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Existence, Compactness, Estimates of Eigenvalues and s-Numbers of a Resolvent for a Linear Singular Operator of the Korteweg-de Vries Type

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Abstract. In this paper, we consider a linear operator of the Korteweg-de Vries type

$$Lu = \frac{\partial u}{\partial y} + R_2(y)\frac{\partial^3 u}{\partial x^3} + R_1(y)\frac{\partial u}{\partial x} + R_0(y)u$$

initially defined on $C^{\infty}_{0,\pi}(\overline{\Omega})$, where $\overline{\Omega} = \{(x, y) : -\pi \le x \le \pi, -\infty < y < \infty\}$. $C^{\infty}_{0,\pi}(\overline{\Omega})$ is a set of infinitely differentiable compactly supported function with respect to a variable y and satisfying the conditions:

$$u_x^{(i)}(-\pi, y) = u_x^{(i)}(\pi, y), \quad i = 0, 1, 2.$$

With respect to the coefficients of the operator L , we assume that these are continuous functions in $R(-\infty, +\infty)$ and strongly growing functions at infinity.

In this paper, we proved that there exists a bounded inverse operator and found a condition that ensures the compactness of the resolvent under some restrictions on the coefficients in addition to the above conditions. Also, two-sided estimates of singular numbers (s-numbers) are obtained and an example is given of how these estimates allow finding estimates of the eigenvalues of the considered operator.

1. Introduction. Formulation of results. Examples

The solvability of boundary value problems for differential equations of odd order and, in particular, for the Korteweg-de Vries equation is devoted to a significant literature [1-9] and the papers cited there.

In this paper, in contrast to those interesting papers, we consider the existence, compactness and estimates of the eigenvalues and s-numbers of the resolvent of a class of the Korteweg-de Vries type linear singular operators in the case of an unbounded domain with strongly increasing coefficients.

In the paper, we consider the differential operator

$$Lu + \mu u = \frac{\partial u}{\partial y} + R_2(y)\frac{\partial^3 u}{\partial x^3} + R_1(y)\frac{\partial u}{\partial x} + R_0(y)u + \mu u \tag{1}$$

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initially defined on $C^{\infty}_{0,\pi}(\overline{\Omega})$, where $\overline{\Omega} = \{(x, y) : -\pi \le x \le \pi, -\infty < y < \infty\}$, $\mu \ge 0$. $C^{\infty}_{0,\pi}(\overline{\Omega})$ is the set of infinitely differentiable and compactly supported functions with respect to the variable y and satisfying the conditions:

$$u_x^{(i)}(-\pi, y) = u_x^{(i)}(\pi, y), \quad i = 0, 1, 2. \tag{2}$$

Further assume that the coefficients $R_0(y), R_1(y), R_2(y)$ satisfy the following conditions:
i) $R_0(y) \ge \delta_0 > 0,\ R_1(y) \ge \delta_1 > 0,\ -R_2(y) \ge \delta_2 > 0$ are continuous functions in $R = (-\infty, +\infty)$;
ii) $\mu_0 = \sup\limits_{|y-t|\le 1} \frac{R_0(y)}{R_0(t)} < \infty,\ \mu_1 = \sup\limits_{|y-t|\le 1} \frac{R_1(y)}{R_1(t)} < \infty,\ \mu_2 = \sup\limits_{|y-t|\le 1} \frac{R_2(y)}{R_2(t)} < \infty.$

The operator $L + \mu I$ admits closure in the space $L_2(\Omega)$, which is also denoted by $L + \mu I$.

The indicated operator generates the so-called periodic problem without initial conditions. As you know, if the boundary regime operates sufficiently long, then due to the friction inherent in any real physical system, the influence of the initial data weakens over time. Thus, we arrive at a problem without initial conditions [10].

Theorem 1.1. *Let the condition i) be fulfilled. Then the operator $L + \mu I$ is continuously invertible in the space $L_2(\Omega)$ for $\mu \ge 0$ and the equality*

$$u(x, y) = (L + \mu I)^{-1} f = \sum_{n=-\infty}^{\infty} (l_n + \mu I)^{-1} f_n(y) e^{inx}, \tag{3}$$

holds, where $f(x, y) \in L_2(\Omega)$, $f(x, y) = \sum\limits_{n=-\infty}^{\infty} f_n(y) \cdot e^{inx}$, $f_n(y) =< f(x, y), \ e^{inx} >$, $i^2 = -1$, $< \cdot, \cdot >$ is a scalar product and

$$(l_n + \mu I)z = z'(y) + (-in^3 R_2(y) + inR_1(y) + R_0(y) + \mu)z, \ \ z \in D(l_n)$$

Defination 1.1 We say the operator L is separable in space $L_2(\Omega)$ if the estimate

$$\left\|\frac{\partial u}{\partial y}\right\|_2 + \left\|R_2(y)\frac{\partial^3 u}{\partial x^3}\right\|_2 + \left\|R_1(y)\frac{\partial u}{\partial x}\right\|_2 + \left\|R_0(y)u\right\|_2 \le C(\|Lu\|_2 + \|u\|_2),$$

holds for $u \in D(L)$, where C is independent of $u(x, y)$, and $\|\cdot\|_2$ is the norm of $L_2(\Omega)$.

