Filomat 36:11 (2022), 3655–3673
https://doi.org/10.2298/FIL2211655S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Approximation by Szász-Baskakov Operators Based on Boas-Buck-Type Polynomials

Melek Sofyalıoğlu[a], **Kadir Kanat**[a]

[a] *Ankara Hacı Bayram Veli University, Polatlı Faculty of Science and Arts, Department of Mathematics, Ankara, Turkey*

Abstract. This paper concerns with a generalization of Szász-Baskakov operators, which includes Boas-Buck-type polynomials. The convergence properties are studied in weighted space and the rate of convergence is obtained by using weighted modulus of continuity. A Voronovskaya-type theorem is investigated. Also, the theoretical results are demonstrated by choosing the particular cases of Boas-Buck-type polynomials, namely Appell polynomials, Hermite polynomials, Gould-Hopper polynomials, Laguerre polynomials and Charlier polynomials.

1. Introduction

Approximation theory concerns with the approximation of functions by using simpler calculated functions. Linear positive operators play a crucial role in this area. One of the well-known linear and positive operators are Szász operators, which is the extension of the Bernstein operators to the infinite interval. In 1950, Szász operators are introduced by Szász [23] which is defined as

$$S_s(f;x) = e^{-sx} \sum_{v=0}^{\infty} \frac{(sx)^v}{v!} f\left(\frac{v}{s}\right), \tag{1}$$

where $s \in \mathbb{N}$, $x \geq 0$ and $f \in C[0,\infty)$. In 1957, Baskakov operators are proposed by Baskakov [9] as follows:

$$L_s(f;x) = \frac{1}{(1+x)^s} \sum_{v=0}^{\infty} \binom{s+v-1}{v} \frac{x^v}{(1+x)^v} f\left(\frac{v}{s}\right), s \in \mathbb{N}^+, x \in [0,\infty).$$

In 1983, Szász-Mirakyan-Baskakov operators are studied by Prasad et al. [22] which is given by

$$D_s(f;x) = (s-1) \sum_{v=0}^{\infty} e^{-sx} \frac{(sx)^v}{v!} \int_0^{\infty} \binom{s+v-1}{v} \frac{t^v}{(1+t)^{s+v}} f(t)\, dt. \tag{2}$$

In 1969, Szász operators are combined with Appell polynomials by Jakimovski and Leviatan [17]. In 1974, Szász operators by utilizing the Sheffer polynomials are studied by [15]. After that, in 2012, Varma et al. [25]

2020 *Mathematics Subject Classification*. Primary 41A25; Secondary 41A36, 47A58
Keywords. Boas-Buck-type polynomials, Weighted space, Voronovskaya-type theorem, Weighted Modulus of Continuity
Received: 23 September 2020; Revised: 07 January 2022; Accepted: 22 June 2022
Communicated by Miodrag Spalević
Corresponding author: Melek Sofyalıoğlu
Email addresses: melek.sofyalioglu@hbv.edu.tr (Melek Sofyalıoğlu), kadir.kanat@hbv.edu.tr (Kadir Kanat)

pioneered a generalization of the Szász operators involving Brenke-type polynomials. They showed that under special choices these polynomials turn to the Appell polynomials and Gould-Hopper polynomials. In the same year, Varma and Taşdelen [26] studied Szász operators with the discrete orthogonal polynomials such as Charlier polynomials. At about the same time, Sucu et al. [24] investigated the linear positive operators, which involve the Boas-Buck-type polynomials as follows:

$$B_s(f;x) = \frac{1}{A(1)B(sxH(1))}\sum_{v=0}^{\infty} p_v(sx) f\left(\frac{v}{s}\right), \; x \geq 0, \; s \in \mathbb{N}. \tag{3}$$

After that, Mursaleen et al. [19] studied Cholodowsky type generalization of Szász operators which include Boas-Buck-type polynomials. Recently, Çekim et al. [12] dealed with Kantorovich-Stancu operators based on Boas-Buck-type polynomials. We also refer some of the important papers in this area [1, 3–6, 8, 10, 11, 18, 20]. The Boas-Buck-type polynomials have generating functions of the form

$$A(u)B(xH(u)) = \sum_{v=0}^{\infty} p_v(x)u^v, \tag{4}$$

where $A(u)$, $B(u)$ and $H(u)$ are analytic functions

$$A(u) = \sum_{r=0}^{\infty} a_r u^r, \quad B(u) = \sum_{r=0}^{\infty} b_r u^r, \quad H(u) = \sum_{r=1}^{\infty} h_r u^r, \tag{5}$$

with the conditions $a_0 \neq 0$, $b_r \neq 0 (r \geq 0)$ and $h_1 \neq 0$ [16]. We assume that the following restrictions are satisfied:

(i) $A(1) \neq 0, H'(1) = 1, p_v(x) \geq 0, v = 0, 1, 2, ..,$

(ii) $B : \mathbb{R} \to (0, \infty)$,

(iii) The power series (4) and (5) converge in the disk $|u| < R$, $(R > 1)$.

