}u\right)\phi\left(\frac{1+\tau}{2}v\right)\right. \\
&\left.+\varphi\left(\frac{1-\tau}{2}v\right)\phi\left(\frac{1-\tau}{2}u\right)+\varphi\left(\frac{1+\tau}{2}v\right)\phi\left(\frac{1+\tau}{2}u\right)\right]d\tau
\end{aligned}$$

and by changing the variables of integration, we have the inequality (44).

Since φ and ϕ are subadditive functions on I, then for $\tau \in [0,1]$, we have

$$\varphi\left(\frac{1-\tau}{2}u+\frac{1+\tau}{2}v\right) \leq \varphi\left(\frac{1-\tau}{2}u\right)+\varphi\left(\frac{1+\tau}{2}v\right), \tag{49}$$

and

$$\phi\left(\frac{1-\tau}{2}u+\frac{1+\tau}{2}v\right) \leq \phi\left(\frac{1-\tau}{2}u\right)+\phi\left(\frac{1+\tau}{2}v\right). \tag{50}$$

From the inequalities (49) and (50), we have

$$\begin{aligned}\varphi\left(\frac{1-\tau}{2}u+\frac{1+\tau}{2}v\right)\phi\left(\frac{1-\tau}{2}u+\frac{1+\tau}{2}v\right) &\leq \varphi\left(\frac{1-\tau}{2}u\right)\phi\left(\frac{1-\tau}{2}u\right)+\varphi\left(\frac{1-\tau}{2}u\right)\phi\left(\frac{1+\tau}{2}v\right) \\ &\quad +\varphi\left(\frac{1+\tau}{2}v\right)\phi\left(\frac{1-\tau}{2}u\right)+\varphi\left(\frac{1+\tau}{2}v\right)\phi\left(\frac{1+\tau}{2}v\right).\end{aligned} \tag{51}$$

Similarly, we have

$$\begin{aligned}\varphi\left(\frac{1+\tau}{2}u+\frac{1-\tau}{2}v\right)\phi\left(\frac{1+\tau}{2}u+\frac{1-\tau}{2}v\right) &\leq \varphi\left(\frac{1+\tau}{2}u\right)\phi\left(\frac{1+\tau}{2}u\right)+\varphi\left(\frac{1+\tau}{2}u\right)\phi\left(\frac{1-\tau}{2}v\right) \\ &\quad +\varphi\left(\frac{1-\tau}{2}v\right)\phi\left(\frac{1+\tau}{2}u\right)+\varphi\left(\frac{1-\tau}{2}v\right)\phi\left(\frac{1-\tau}{2}v\right).\end{aligned} \tag{52}$$

Adding the inequalities (51) and (52), we have

$$\begin{aligned}&\varphi\left(\frac{1-\tau}{2}u+\frac{1+\tau}{2}v\right)\phi\left(\frac{1-\tau}{2}u+\frac{1+\tau}{2}v\right)+\varphi\left(\frac{1+\tau}{2}u+\frac{1-\tau}{2}v\right)\phi\left(\frac{1+\tau}{2}u+\frac{1-\tau}{2}v\right) \\ \leq\ &\left[\varphi\left(\frac{1-\tau}{2}u\right)\phi\left(\frac{1-\tau}{2}u\right)+\varphi\left(\frac{1+\tau}{2}u\right)\phi\left(\frac{1+\tau}{2}u\right)+\varphi\left(\frac{1+\tau}{2}v\right)\phi\left(\frac{1+\tau}{2}v\right)+\varphi\left(\frac{1-\tau}{2}v\right)\phi\left(\frac{1-\tau}{2}v\right)\right] \\ &+\left[\varphi\left(\frac{1-\tau}{2}u\right)\phi\left(\frac{1+\tau}{2}v\right)+\varphi\left(\frac{1+\tau}{2}u\right)\phi\left(\frac{1-\tau}{2}v\right)+\varphi\left(\frac{1+\tau}{2}v\right)\phi\left(\frac{1-\tau}{2}u\right)+\varphi\left(\frac{1-\tau}{2}v\right)\phi\left(\frac{1+\tau}{2}u\right)\right].\end{aligned} \tag{53}$$