Theorem 1.2. *Let the conditions i) - ii) be fulfilled. Then the operator L is separable.*

Example 1. Let $R_0(y) = |y| + 1$, $R_1(y) = e^{|y|}$, $R_2(y) = -10 \cdot e^{|y|}$, $-\infty < y < \infty$. It is easy to verify that all the conditions of Theorem 1.2 are satisfied. Consequently, the operator L is separable, i.e.

$$\left\|\frac{\partial u}{\partial y}\right\|_2 + \left\|-10 \cdot e^{|y|}\frac{\partial^3 u}{\partial x^3}\right\|_2 + \left\|e^{|y|}\frac{\partial u}{\partial x}\right\|_2 + \left\|(|y| + 1)u\right\|_2 \le C(\|Lu\|_2 + \|u\|_2),$$

where C is a constant.

Theorem 1.3. *Let the conditions i)- ii) be fulfilled. Then the resolvent of the operator $L + \lambda I$, $\lambda \ge 0$ is compact if and only if*

$$\lim_{|y|\to\infty} R_0(y) = \infty. \tag{*}$$

Definition [11]. Let A be a linear completely continuous operator and let $|A| = \sqrt{A * A}$. The eigenvalues of $|A|$ are called the s-numbers of the operator A.

The non-zero s-numbers of the operator $(L + \mu I)^{-1}$ be numbered according to decreasing magnitude and observing their multiplicities and so

$$S_k((L + \mu I)^{-1}) = \lambda_k(|(L + \mu I)^{-1}|), \ \ k = 1, 2, ...$$

We introduce the counting function $N(\lambda) = \sum\limits_{S_k > \lambda} 1$, of those S_k greater than $\lambda > 0$.

Theorem 1.4. *Let the conditions i) - ii) be fulfilled. Then the estimates*

$$c^{-1}\sum_{n=-\infty}^{\infty}\lambda^{-1}mes(y\in\mathbb{R}:Q_n(y)\le c^{-1}\lambda^{-1})\le N(\lambda)\le c\sum_{n=-\infty}^{\infty}\lambda^{-1}mes(y\in\mathbb{R}:Q_n(y)\le c^{-1}\lambda^{-1})$$

hold, where $Q_n(y)=|-in^3R_2(y)+inR_1(y)+R_0(y)|$ *and* $c>0$ *is a constant not depending on* $Q_n(y)$ *and* λ.

Example 2. In this example we will show how Theorem 1.4 allows to obtain estimates of the eigenvalues for the operator $(L+\mu I)^{-1}$.

Consider the operator:

$$(L+\mu I)u=\frac{\partial u}{\partial y}+(-|y|+1)\frac{\partial^3 u}{\partial x^3}+(|y|+1)\frac{\partial u}{\partial x}+(|y|+1)u+\mu u \tag{4}$$

$u\in D(L)$, $\mu\ge 0$. From equality (3) it follows that if s is a singular point of the operator $(L+\mu I)^{-1}$, then s is a singular number of one of the operators $(l_n+\mu I)^{-1}$ ($n=0,\pm1,\pm2,...$) , and vice versa. Therefore, taking this into account, further, we denote by $S_{k,n}(k=1,2,...)$ the singular values of the operator $(l_n+\mu I)^{-1}$ ($n=0,\pm1,\pm2,...$) for $\mu\ge 0$. According to Theorem 1.4, we have

$$\frac{c^{-1}}{(|n|+1)^{3/2}k^{1/2}}\le S_{k,n}\le\frac{c}{(|n|+1)^{3/2}k^{1/2}},\quad k=1,2,...,\quad n=0,\pm1,\pm2,... \tag{5}$$

Now, suppose that the operator $(l_n+\mu I)^{-1}$ has an infinite number of eigenvalues, then from estimate (5) and Weyl's inequality [11], we obtained that

$$|\lambda_{k,n}|^k\le\prod_{j=1}^{k}|\lambda_{j,n}|\le\prod_{j=1}^{k}S_{j,n}\le c^k(k!)^{-\frac{1}{2}}\cdot\frac{1}{(|n|+1)^{\frac{3}{2}}}$$

Further, using the inequality $e^k\cdot k!\ge k^k$ ($k=1,2,...$), we obtain the estimate of the eigenvalues:

$$|\lambda_{k,n}|\le\frac{c\cdot e^{\frac{1}{2}}k^{-\frac{1}{2}}}{(|n|+1)^{\frac{3}{2}}},\quad k=1,2,...$$

2. The existence of the resolvent. Proof of Theorem 1.1

Lemma 2.1. *Let the condition i) be fulfilled and* $\mu\ge 0$. *Then the inequality*

$$\left\|(L+\mu I)u\right\|_{L_2(\Omega)}\ge(\delta_0+\mu)\|u\|_{L_2(\Omega)},$$

holds for all $u\in D(L)$, *where* $\delta_0>0$.

Proof. Compose the scalar product $<(L+\lambda I)u,u>$, $u\in C^\infty_{0,\pi}(\Omega)$. Integrating by parts and taking into account that terms outside the integral vanish by virtue of $u\in C^\infty_{0,\pi}(\Omega)$, we obtain

$$\left\|(L+\mu I)u\right\|_{L_2(\Omega)}\ge(\delta_0+\mu)\|u\|_{L_2(\Omega)}.$$

Since the norm is continuous, the last estimate holds for all $u\in D(L)$. Lemma 2.1 is proved.

It is easy to verify by direct computations that the operator (1) in $L_2(\Omega)$ can be reduced using the Fourier method to the study of the following operator

$$(l_n+\mu I)z=z'(y)+(-in^3R_2(y)+inR_1(y)+R_0(y)+\mu)z(y), \tag{6}$$

$z \in D(l_n)$, $n = 0, \pm 1, \pm 2, \ldots$.