In (3), by choosing $A(u) = 1$, $H(u) = u$ and $B(u) = e^u$, in view of the generating functions (4), we obtain $p_v(sx) = \frac{(sx)^v}{v!}$ and the operators (3) reduce to the Szász operators (1). In the current paper, we aim to construct a generalization of the Szász-Mirakyan-Baskakov operators [22] by utilizing the Boas-Buck-type polynomials. Then we show that the Boas-Buck-type polynomials include the Appell polynomials, Hermite polynomials, Gould-Hopper polynomials, Laguerre polynomials and Charlier polynomials as special cases. We consider the Szász-Baskakov operators including Boas-Buck-type polynomials as follows:

$$D_s^*(f;x) = \frac{(s-1)}{A(1)B(sxH(1))}\sum_{v=0}^{\infty} p_v(sx)\int_0^{\infty}\binom{s+v-1}{v}\frac{t^v}{(1+t)^{s+v}} f(t)\,dt, \tag{6}$$

where $s \in \mathbb{N}$, $x \geq 0$ and $f \in C[0, \infty)$.

2. Approximation properties of D_s^*

Lemma 2.1. *For every $x \in [0, \infty)$, we write*

$$\begin{aligned} D_s^*(1;x) &= 1, \\ D_s^*(e_1;x) &= \frac{sB'(sxH(1))}{(s-2)B(sxH(1))}x + \frac{A(1)+A'(1)}{(s-2)A(1)}, s > 2, \end{aligned}$$

$$\begin{aligned}
D_s^*(e_2;x) &= \frac{s^2B''(sxH(1))}{(s-2)(s-3)B(sxH(1))}x^2+\frac{s\,(4A(1)+2A'(1)+A(1)H''(1))\,B'(sxH(1))}{(s-2)(s-3)A(1)B(sxH(1))}x \\
&\quad +\frac{2A(1)+4A'(1)+A''(1)}{(s-2)(s-3)A(1)}, s>3, \\
D_s^*(e_3;x) &= \frac{s^3B'''(sxH(1))}{(s-2)(s-3)(s-4)B(sxH(1))}x^3+\frac{s^2\,(9A(1)+3A'(1)+3A(1)H''(1))\,B''(sxH(1))}{(s-2)(s-3)(s-4)A(1)B(sxH(1))}x^2 \\
&\quad +\frac{s\,(18A(1)+18A'(1)+9A(1)H''(1)+3A''(1)+3A'(1)H''(1)+A(1)H'''(1))\,B'(sxH(1))}{(s-2)(s-3)(s-4)A(1)B(sxH(1))}x \\
&\quad +\frac{6A(1)+18A'(1)+9A''(1)+A'''(1)}{(s-2)(s-3)(s-4)A(1)}, s>4, \\
D_s^*(e_4;x) &= \frac{s^4B^{(iv)}(sxH(1))}{(s-2)(s-3)(s-4)(s-5)B(sxH(1))}x^4+\frac{s^3\,(16A(1)+4A'(1)+6A(1)H''(1))\,B'''(sxH(1))}{(s-2)(s-3)(s-4)(s-5)A(1)B(sxH(1))}x^3 \\
&\quad +\frac{s^2B''(sxH(1))}{(s-2)(s-3)(s-4)(s-5)A(1)B(sxH(1))}\{72A(1)+48A'(1)+48A(1)H''(1)+6A''(1) \\
&\quad +12A'(1)H''(1)+4A(1)H'''(1)+3A(1)(H''(1))^2\Big\}x^2 \\
&\quad +\frac{sB'(sxH(1))}{(s-2)(s-3)(s-4)(s-5)A(1)B(sxH(1))}\{96A(1)+144A'(1)+48A''(1)+4A'''(1) \\
&\quad +72A(1)H''(1)+48A'(1)H''(1)+16A(1)H'''(1)+6A''(1)H''(1)+4A'(1)H'''(1)+A(1)H^{(iv)}(1)\Big\}x \\
&\quad +\frac{24A(1)+96A'(1)+72A''(1)+16A'''(1)+A^{(iv)}(1)}{(s-2)(s-3)(s-4)(s-5)A(1)}, s>5,
\end{aligned}$$

where $e_m=t^m$ *for* $m=1,2,3,4$.

Proof. One can easily check that

$$\sum_{v=0}^{\infty}p_v(sx)=1,\quad \int_0^{\infty}p_v(sx)dx=\frac{1}{s} \tag{7}$$

and

$$\sum_{v=0}^{\infty}\binom{s+v-1}{v}\frac{t^v}{(1+t)^{s+v}}=1,\quad \int_0^{\infty}\binom{s+v-1}{v}\frac{t^v}{(1+t)^{s+v}}dt=\frac{1}{s-1}. \tag{8}$$