Multiplying both sides of the above inequality by $\tau^{\alpha-1}$ and integrating the resultant inequality with respect to τ over $[0,1]$, we have

$$\begin{aligned}&\int_0^1\tau^{\alpha-1}\left[\varphi\left(\frac{1-\tau}{2}u+\frac{1+\tau}{2}v\right)\phi\left(\frac{1-\tau}{2}u+\frac{1+\tau}{2}v\right)+\varphi\left(\frac{1+\tau}{2}u+\frac{1-\tau}{2}v\right)\phi\left(\frac{1+\tau}{2}u+\frac{1-\tau}{2}v\right)\right]d\tau \\ \leq\ &\int_0^1\tau^{\alpha-1}\left[\varphi\left(\frac{1-\tau}{2}u\right)\phi\left(\frac{1-\tau}{2}u\right)+\varphi\left(\frac{1+\tau}{2}u\right)\phi\left(\frac{1+\tau}{2}u\right)+\varphi\left(\frac{1+\tau}{2}v\right)\phi\left(\frac{1+\tau}{2}v\right)+\varphi\left(\frac{1-\tau}{2}v\right)\phi\left(\frac{1-\tau}{2}v\right)\right]d\tau \\ &+\int_0^1\tau^{\alpha-1}\left[\varphi\left(\frac{1-\tau}{2}u\right)\phi\left(\frac{1+\tau}{2}v\right)+\varphi\left(\frac{1+\tau}{2}u\right)\phi\left(\frac{1-\tau}{2}v\right)+\varphi\left(\frac{1+\tau}{2}v\right)\phi\left(\frac{1-\tau}{2}u\right)+\varphi\left(\frac{1-\tau}{2}v\right)\phi\left(\frac{1+\tau}{2}u\right)\right]\end{aligned}$$

and changing the variables of integration, we get the inequality (45). □

Corollary 2.16. *Under the assumptions of Theorem 2.15, if we take $\varphi(\tau\varkappa)\leq\tau\varphi(\varkappa)$, then we get*

$$\begin{aligned}2\varphi\left(\frac{u+v}{2}\right)\phi\left(\frac{u+v}{2}\right) &\leq \frac{1}{2}\varphi(u+v)\phi(u+v) \\ &\leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(v-u)^{\alpha}}\left[J_{u+}^{\alpha}\varphi\left(\frac{u+v}{2}\right)\phi\left(\frac{u+v}{2}\right)+J_{v-}^{\alpha}\varphi\left(\frac{u+v}{2}\right)\phi\left(\frac{u+v}{2}\right)\right] \\ &\quad +M(u,v)\frac{\alpha}{(\alpha+1)(\alpha+2)}+N(u,v)\left(\frac{\alpha}{\alpha+2}-\frac{\alpha}{\alpha+1}+\frac{1}{2}\right)\end{aligned}$$

and

$$\begin{aligned}&\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(v-u)^{\alpha}}\left[J_{u+}^{\alpha}\varphi\left(\frac{u+v}{2}\right)\phi\left(\frac{u+v}{2}\right)+J_{v-}^{\alpha}\varphi\left(\frac{u+v}{2}\right)\phi\left(\frac{u+v}{2}\right)\right] \\ \leq\ &M(u,v)\left(\frac{\alpha}{\alpha+2}-\frac{\alpha}{\alpha+1}+\frac{1}{2}\right)+N(u,v)\frac{\alpha}{(\alpha+1)(\alpha+2)},\end{aligned}$$

where $M(u,v)$ and $N(u,v)$ are defined by (10).

3. Concluding Remarks

In this investigation, we proved the fractional Hermite-Hadamard inequalities and related inequalities for subadditive functions by using the Riemann-Liouville fractional integrals. We also prove that the results given in the current research are transformed into some existing results by assuming $\alpha = 1$ and $\varphi(\tau\varkappa) \leq \tau\varphi(\varkappa)$ in the main results. It is an interesting and new problem that the upcoming researchers can obtain similar inequalities for different type of fractional integral operators in their future work.

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