Here we present a number of statements that reduce the existence and compactness of the resolvent of the operator l_n with strongly increasing coefficients to the case of an operator with periodic bounded coefficients.

Consider the operator

$$(l_{n,j} + \mu I)z = z'(y) + (-in^3 R_{2,j}(y) + inR_{1,j}(y) + R_{0,j}(y) + \mu)z(y),$$

where $R_{2,j}(y),\ R_{1,j}(y),\ R_{0,j}(y)$ are bounded periodic functions of the same period, obtained by the continuation of $R_2(y),\ R_1(y),\ R_0(y)$, from $\Delta_j = (j-1, j+1),\ j \in Z$ to all $R = (-\infty, \infty)$.

The operator $l_{n,j} + \mu I$ admits closure in $L_2(\Omega)$, which is also denote by $l_{n,j} + \mu I$.

Lemma 2.2. *Let the condition i) be fulfilled. Then the estimate*

$$\left\|(l_{n,j} + \mu I)z\right\|_2 \geq (\delta_0 + \mu)\left\|z\right\|_2$$

holds for all $z(y) \in D(l_{n,j} + \mu I)$, where $\|\cdot\|_2$ is the norm of $L_2(R)$.

Proof. Let $z(y) \in C_0^\infty(R),\ z(y) = u(y) + i\vartheta(y)$. Then the equality

$$< (l_{n,j} + \mu I)z, z >= i(2\int_{-\infty}^{\infty} u\vartheta' dy + \int_{-\infty}^{\infty} (-n^3 R_{2,j}(y) + nR_{1,j}(y))|z|^2 dy) + \int_{-\infty}^{\infty} (R_{0,j}(y) + \mu)|z|^2 dy$$

holds. Hence, using the properties of complex numbers and taking into account that $R_{0,j}(y)$ does not change sign, we obtain

$$| < (l_{n,j} + \mu I)z, z > | \geq \int_{-\infty}^{\infty} |R_{0,j}(y) + \mu|\ |z|^2 dy.$$

Hence, using the continuity of the scalar product for all $z(y) \in D(l_{n,j} + \mu I)$, we have

$$\left\|(l_{n,j} + \mu I)z\right\|_2 \geq (\delta_0 + \mu)\left\|z\right\|_2.$$

Lemma is proved.

Lemma 2.3. *Let the condition i) be fulfilled.Then the operator $(l_{n,j} + \mu I)$ has a continuous inverse operator $(l_{n,j} + \mu I)^{-1}$ defined on the whole $L_2(R)$.*

Proof. Lemma 2.3 is proved in the same way as Lemma 2.2 of [12].

Let $\{\varphi_j\}_{j=-\infty}^{\infty} \in C_0^\infty(R)$ be a set of functions such that $\varphi_j(y) \geq 0$, $\sup p\varphi_j \subseteq \Delta_j (j \in Z)$, $\Delta_j = (j-1, j+1)$, $\sum\limits_{j=-\infty}^{\infty} \varphi_j^2(y) = 1$.

Here we note immediately that any point $y \in R$ can belong to no more than three segments from the system of segments $\{supp\varphi_j\}$[13,14].

Assume

$$K_\mu f = \sum_{j=-\infty}^{\infty} \varphi_j(y)(l_{n,j} + \mu I)^{-1}\varphi_j f,$$

$$B_\mu f = \sum_{j=-\infty}^{\infty} \varphi_j'(y)(l_{n,j} + \mu I)^{-1}\varphi_j f, \quad f \in C_0^\infty(R), \quad \mu \geq 0.$$

It is easy to verify that

$$(l_n + \mu I)K_\mu f = f + \sum_j \varphi_j'(y)(l_{n,j} + \mu I)^{-1}\varphi_j f, \tag{7}$$

where $(l_n + \mu I)z = -z'(y) + (-in^3 R_2(y) + inR_1(y) + R_0(y) + \mu)z,\ z \in D(l_n)$.

Lemma 2.4. *Let the condition i) be fulfilled. Then there is a number* $\mu_0 > 0$ *such that* $\left\|B_\mu\right\|_{2\to 2} < 1$ *for all* $\mu \geq \mu_0$.

Proof. Repeating the computations and arguments used in the proof of Lemma 3.2 of [12], we obtain the proof of Lemma 2.4.

Lemma 2.5. *Let the condition i) be fulfilled. Then the estimate*

$$\left\|(l_n + \mu I)z\right\|_2 \geq (\delta_0 + \mu)\,\|z\|_2\,.$$

holds for all $z \in D(l_n)$.

Lemma 2.5 is proved in exactly the same way as Lemma 2.2.

Lemma 2.6. *Let the condition i) be fulfilled. Then the operator* $l_n + \mu I$ *for* $\mu \geq \mu_0$ *is boundedly invertible and the equality*

$$(l_n + \mu I)^{-1} = K_\mu (I - B_\mu)^{-1}. \tag{8}$$

holds. Here μ_0 *is the number in Lemma 2.4.*

The proof of Lemma 2.6 follows from the representation (7) and Lemmas 2.4 and 2.5.