The lemma given above is proved by using the generating functions of the Boas-Buck-type polynomials. By taking into consideration (4), we get

$$\begin{aligned}
\sum_{v=0}^{\infty}p_v(sx) &= A(1)B(sxH(1)), \\
\sum_{v=0}^{\infty}vp_v(sx) &= A'(1)B(sxH(1))+sxA(1)B'(sxH(1)), \\
\sum_{v=0}^{\infty}v^2p_v(sx) &= (A'(1)+A''(1))\,B(sxH(1))+sx\,(A(1)+2A'(1)+A(1)H''(1))\,B'(sxH(1)) \\
&\quad +s^2x^2A(1)B''(sxH(1)), \\
\sum_{v=0}^{\infty}v^3p_v(sx) &= (A'(1)+3A''(1)+A'''(1))\,B(sxH(1))+sx\,(A(1)+6A'(1)+3A(1)H''(1) \\
&\quad +3A''(1)+3A'(1)H''(1)+A(1)H'''(1))\,B'(sxH(1)) \\
&\quad +s^2x^2\,(3A(1)+3A'(1)+3A(1)H''(1))\,B''(sxH(1))+s^3x^3A(1)B'''(sxH(1)),
\end{aligned}$$

$$
\begin{aligned}
\sum_{v=0}^{\infty} v^4 p_v(sx) &= \left(A'(1) + 7A''(1) + 6A'''(1) + A^{(iv)}(1)\right) B(sxH(1)) \\
&\quad + sx\,(A(1) + 14A'(1) + 7A(1)H''(1) + 18A''(1) + 18A'(1)H''(1) + 6A(1)H'''(1) \\
&\quad + 4A'''(1) + 6A''(1)H''(1) + 4A'(1)H'''(1) + A(1)H^{(iv)}(1)\Big) B'(sxH(1)) \\
&\quad + s^2x^2\,(7A(1) + 18A'(1) + 18A(1)H''(1) + 6A''(1) + 12A'(1)H''(1) + 4A(1)H'''(1) \\
&\quad + 3A(1)(H''(1))^2\Big) B''(sxH(1)) + s^3x^3\,(6A(1) + 4A'(1) + 6A(1)H''(1))\, B'''(sxH(1)) \\
&\quad + s^4x^4 A(1) B^{(iv)}(sxH(1)).
\end{aligned}
$$

Having regard to these equalities, we achieve the assertions of the lemma. □

Let the class of K be

$$K := \left\{ f : x \in [0,\infty),\ |f(x)| \le ae^{bx},\ \ \text{a, b positive and finite} \right\}. \tag{9}$$

Also, assume that

$$\lim_{s\to\infty} \frac{B'(s)}{B(s)} = 1,\quad \lim_{s\to\infty} \frac{B''(s)}{B(s)} = 1,\quad \lim_{s\to\infty} \frac{B'''(s)}{B(s)} = 1,\quad \lim_{s\to\infty} \frac{B^{(iv)}(s)}{B(s)} = 1. \tag{10}$$

Theorem 2.2. *Let $f \in C[0,\infty) \cap K$ and Eqn. (10) be satisfied. Then*

$$\lim_{s\to\infty} D_s^*(f;x) = f(x), \tag{11}$$

and in each compact subset of $[0,\infty)$ *the operators* D_s^* *converge uniformly.*

Proof. By taking into consideration Lemma 2.1 and Eqn. (10), we obtain

$$\lim_{s\to\infty} D_s^*(e_m;x) = e_m(x),\ \ m = 0,1,2. \tag{12}$$

Thusly, the new constructed operators D_s^* converge uniformly in each compact subset of $[0,\infty)$. Then we complete the proof by applying universal Korovkin-type property with respect to positive linear operators ([2], Theorem 4.1.4). □

Lemma 2.3. *For every $x \in [0,\infty)$, we have*

$$
\begin{aligned}
D_s^*(t-x;x) &= \left(\frac{sB'(sxH(1))}{(s-2)B(sxH(1))} - 1 \right) x + \frac{A(1) + A'(1)}{(s-2)A(1)},\ \ s > 2, \\
D_s^*((t-x)^2;x) &= \left(\frac{s^2B''(sxH(1))}{(s-2)(s-3)B(sxH(1))} - \frac{2sB'(sxH(1))}{(s-2)B(sxH(1))} + 1 \right) x^2 \\
&\quad + \left(\frac{s\,(4A(1) + 2A'(1) + A(1)H''(1))\, B'(sxH(1))}{(s-2)(s-3)A(1)B(sxH(1))} - \frac{2A(1) + 2A'(1)}{(s-2)A(1)} \right) x \\
&\quad + \frac{2A(1) + 4A'(1) + A''(1)}{(s-2)(s-3)A(1)},\ \ s > 3,
\end{aligned}
$$

$$
\begin{aligned}
D_s^*((t-x)^4;x) &= \left(\frac{s^4B^{(iv)}(sxH(1))}{(s-2)(s-3)(s-4)(s-5)B(sxH(1))} - \frac{4s^3B'''(sxH(1))}{(s-2)(s-3)(s-4)B(sxH(1))}\right.\\
&\left. + \frac{6s^2B''(sxH(1))}{(s-2)(s-3)B(sxH(1))} - \frac{4sB'(sxH(1))}{(s-2)B(sxH(1))} + 1\right)x^4\\
&+\left(\frac{s^3\left(16A(1)+4A'(1)+6A(1)H''(1)\right)B'''(sxH(1))}{(s-2)(s-3)(s-4)(s-5)A(1)B(sxH(1))}\right.\\
&-\frac{4s^2\left(9A(1)+3A'(1)+3A(1)H''(1)\right)B''(sxH(1))}{(s-2)(s-3)(s-4)A(1)B(sxH(1))}\\
&\left.+\frac{6s\left(4A(1)+2A'(1)+A(1)H''(1)\right)B'(sxH(1))}{(s-2)(s-3)A(1)B(sxH(1))} - \frac{4A(1)+4A'(1)}{(s-2)A(1)}\right)x^3\\
&+\left(\frac{s^2B''(sxH(1))}{(s-2)(s-3)(s-4)(s-5)A(1)B(sxH(1))}\{72A(1)+48A'(1)+48A(1)H''(1)+6A''(1)\right.\\
&+12A'(1)H''(1)+4A(1)H'''(1)+3A(1)(H''(1))^2\Big\}\\
&-\frac{4s\left(18A(1)+18A'(1)+9A(1)H''(1)+3A''(1)+3A'(1)H''(1)+A(1)H'''(1)\right)B'(sxH(1))}{(s-2)(s-3)(s-4)A(1)B(sxH(1))}\\
&\left.+\frac{12A(1)+24A'(1)+6A''(1)}{(s-2)(s-3)A(1)}\right)x^2\\
&+\left(\frac{sB'(sxH(1))}{(s-2)(s-3)(s-4)(s-5)A(1)B(sxH(1))}\{96A(1)+144A'(1)+48A''(1)+4A'''(1)\right.\\
&+72A(1)H''(1)+48A'(1)H''(1)+16A(1)H'''(1)+6A''(1)H''(1)+4A'(1)H'''(1)+A(1)H^{(iv)}(1)\Big\}\\
&\left.-\frac{24A(1)+72A'(1)+36A''(1)+4A'''(1)}{(s-2)(s-3)(s-4)A(1)}\right)x\\
&+\frac{24A(1)+96A'(1)+72A''(1)+16A'''(1)+A^{(iv)}(1)}{(s-2)(s-3)(s-4)(s-5)A(1)},\ \ s>5.
\end{aligned}
$$

Proof. By the help of

$$
\begin{aligned}
D_s^*(t-x;x) &= D_s^*(e_1;x) - xD_s^*(1;x),\ \ s>2,\\
D_s^*((t-x)^2;x) &= D_s^*(e_2;x) - 2xD_s^*(e_1;x) + x^2D_s^*(1;x),\ \ s>3,\\
D_s^*((t-x)^4;x) &= D_s^*(e_4;x) - 4xD_s^*(e_3;x) + 6x^2D_s^*(e_2;x) - 4x^3D_s^*(e_1;x) + x^4D_s^*(1;x),\ \ s>5
\end{aligned}
$$

we get the desired result of the lemma. □

Lemma 2.4. *We have following results*

$$
\lim_{s\to\infty} sD_s^*(t-x;x) = x\mu_1(x) + \frac{A(1)+A'(1)}{A(1)} \tag{13}
$$

$$
\lim_{s\to\infty} sD_s^*((t-x)^2;x) = x^2\mu_2(x) + x(2+H''(1)) \tag{14}
$$

$$
\lim_{s\to\infty} s^2D_s^*((t-x)^4;x) = x^4\mu_3(x) + x^3\mu_4(x) + x^2\left(12+12H''(1)+3(H''(1))^2\right), \tag{15}
$$

where

$$
\mu_1(x) = \lim_{s\to\infty} s\left(\frac{sB'(sxH(1)) - (s-2)B(sxH(1))}{(s-2)B(sxH(1))}\right) \tag{16}
$$

$$
\mu_2(x) = \lim_{s\to\infty} s\left(\frac{s^2B''(sxH(1)) - 2s(s-3)B'(sxH(1)) + (s-2)(s-3)B(sxH(1))}{(s-2)(s-3)B(sxH(1))}\right) \tag{17}
$$

$$\mu_3(x) = \lim_{s\to\infty} s^2 \left(\frac{s^4 B^{(iv)}(sxH(1)) - 4s^3(s-5)B'''(sxH(1) + 6s^2(s-4)(s-5)B''(sxH(1))}{(s-2)(s-3)(s-4)(s-5)B(sxH(1))} - \frac{4s(s-3)(s-4)(s-5)B'(sxH(1))}{(s-2)(s-3)(s-4)(s-5)B(sxH(1))} + 1 \right) \quad (18)$$

$$\begin{aligned}\mu_4(x) = \lim_{s\to\infty} s^2 \Bigg(& \frac{s^3\left(16A(1) + 4A'(1) + 6A(1)H''(1)\right)B'''(sxH(1))}{(s-2)(s-3)(s-4)(s-5)B(sxH(1))} \\ & - \frac{4s^2(s-5)\left(9A(1) + 3A'(1) + 3A(1)H''(1)\right)B''(sxH(1))}{(s-2)(s-3)(s-4)(s-5)A(1)B(sxH(1))} \\ & + \frac{6s(s-4)(s-5)\left(4A(1) + 2A'(1) + A(1)H''(1)\right)B'(sxH(1))}{(s-2)(s-3)(s-4)(s-5)A(1)B(sxH(1))} \\ & - \frac{(s-3)(s-4)(s-5)(4A(1) + 4A'(1))B(sxH(1))}{(s-2)(s-3)(s-4)(s-5)A(1)B(sxH(1))} \Bigg). \quad (19)\end{aligned}$$