Lemma 2.7. *[15]. Let the operator* $L + \mu_0 I$, $(\mu_0 > 0)$ *is boundedly invertible in* $L_2(\Omega)$ *and the estimate* $\left\|(L + \mu I)u\right\|_{L_2(\Omega)} \geq c\,\|u\|_{L_2(\Omega)}$, $u \in D(L + \mu I)$ *holds for* $\mu \in [0, \mu_0]$. *Then the operator* $L : L_2(\Omega) \to L_2(\Omega)$ *is also boundedly invertible.*

Proof of Theorem 1.1. Lemma 2.6 implies that

$$u_k(x,y) = \sum_{n=-k}^{k} (l_n + \mu I)^{-1} f_n(y) e^{inx} \tag{9}$$

is the solution of the problem

$$(L + \mu I)u_k(x,y) = f_k(x,y),$$

$$u_k^{(i)}(-\pi, y) = u_k^{(i)}(\pi, y), \;\; i = 0,1,2,$$

where $f_k(x,y) \xrightarrow{L_2} f(x,y)$, $f_k(x,y) = \sum\limits_{n=-k}^{k} f_n(y)\cdot e^{inx}$, $i^2 = -1$, $(l_n + \mu I)^{-1}$ is the inverse operator to the operator $l_n + \mu I$. Using Lemma 2.1, we obtain

$$\|u_k - u_m\|_{L_2(\Omega)} \leq \frac{1}{\delta_0 + \mu}\left\|f_k - f_m\right\|_{L_2(\Omega} \to 0, \;\; as \;\; k,m \to \infty.$$

Hence, by the completeness of the space $L_2(\Omega)$, it follows that there exist a unique function $u \in L_2(\Omega)$ such that

$$u_k(x,y) \xrightarrow{L_2} u_m(x,y), \; as \;\; k,m \to \infty \tag{10}$$

It follows from (9) and (10) that

$$u(x,y) = (L + \mu I)^{-1} f(x,y) = \sum_{n=-\infty}^{\infty} (l_n + \mu I)^{-1} f_n(y)\cdot e^{inx} \tag{11}$$

is a strong solution for the problem

$$(L + \mu I)u = f \tag{12}$$

$$u_x^{(i)}(-\pi, y) = u_x^{(i)}(\pi, y), \quad i = 0,1,2. \tag{13}$$

for any $f \in L_2(\Omega)$.

Let us recall the definition of a strong solution. The function $u \in L_2(\Omega)$ is called a strong solution of the problem (12)-(13) if there exists a sequence $\{u_k\}_{k=1}^{\infty} \subset C_{0,\pi}^{\infty}(\Omega)$ such that

$$\|u_k - u\|_2 \to 0, \ \ \left\|(L+\mu I)u_k - f\right\|_2 \to 0, \ \ as \ \ k \to \infty.$$

Hence, it is easy to verify that formula (11) is the inverse operator to the closed operator $L + \mu I$.

Now it follows from Lemmas 2.1, 2.7 and equality (11) that Theorem 1.1 is holds for all $\mu \geq 0$. Theorem 1.1 is completely proved.

3. Proof of Theorem 1.2.

In order to prove separability (maximal regularity of solutions), we first give a series of lemmas that reduce the question of separability of an operator with unbounded coefficients to the case of an operator with periodic bounded coefficients.

Lemma 3.1. *Let* $z(y) \in D(l_{n,j} + \mu I)$ *and* $z(y) = u(y) + i\vartheta(y)$*, then* $in^3 R_2(y) z(y) \in L_2(R)$ *if and only if* $n^3 R_2(y) u(y) \in L_2(R)$ *and* $n^3 R_2(y)\vartheta(y) \in L_2(R)$*.*

Proof. The proof follows from the property of complex numbers. Let $in^3 R_2(y) z(y) \in L_2(R)$. Then

$$\left\|in^3 R_2(y) z(y)\right\|_2^2 = \left\|n^3 R_2(y) u\right\|_2^2 + \left\|n^3 R_2(y)\vartheta\right\|_2^2$$

Remark. This Lemma is also holds for $inR_1(y)z(y)$.

By virtue of Lemma 3.1 we consider the operator

$$(l_{n,j} + \mu I)u = u'(y) + (-in^3 R_{2,j}(y) + inR_{1,j}(y) + R_{0,j}(y) + \mu)u,$$

in the set of infinitely differentiable, compactly supported and real-valued functions and the set is also denoted by $C_0^{\infty}(R)$, where $R_{0,j}(y)$, $R_{1,j}(y)$, $R_{2,j}(y)$ are bounded periodic coefficients of the same period $\Delta_j = (j-1, j+1)$, $j = \pm 0, \pm 1, \pm 2....$

Lemma 3.2. *Let the condition i) be fulfilled. Then the estimates:*

$$\left\|(l_{n,j} + \mu I)u(y)\right\|_2 \geq R_0(y_j)\, \|u\|_2\,, \ \ n = 0, \pm 1, \pm 2..., \ \ where \ \ R_0(y_j) = \min_{y \in \overline{\Delta}_j} R_{0,j}(y);$$

$$\left\|(l_{n,j} + \mu I)u(y)\right\|_2 \geq |n| R_1(\overline{y_j})\, \|u\|_2\,, \quad n = \pm 1, \pm 2..., \ \ where \ \ R_1(\overline{y_j}) = \min_{y \in \overline{\Delta}_j} R_{1,j}(y);$$

$$\left\|(l_{n,j} + \mu I)u(y)\right\|_2 \geq |n|^3 R_2(\overline{\overline{y_j}})\, \|u\|_2 \quad n = \pm 1, \pm 2..., \ \ where \ \ R_2(\overline{\overline{y_j}}) = \min_{y \in \overline{\Delta}_j} |R_{2,j}(y)|,$$

hold for all $u(y) \in D(l_{n,j} + \mu I)$*.*

Proof. Taking the fulfillment of the equality $\int_{-\infty}^{\infty} u'(y)u(y)dy = 0$ for all $u(y) \in C_0^{\infty}(R)$ into account, we find that

$$| < (l_{n,j} + \mu I)u, u > | = | \int_{-\infty}^{\infty} (-in^3 R_{2,j}(y) + inR_{1,j}(y) + R_{0,j}(y) + \mu)|u|^2 dy|,$$