3. Weighted approximation

We need weighted spaces in order to compute the rate of convergence of the boundless function defined on $[0,\infty)$. Here, we investigate the approximation properties of the new constructed operators D_s^* on the weighted spaces of functions with exponential growth on $[0,\infty)$. Firstly, we recall the notations of the weighted spaces. Let R_f be a positive constant and $\rho(x) = 1 + x^2$ be the weighted function.
$B_\rho([0,\infty)) = \left\{ f : [0,\infty) \to \mathbb{R} \mid |f(x)| \le R_f \rho(x) \right\}$ is a normed linear space endowed with $\|f\| = \sup_{x\in[0,\infty)} \frac{|f(x)|}{\rho(x)}$.
$C_\rho([0,\infty)) = \left\{ f \in B_\rho([0,\infty)) \mid \text{f is continuous} \right\}$, $C_\rho^*([0,\infty)) = \left\{ f \in C_\rho([0,\infty)) \mid \lim_{x\to\infty} \frac{f(x)}{\rho(x)} < \infty \right\}$. The relation between these spaces can be shown as $C_\rho^*([0,\infty)) \subset C_\rho([0,\infty)) \subset B_\rho([0,\infty))$.

Lemma 3.1. *[28] The sequence of linear and positive operators (B_s), $s \ge 1$ act from $C_\rho([0,\infty))$ to $B_\rho([0,\infty))$ if and only if there exists a positive constant R such that*

$$\|D_s^*(\rho; x)\|_\rho \le R. \quad (20)$$

Lemma 3.2. *Let the weight function be $\rho(x) = 1 + x^2$ and*

$$\lim_{s\to\infty} \frac{B'(s)}{B(s)} = 1, \quad \lim_{s\to\infty} \frac{B''(s)}{B(s)} = 1$$

be satisfied. If $f \in C_\rho([0,\infty))$, then

$$\|D_s^*(\rho; x)\|_\rho \le R, \quad (21)$$

where R is positive constant.

Proof. Using Lemma 2.1 and substituting $\rho(x) = 1 + x^2$ we have

$$\begin{aligned} D_s^*(\rho; x) &= D_s^*(1; x) + D_s^*(e_2; x) \\ &= 1 + \frac{s^2 B''(sxH(1))}{(s-2)(s-3)B(sxH(1))} x^2 + \frac{s\left(4A(1) + 2A'(1) + A(1)H''(1)\right)B'(sxH(1))}{(s-2)(s-3)A(1)B(sxH(1))} x \\ &\quad + \frac{2A(1) + 4A'(1) + A''(1)}{(s-2)(s-3)A(1)}, s > 3. \end{aligned}$$

Therefore, we write

$$\begin{aligned}\|D_s^*(\rho;x)\|_\rho &= \sup_{x\geq 0}\left\{\left|\frac{s^2B''(sxH(1))}{(s-2)(s-3)B(sxH(1))}\right|\frac{x^2}{1+x^2}\right.\\ &+\left|\frac{s\,(4A(1)+2A'(1)+A(1)H''(1))\,B'(sxH(1))}{(s-2)(s-3)A(1)B(sxH(1))}\right|\frac{x}{1+x^2}\\ &+\left.\left|\left(\frac{2A(1)+4A'(1)+A''(1)}{(s-2)(s-3)A(1)}+1\right)\right|\frac{1}{1+x^2}\right\}, s>3.\end{aligned}$$

Since

$$\sup_{x\geq 0}\frac{1}{1+x^2}=1,\quad \sup_{x\geq 0}\frac{x}{1+x^2}=\frac{1}{2},\quad \sup_{x\geq 0}\frac{x^2}{1+x^2}=1, \tag{22}$$

we have

$$\begin{aligned}\|D_s^*(\rho;x)\|_\rho &\leq \frac{s^2B''(sxH(1))}{(s-2)(s-3)B(sxH(1))}+\frac{s\,(4A(1)+2A'(1)+A(1)H''(1))\,B'(sxH(1))}{2(s-2)(s-3)A(1)B(sxH(1))}\\ &+\frac{2A(1)+4A'(1)+A''(1)}{(s-2)(s-3)A(1)}+1,\ \ s>3.\end{aligned}$$

By using the condition (10), there exist a positive constant R

$$\|D_s^*(\rho;x)\|_\rho\leq R.$$

□

It can be understood from Lemma 3.2 that the operators D_s^* act from $C_\rho([0,\infty))$ to $B_\rho([0,\infty))$.

Theorem 3.3. *[27], [28] The sequence of linear and positive operators (B_s), $s\geq 1$ act from $C_\rho([0,\infty))$ to $B_\rho([0,\infty))$ such that*

$$\underset{s\to\infty}{lim}\|D_s^*(t^m;x)-x^m\|_\rho=0,\ \ m=0,1,2. \tag{23}$$

Then for any function $f\in C_\rho^R([0,\infty))$,

$$\underset{s\to\infty}{lim}\|D_s^*f-f\|_\rho=0. \tag{24}$$

Theorem 3.4. *Let the weight function be $\rho(x)=1+x^2$ and*

$$\lim_{s\to\infty}\frac{B'(s)}{B(s)}=1,\quad \lim_{s\to\infty}\frac{B''(s)}{B(s)}=1$$

be satisfied. Then for every $f\in C_\rho^R([0,\infty))$,

$$\lim_{s\to\infty}\|D_s^*f-f\|_\rho=0. \tag{25}$$

Proof. In order to prove this theorem, it is enough to show the conditions of the weighted Korovkin theorem for $m=0,1,2$

$$\lim_{s\to\infty}\|D_s^*(t^m;x)-x^m\|_\rho=0. \tag{26}$$

It can be easily seen from Lemma 2.1

$$\lim_{s\to\infty}\|D_s^*(1;x)-1\|_\rho=0. \tag{27}$$

After that, by using Lemma 2.1 we have

$$\begin{aligned}\|D_s^*(e_1;x)-e_1(x)\|_\rho &= \sup_{x\geq 0}\left\{\left|\frac{sB'(sxH(1))}{(s-2)B(sxH(1))}-1\right|\frac{x}{1+x^2}+\left|\frac{A(1)+A'(1)}{(s-2)A(1)}\right|\frac{1}{1+x^2}\right\}\\ &\leq \frac{1}{2}\left|\frac{sB'(sxH(1))}{(s-2)B(sxH(1))}-1\right|+\left|\frac{A(1)+A'(1)}{(s-2)A(1)}\right|\end{aligned}$$

for $s>2$. Thusly, we obtain

$$\lim_{s\to\infty}\|D_s^*(e_1;x)-e_1(x)\|_\rho = 0. \tag{28}$$

In the same manner,

$$\begin{aligned}\|D_s^*(e_2;x)-e_2(x)\|_\rho &= \sup_{x\geq 0}\left\{\left|\frac{s^2B''(sxH(1))}{(s-2)(s-3)B(sxH(1))}-1\right|\frac{x^2}{1+x^2}\right.\\ &\quad +\left|\frac{s\,(4A(1)+2A'(1)+A(1)H''(1))\,B'(sxH(1))}{(s-2)(s-3)A(1)B(sxH(1))}\right|\frac{x}{1+x^2}\\ &\quad \left.+\left|\frac{2A(1)+4A'(1)+A''(1)}{(s-2)(s-3)A(1)}\right|\frac{1}{1+x^2}\right\}\\ &\leq \left|\frac{s^2B''(sxH(1))}{(s-2)(s-3)B(sxH(1))}-1\right|+\left|\frac{s\,(4A(1)+2A'(1)+A(1)H''(1))\,B'(sxH(1))}{2(s-2)(s-3)A(1)B(sxH(1))}\right|\\ &\quad +\left|\frac{2A(1)+4A'(1)+A''(1)}{(s-2)(s-3)A(1)}\right|\end{aligned}$$

for $s>3$. So, we have

$$\lim_{s\to\infty}\|D_s^*(e_2;x)-e_2(x)\|_\rho = 0. \tag{29}$$

Therefore, we get

$$\lim_{s\to\infty}\|D_s^*(t^m;x)-x^m\|_\rho = 0 \tag{30}$$

for $m=0,1,2$. Finally from Theorem 3.3, we obtain the desired result

$$\lim_{s\to\infty}\|D_s^*f-f\|_\rho = 0. \tag{31}$$

□

4. Weighted modulus of continuity

If f is not uniformly continuous on $[0,\infty)$, then the modulus of continuity $\omega(f,\delta)$ does not tend to 0 as $\delta\to 0$. Thus, the weighted modulus of continuity is defined by Gadjieva and Doğru [27] in 1998 as follows:

$$\Omega(f;\delta)=\sup_{x\geq 0,\,|h|\leq\delta}\frac{|f(x+h)-f(x)|}{(1+x^2)(1+h^2)}.$$

In 2006, Yüksel and Ispir [29] gave the definiton of weighted modulus of continuity as follows:

$$\Omega(f,\delta)=\sup_{x\geq 0}\ \sup_{0<h\leq\delta}\frac{|f(x+h)-f(x)|}{1+(x+h)^2},$$

where $f\in C_\rho^*[0,\infty)$. In the next lemma, we will present the properties of $\Omega(.,.)$.

Lemma 4.1. *[29] If $f \in C^*_\rho[0,\infty)$ then*
(i) *$\Omega(f,x)$ is monotone increasing function of δ,*
(ii) $\lim\limits_{\delta\to 0^+}\Omega(f,x)=0$,
(iii) *for any $\lambda\in[0,\infty)$, $\Omega(f,\lambda x)\le(1+\lambda)\Omega(f,x)$.*

Now, we will obtain the rate of convergence for $f \in C^*_\rho[0,\infty)$ by using the weighted modulus of continuity.