Hence, using the Cauchy inequality, we have

$$\left\|l_{n,j} + \mu I)u(y)\right\|_2 \geq R_0(y_j)\, \|u\|_2\,, \ \ n = 0, \pm 1, \pm 2...;$$

$$\left\|(l_{n,j} + \mu I)u(y)\right\|_2 \geq |n| R_1(\overline{y_j})\, \|u\|_2\,, \quad n = \pm 1, \pm 2...;$$

$$\left\|(l_{n,j} + \mu I)u(y)\right\|_2 \geq |n|^3 R_2(\overline{\overline{y_j}})\, \|u\|_2 \quad n = \pm 1, \pm 2....$$

Here we take into account that the coefficients $R_0(y)$, $R_1(y)$, $R_2(y)$ do not change sign for $y \in R$. Lemma 3.2. is proved.

Lemma 3.3. *Let the condition i) be fulfilled and $\mu \geq \mu_0$, $\alpha = 0, 1, 2, 3$, $p(y)$ be a continuous function defined on R. Then the estimate*

$$\left\|p(y)|n|^{\alpha}(l_n + \mu I)^{-1}\right\|^2_{L_2(R)\to L_2(R)} \leq c(\mu) \sup_{j\in Z} \left\|p(y)|n|^{\alpha}\varphi_j(l_{n,j} + \mu I)^{-1}\right\|^2_{L_2(\Delta_j)\to L_2(\Delta_j)}. \tag{14}$$

holds.

Proof. Let $f \in C_0^{\infty}(R)$. From the representation (8), taking the properties of the functions φ_j $(j \in Z)$ into account, we have:

$$\left\|p(y)|n|^{\alpha}(l_n + \mu I)^{-1} f\right\|_{L_2(R)} \leq \sum_{j=-\infty}^{\infty} \int_{-\infty}^{\infty} |\sum_{k=j-1}^{j+1} p(y)|n|^{\alpha}\varphi_k(l_{n,k} + \mu I)^{-1}\varphi_k(I - B_{\mu})^{-1} f|^2 dy.$$

Hence, applying the inequality $(a + b + c)^2 \leq 3(a^2 + b^2 + c^2)$ once again and using Lemma 2.6, we obtain the estimate (14). Lemma 3.3 is proved.

Lemma 3.4. *Let the conditions i)-ii) be fulfilled. Then the following estimates:*

$$\left\|R_0(y)(l_n + \mu I)^{-1}\right\|_{L_2(R)\to L_2(R)} \leq C < \infty; \quad n = 0, \pm 1, \pm 2...;$$

$$\left\|R_1(y)|n|(l_n + \mu I)^{-1}\right\|_{L_2(R)\to L_2(R)} \leq C_1 < \infty; \quad n = \pm 1, \pm 2...;$$

$$\left\|R_2(y)|n|^3(l_n + \mu I)^{-1}\right\|_{L_2(R)\to L_2(R)} \leq C_2 < \infty; \quad n = \pm 1, \pm 2...;$$

hold, where C_0, C_1, C_2 are independent of n $(n = 0, \pm 1, \pm 2...)$.

The proof of Lemma 3.4 follows from Lemmas 3.2 and 3.3.

Proof of Theorem 1.2 Using the representation (11), we find that

$$\left\|R_0(y)(L + \mu I)^{-1} f\right\|^2_{L_2(\Omega)} = \left\|\sum_{n=-\infty}^{\infty} R_0(y)(l_n + \mu I)^{-1} f_n(y) \cdot e^{inx}\right\|^2_{L_2(\Omega)} \leq 2\pi \sum_{n=-\infty}^{\infty} \left\|R_0(y)(l_n + \mu I)^{-1} f_n(y)\right\|^2_{L_2(\Omega)} \leq$$

$$\leq 2\pi \sum_{n=-\infty}^{\infty} \left\|R_0(y)(l_n + \mu I)^{-1}\right\|^2_{L_2(\Omega)\to L_2(\Omega)} \cdot \left\|f_n(y)\right\|^2_{L_2(\Omega)}.$$

Using the last inequality and Lemma 3.4, we obtain

$$\left\|R_0(y)u(x,y)\right\|^2_{L_2(\Omega)} = \left\|R_0(y)(L + \mu I)^{-1} f\right\|^2_{L_2(\Omega)} \leq 2\pi \cdot C^2 \sum_{n=-\infty}^{\infty} \left\|f_n(y)\right\|^2_{L_2(\Omega)} \leq C \left\|f\right\|^2_{L_2(\Omega)}.$$

From here we finally have

$$\left\|R_0(y)u(x,y)\right\|^2_{L_2(\Omega)} \leq C \left\|(L + \mu I)u\right\|^2_{L_2(\Omega)}, \tag{15}$$

where $(L + \mu I)u = f(x,y)$.

Similarly, using Lemma 3.4, we have

$$\left\|R_1(y)\frac{\partial u}{\partial x}\right\|^2_{L_2(\Omega)} \leq C_1 \left\|(L + \mu I)u\right\|^2_{L_2(\Omega)}; \tag{16}$$

$$\left\|R_2(y)\frac{\partial^3 u}{\partial x^3}\right\|^2_{L_2(\Omega)} \leq C_2 \left\|(L + \mu I)u\right\|^2_{L_2(\Omega)}. \tag{17}$$

It is easy to verify that

$$\left\|\frac{\partial u}{\partial y}\right\|_{L_2(\Omega)} = \left\|(L+\mu I)u - R_2(y)\frac{\partial^3 u}{\partial x^3} - R_1(y)\frac{\partial u}{\partial x} - R_0(y)u - \mu u\right\|_{L_2(\Omega)}.$$

Hence, from inequalities (15)-(17) we obtain

$$\left\|\frac{\partial u}{\partial y}\right\|_{L_2(\Omega)} \le C_3 \left\|(L+\mu I)u\right\|_{L_2(\Omega)}, \tag{18}$$

where $C_3 > 0$ is a constant number.

The inequalities (15)-(18) prove Theorem 1.2.

4. Proof of Theorem 1.3. Compactness of the resolvent

Lemma 4.1. *Let conditions i)-ii) be fulfilled. Then the resolvent of $l_n (n = 0, \pm 1, \pm 2, ...)$ is compact if and only if*

$$\lim_{|y|\to\infty} R_0(y) = \infty.$$

Lemmas 2.5 and 2.6 imply that the resolvent of the operator $(l_n + \mu I)u$ exists for all $\mu \ge 0$. Therefore, it suffices to prove the compactness of the inverse operator l_n^{-1}.

To prove Lemma 4.1, first consider the case $n = 0$. In this case, the operator l_0 will take the form:

$$l_0 z(y) = z'(y) + R_0(y)z(y), \;\; z(y) \in D(l_n).$$

It follows from Lemma 3.4 that the domain of the operator coincides with the space $W^1_{2,R_0(y)}(R)$. $W^1_{2,R_0(y)}(R)$ is the Sobolev space with weight obtained by completing $C_0^\infty(R)$ with respect to the norm:

$$||u||_{W^1_{2,R_0(y)}(R)} = (\int_R |z'(y)|^2 + R_0^2(y)|z(y)|^2 dy)^{\frac{1}{2}}.$$

It is easy to note that the range of the operator l_0^{-1} coincides with the space $W^1_{2,R_0(y)}(R)$. Therefore, it remains to prove the compactness of the embedding of the space $W^1_{2,R_0(y)}(R)$ into the space $L_2(R)$.

The space $W^1_{2,R_0(y)}(R)$ is compactly embedded in $L_2(R)$ according to the result of [16] (Theorem 6.1) if and only if

$$R_0^*(y) \to \infty, \;\; as \;\; y \to \infty, \tag{19}$$

where $R_0^*(y)$ is a special averaging of functions $R_0(y)$, where

$$R_0^*(y) = \inf\{d^{-1} : d^{-1} \ge \int_{y-\frac{d}{2}}^{y+\frac{d}{2}} R_0^2(t)dt\}.$$

To complete the proof of Lemma 4.1, we need the following lemma.

Lemma 4.2. *Let conditions i)-ii) be fulfilled. Then*

$$c^{-1}R_0(y) \le R_0^*(y) \le cR_0(y), \;\; for\ all \;\; y \in R, \tag{20}$$

where $c > 0$ is a constant.

Lemma 4.2 is proved in the same way as Lemma 12 of [17] and Lemma 2.7 of [18].

From (19) and (20) the proof of Lemma 4.1 follows for the case $n = 0$.

Now it remains to prove Lemma 4.1 for the case $n \neq 0$. Note that according to Lemma 3.4 for $\mu \geq 0$

$$\lim_{|n|\to\infty} \left\|(l_n + \mu I)^{-1}\right\|_{L_2(R)\to L_2(R)} = 0,$$

where $(l_n + \mu I)z = z'(y) + (-in^3R_2(y) + inR_1(y) + R_0(y) + \mu)z(y), \quad z(y) \in D(l_n)$.

Hence it follows that it is sufficient to prove the compactness of the operator l_n^{-1} for any finite $n \neq 0$.

Taking this into account, to complete the proof of Lemma 4.1 for the case $n \neq 0$, we repeat all computations and arguments used in [12] to the proof of Theorems 1.2–1.3.

Proof of Theorem 1.3. Theorem 1.1 and equality (11) imply that the resolvent of the operator $(L + \mu I)$ has the form:

$$u(x, y) = (L + \mu I)^{-1} f = \sum_{n=-\infty}^{\infty} (l_n + \mu I)^{-1} f_n(y) \cdot e^{inx}, \tag{21}$$

where $f(x, y) \in L_2(\Omega)$, $f(x, y) = \sum\limits_{n=-\infty}^{\infty} f_n(y) \cdot e^{inx}$.

According to Lemma 3.4

$$\lim_{|n|\to\infty} \left\|(l_n + \mu I)^{-1}\right\| = 0.$$

It follows from this and from (21) that the operator $(L + \mu I)^{-1}$ is completely continuous if and only if $(l_n + \mu I)^{-1}$ is completely continuous. Now Theorem 1.3 being proved follows from Lemma 4.1. Theorem 1.3 is proved.