Theorem 4.2. *If $f \in C^*_\rho[0,\infty)$, then*

$$\sup_{x\in[0,\infty)}\frac{|D^*_s(f;x)-f(x)|}{(1+x^2)^{\frac{5}{2}}} \le 2\left(2+M^*_0(s)+\sqrt{M^*_1(s)}\right)\Omega\left(f,\sqrt{M^*_0(s)}\right),$$

where

$$\begin{aligned}
M^*_0(s) &= \left(\frac{s^2B''(sxH(1))}{(s-2)(s-3)B(sxH(1))}-\frac{2sB'(sxH(1))}{(s-2)B(sxH(1))}+1\right)\\
&+\frac{1}{2}\left(\frac{s\left(4A(1)+2A'(1)+A(1)H''(1)\right)B'(sxH(1))}{(s-2)(s-3)A(1)B(sxH(1))}-\frac{2A(1)+2A'(1)}{(s-2)A(1)}\right)\\
&+\frac{2A(1)+4A'(1)+A''(1)}{(s-2)(s-3)A(1)},\quad s>3 \qquad (32)
\end{aligned}$$

$$\begin{aligned}
M^*_1(s) &= \left(\frac{s^4B^{(iv)}(sxH(1))}{(s-2)(s-3)(s-4)(s-5)B(sxH(1))}-\frac{4s^3B'''(sxH(1))}{(s-2)(s-3)(s-4)B(sxH(1))}\right.\\
&\left.+\frac{6s^2B''(sxH(1))}{(s-2)(s-3)B(sxH(1))}-\frac{4sB'(sxH(1))}{(s-2)B(sxH(1))}+1\right)+\frac{3\sqrt{3}}{16}\left(\frac{s^3\left(16A(1)+4A'(1)+6A(1)H''(1)\right)B'''(sxH(1))}{(s-2)(s-3)(s-4)(s-5)A(1)B(sxH(1))}\right.\\
&-\frac{4s^2\left(9A(1)+3A'(1)+3A(1)H''(1)\right)B''(sxH(1))}{(s-2)(s-3)(s-4)A(1)B(sxH(1))}\\
&\left.+\frac{6s\left(4A(1)+2A'(1)+A(1)H''(1)\right)B'(sxH(1))}{(s-2)(s-3)A(1)B(sxH(1))}-\frac{4A(1)+4A'(1)}{(s-2)A(1)}\right)\\
&+\frac{1}{4}\left(\frac{s^2B''(sxH(1))}{(s-2)(s-3)(s-4)(s-5)A(1)B(sxH(1))}\{72A(1)+48A'(1)+48A(1)H''(1)+6A''(1)\right.\\
&+12A'(1)H''(1)+4A(1)H'''(1)+3A(1)(H''(1))^2\Big\}\\
&-\frac{4s\left(18A(1)+18A'(1)+9A(1)H''(1)+3A''(1)+3A'(1)H''(1)+A(1)H'''(1)\right)B'(sxH(1))}{(s-2)(s-3)(s-4)A(1)B(sxH(1))}\\
&\left.+\frac{12A(1)+24A'(1)+6A''(1)}{(s-2)(s-3)A(1)}\right)\\
&+\frac{3\sqrt{3}}{16}\left(\frac{sB'(sxH(1))}{(s-2)(s-3)(s-4)(s-5)A(1)B(sxH(1))}\{96A(1)+144A'(1)+48A''(1)+4A'''(1)\right.\\
&+72A(1)H''(1)+48A'(1)H''(1)+16A(1)H'''(1)+6A''(1)H''(1)+4A'(1)H'''(1)+A(1)H^{(iv)}(1)\Big\}\\
&\left.-\frac{24A(1)+72A'(1)+36A''(1)+4A'''(1)}{(s-2)(s-3)(s-4)A(1)}\right)\\
&+\frac{24A(1)+96A'(1)+72A''(1)+16A'''(1)+A^{(iv)}(1)}{(s-2)(s-3)(s-4)(s-5)A(1)},\quad s>5. \qquad (33)
\end{aligned}$$

Proof. From the definition of the weighted modulus of continuity and Lemma 4.1, we write

$$\begin{aligned}
\left|f(t)-f(x)\right| &\le \left(1+(x+|t-x|)^2\right)\left(1+\frac{|t-x|}{\delta}\right)\Omega(f,\delta)\\
&\le 2(1+x^2)\left(1+(t-x)^2\right)\left(1+\frac{|t-x|}{\delta}\right)\Omega(f,\delta).
\end{aligned}$$

Furthermore, applying D_s^* for both sides, we obtain

$$|D_s^*(f;x)-f(x)| \leq 2(1+x^2)\left(1+D_s^*\left((t-x)^2;x\right)+D_s^*\left(\left(1+(t-x)^2\right)\frac{|t-x|}{\delta};x\right)\right)\Omega(f,\delta).$$