5. Estimates of singular numbers (s-numbers). The proof of Theorem 1.4

To prove Theorem 1.4, we need the following lemmas below. We introduce the following sets:

$$M = \{u \in L_2(R) : \|l_n u\|_2^2 + \|u\|_2^2 \leq 1\},$$

where $\|\cdot\|$ norm in $L_2(R)$.

$$\widetilde{M}_{C_0} = \{u \in L_2(R) : \|u'\|_2^2 + \left\|-in^3R_2(y)u\right\|_2^2 + \left\|inR_1(y)u\right\|_2^2 + \left\|R_0(y)u\right\|_2^2 \leq C_0\};$$

$$\widetilde{M}_{C_0^{-1}} = \{u \in L_2(R) : \|u'\|_2^2 + \left\|-in^3R_2(y)u\right\|_2^2 + \left\|inR_1(y)u\right\|_2^2 + \left\|R_0(y)u\right\|_2^2 \leq C_0^{-1}\},$$

where $C_0 > 0$ is a constant independent of $u(y)$ and n.

Lemma 5.1. *Let the conditions i)-ii) be fulfilled. Then the inclusions are valid*

$$\widetilde{M}_{C_0^{-1}} \subseteq M \subseteq \widetilde{M}_C$$

where $C_0 > 0, C > 0$ *are constant numbers independent of* $u(y)$ *and* $n (n = 0, \pm 1, \pm 2, ...)$.

Proof. Let $u \in \widetilde{M}_{C_0^{-1}}$. Then we have

$$\|l_n u\|_2^2 + \|u\|_2^2 \leq \|u'\|_2^2 + \left\|-in^3R_2(y)u\right\|_2^2 + \left\|inR_1(y)u\right\|_2^2 + \left\|R_0(y)u\right\|_2^2 + \|u\|_2^2. \tag{22}$$

By virtue of condition *i*), we find:

$$\left\|R_0(y)u\right\|_2^2 + \|u\|_2^2 \leq \left\|R_0(y)u\right\|_2^2 + \frac{1}{\delta}\left\|R_0(y)u\right\|_2^2 \leq (1 + \frac{1}{\delta^2})\left\|R_0(y)u\right\|_2^2. \tag{23}$$

Using (23), we obtain from inequality (22)

$$\|l_n u\|_2^2 + \|u\|_2^2 \le \|u'\|_2^2 + \left\|-in^3R_2(y)u\right\|_2^2 + \left\|inR_1(y)u\right\|_2^2 + \left\|R_0(y)u\right\|_2^2 + \|u\|_2^2 \le$$

$$\le \|u'\|_2^2 + \left\|-in^3R_2(y)u\right\|_2^2 + \left\|inR_1(y)u\right\|_2^2 + (1+\frac{1}{\delta^2})\left\|R_0(y)u\right\|_2^2 \le$$

$$\le C_0(\|u'\|_2^2 + \left\|-in^3R_2(y)u\right\|_2^2 + \left\|inR_1(y)u\right\|_2^2 + \left\|R_0(y)u\right\|_2^2),$$

where $C_0 = max\{1, 1+\frac{1}{\delta^2}\}$.

Since $u \in \widetilde{M}_{C_0^{-1}}$, the last inequality implies that

$$\|l_n u\|_2^2 + \|u\|_2^2 \le C_0 \cdot C_0^{-1} \le 1. \tag{24}$$

The inequality (24) implies that $u \in M$, i.e. $\widetilde{M}_{C_0^{-1}} \subset M$. The left inclusion is proved.

Now, let us prove the right inclusion. Let $u \in M$. This means that $u \in D(l_n + \mu I)$. Therefore, by Lemma 3.4, we have:

$$\left\|R_0(y)u\right\|_2^2 = \left\|R_0(y)(l_n+\mu I)^{-1}(l_n+\mu I)u\right\|_2^2 \le C^2\left\|(l_n+\mu I)u\right\|_2^2.$$

Hence, we get that

$$\left\|R_0(y)u\right\|_2^2 \le C^2\left\|(l_n+\mu I)u\right\|_2^2 \tag{25}$$

Similarly, repeating the above computations, we get

$$\left\|inR_1(y)u\right\|_2^2 \le C_1^2\left\|(l_n+\mu I)u\right\|_2^2 \tag{26}$$

$$\left\|-in^3R_2(y)u\right\|_2^2 \le C_2^2\left\|(l_n+\mu I)u\right\|_2^2 \tag{27}$$

where C, C_1 and C_2 from Lemma 3.4.

Now compute the norm $\|u'\|$:

$$\|u'\|_2 = \left\|(l_n+\mu I)u - (-in^3R_2(y) + inR_1(y) + R_0(y) + \mu)u\right\|_2 \le \left\|(l_n+\mu I)u\right\|_2 +$$

$$+C_2\left\|(l_n+\mu I)u\right\|_2 + C_1\left\|(l_n+\mu I)u\right\|_2 + C\left\|(l_n+\mu I)u\right\|_2 + \left\|(l_n+\mu I)u\right\|_2 \le C_3\left\|(l_n+\mu I)u\right\|_2.$$

Hence, we have

$$\|u'\|_2 \le C_3\left\|(l_n+\mu I)u\right\|_2, \tag{28}$$

where $C_3 = max\{1, C_2, C_1, C\}$.

In order to estimate the norm $\left\|\mu u\right\|_2$, we used Lemma 2.5 here, i.e. the inequality

$$\left\|\mu u\right\|_2 \le \left\|(l_n+\mu I)u\right\|_2.$$

From inequalities (25) – (28) we find

$$\|u'\|_2^2 + \left\|-in^3R_2(y)u\right\|_2^2 + \left\|inR_1(y)u\right\|_2^2 + \left\|R_0(y)u\right\|_2^2 \le C_4^2\left\|(l_n+\mu)u\right\|_2^2, \tag{29}$$

where $C_4 = max\{C, C_1, C_2, C_3\}$.

From inequality (29) we find

$$\|u'\|_2^2 + \left\|-in^3R_2(y)u\right\|_2^2 + \left\|inR_1(y)u\right\|_2^2 + \left\|R_0(y)u\right\|_2^2 \le C_4^2\left\|(l_n+\mu)u\right\|_2^2 \le$$

$$\le 2\cdot C_4^2\|l_n u\|_2^2 + 2\cdot C_4^2\cdot\mu^2\|u\|_2^2.$$

Hence, we obtain

$$\|u'\|_2^2 + \left\|-in^3R_2(y)u\right\|_2^2 + \left\|inR_1(y)u\right\|_2^2 + \left\|R_0(y)u\right\|_2^2 \le C_0(\|l_nu\|_2^2 + \|u\|_2^2),$$

where $C_0 = max\{2 \cdot C_4^2, 2 \cdot \mu \cdot C_4^2\}$.

Since $u \in M$, the last inequality implies that

$$\|u'\|_2^2 + \left\|-in^3R_2(y)u\right\|_2^2 + \left\|inR_1(y)u\right\|_2^2 + \left\|R_0(y)u\right\|_2^2 \le C_0(\|l_nu\|_2^2 + \|u\|_2^2) \le C_0.$$

This implies that $u \in \widetilde{M}_{C_0}$, i.e. $M \subseteq \widetilde{M}_{C_0}$. Lemma 5.1 is completely proved.

Definition 5.1 [11]. The magnitude

$$d_k = \inf_{\{Y_k\}} \sup_{u \in M} \inf_{v \in Y_k} \|u - v\|_2$$

is called Kolmogorov k-widths (diameters) of the set M, where $\{Y_k\}$ is the set of all subspaces in $L_2(R)$ whose dimensions do not exceed k.

The following lemmas are valid.

Lemma 5.2. *Let the conditions i)-ii) be fulfilled. Then the estimates*

$$c^{-1}\widetilde{d_k} \le S_{k+1} \le c\widetilde{d_k}, k = 1, 2, .., $$

hold, where $c > 0$ is a constant, S_k are the s numbers of the operator l_n^{-1}, d_k , $\widetilde{d_k}$ are the Kolmogorov widths of the corresponding sets M, $\widetilde{M}$.

Lemma 5.3. *Let the conditions i)-ii) be fulfilled. Then the estimates*

$$\widetilde{N}(c\lambda) \le N(\lambda) \le \widetilde{N}(c^{-1}\lambda),$$

hold, where $N(\lambda) = \sum\limits_{S_{k+1} > \lambda} 1$ is the counting function of those S_{k+1} of the operator l_n^{-1} that are greater than $\lambda > 0$, $\widetilde{N}(\lambda) = \sum\limits_{d_k > \lambda} 1$ is the counting function of those $\widetilde{d_k}$ greater than $\lambda > 0$.

Lemmas 5.2 and 5.3 are proved in exactly the same way as Lemmas 4.3 and 4.4 of [12].

Lemma 5.4. *Let the conditions i)-ii) be fulfilled. Then the estimates*

$$c^{-1}\lambda^{-1}mes(y \in \mathbb{R} : Q_n(y) \le c^{-1}\lambda^{-1}) \le N(\lambda) \le c\lambda^{-1}mes(y \in \mathbb{R} : Q_n(y) \le c\lambda^{-1}),$$

hold, where $Q_n(y) = | - in^3R_2(y) + inR_1(y) + R_0(y)|$ and $c > 0$ is a constant not depending on $Q_n(y)$ and λ.

To prove Lemma 5.4, we first prove the following lemma.

Lemma 5.5. *Let the conditions i) - ii) be fulfilled. Then the estimates*

$$c^{-1}Q_n(y) \le Q_n^*(y) \le cQ_n(y), \tag{30}$$

hold, where $c > 0$ is a constant and $Q_n^*(y)$ is a special averaging of the functions $Q_n(y)$ (see Lemmas 4.1 - 4.2).

Lemma 5.5 is proved in the same way as Lemma 4.2.

Proof of Lemma 5.4. We denote by $L_2^1(R, Q_n(y))$ the space obtained by completing $C_0^\infty(R)$ with respect to the norm

$$\|u\|_{L_2^1(R,Q_n(y))} = (\int |u'|^2 + Q_n^2(y)|u|^2 dy)^{\frac{1}{2}}.$$

It is easy to verify that $\widetilde{M} \subset L_2^1(R, Q_n(y))$. Hence, using Lemma 5.5 and also repeating the computations and arguments used in the proof of Theorem 1.4 from [12], we obtain the proof of Lemma 5.4.

Proof of Theorem 1.4. Theorem 1.1 and the equality (11) imply that

$$u(x,y) = (L+\mu I)^{-1} f(x,y) = \sum_{n=-\infty}^{\infty} (l_n + \mu I)^{-1} f_n(y) \cdot e^{inx}, \quad \mu \geq 0. \tag{31}$$

The representation (31) implies that if s is a singular point of $(L+\mu I)^{-1}$ then s is a singular number of one of $(l_n + \mu I)^{-1}(n = 0, \pm 1, \pm 2, ...)$, and vice versa, if s is a singular number of one of $(l_n + \mu I)^{-1}$ then s is a singular point of $(L+\mu I)^{-1}$. The proof of Theorem 1.4 follows from this and Lemma 5.4.

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