By using the Cauchy-Schwarz inequality in the term $D_s^*\left(\left(1+(t-x)^2\right)\frac{|t-x|}{\delta};x\right)$ we obtain

$$\begin{aligned}|D_s^*(f;x)-f(x)| &\leq 2(1+x^2)\Big\{1+D_s^*\left((t-x)^2;x\right)\\ &\quad+\frac{1}{\delta}\sqrt{D_s^*\left((t-x)^2;x\right)}+\frac{1}{\delta}\sqrt{D_s^*\left((t-x)^4;x\right)}\sqrt{D_s^*\left((t-x)^2;x\right)}\Big\}\Omega(f,\delta).\end{aligned}$$

From Lemma 2.3, we can write $D_s^*\left((t-x)^2;x\right)\leq M_0^*(s)(1+x^2)$ and $D_s^*\left((t-x)^4;x\right)\leq M_1^*(s)(1+x^2)^2$. Here, $M_0^*(s)$ and $M_1^*(s)$ are given by (38) and (39), respectively. Thusly, when we choose $\delta=M_0^*(s)$, we have

$$\begin{aligned}|D_s^*(f;x)-f(x)| &\leq 2(1+x^2)\left\{1+M_0^*(s)(1+x^2)+(1+x^2)^{1/2}+\sqrt{M_1^*(s)}(1+x^2)^{3/2}\right\}\Omega(f,\delta)\\ &\leq 2(1+x^2)^{5/2}\left\{2+M_0^*(s)+\sqrt{M_1^*(s)}\right\}\Omega(f,\delta).\end{aligned}$$

Finally, we achieve the desired result

$$\sup_{x\in[0,\infty)}\frac{|D_s^*(f;x)-f(x)|}{(1+x^2)^{\frac{5}{2}}} \leq 2\left(2+M_0^*(s)+\sqrt{M_1^*(s)}\right)\Omega\left(f,\sqrt{M_0^*(s)}\right).$$

Here, $M_0^*(s)$ and $M_1^*(s)$ as given in (32) and (33), respectively. □

5. Voronovskaya-type theorem

Theorem 5.1. *For $f,f',f''\in C[0,\infty)\cap K$ and $x\in[0,\infty)$, we get*

$$\lim_{s\to\infty}s\left(D_s^*(f;x)-f(x)\right)=\left(x\mu_1(x)+\frac{A(1)+A'(1)}{A(1)}\right)f'(x)+\frac{1}{2}\left(x^2\mu_2(x)+x(2+H''(1))\right)f''(x)$$

uniformly in each compact subset of $[0,\infty)$, where $\mu_1(x)$ and $\mu_2(x)$ are respectively given by Eqn. (16) and Eqn. (17).

Proof. By using the classical Taylor expansion of the function f, we have

$$f(t)=f(x)+(t-x)f'(x)+\frac{(t-x)^2}{2}f''(x)+r(t,x)(t-x)^2, \tag{34}$$

where the remainder term $r(t,x)\in C[0,\infty)\cap K$ and $\lim_{t\to x}k(t,x)=0$. Applying the D_s^* operators to the both sides of Eqn. (34), we obtain

$$D_s^*(f;x)=f(x)+f'(x)D_s^*(t-x;x)+\frac{f''(x)}{2}D_s^*((t-x)^2;x)+D_s^*(r(t,x)(t-x)^2;x).$$

Then

$$\begin{aligned}\lim_{s\to\infty}s\left(D_s^*(f;x)-f(x)\right) &= f'(x)\lim_{s\to\infty}sD_s^*(t-x;x)+\frac{f''(x)}{2}\lim_{s\to\infty}sD_s^*((t-x)^2;x)\\ &\quad+\lim_{s\to\infty}sD_s^*(r(t,x)(t-x)^2;x).\end{aligned}$$

When we use the Cauchy-Schwarz inequality for $sD_s^*(r(t,x)(t-x)^2;x)$, we obtain

$$sD_s^*(r(t,x)(t-x)^2;x)\leq\sqrt{D_s^*(r^2(t,x);x)}\sqrt{s^2D_s^*((t-x)^4;x)}.$$

Since $r(t,x) \to 0$ as $t \to x$,

$$\lim_{s\to\infty} D_s^*(r^2(t,x);x) = r^2(x,x) = 0 \tag{35}$$

is verified uniformly in each compact subset of $[0,\infty)$. Thusly, from (13), (14) and (35) we achieve the desired result. □

6. Special cases of the operators D_s^*

In this section, we obtain some special polynomials under particular choices of analytic functions A, B and H.

6.1. Appell polynomials

Appell [7] introduced the sequences of s-degree polynomials R_s, $s = 1, 2, \ldots$ satisfying the recursive relation

$$R_s'(x) = sR_{s-1}(x), \;\; s = 1, 2, \ldots \tag{36}$$

There exists a power series $A(u) = \sum_{m=0}^{\infty} a_m u^m$, $(a_0 \neq 0)$ such that

$$A(u)e^{ux} = \sum_{s=0}^{\infty} R_s(x)u^s. \tag{37}$$

If we choose $B(t) = e^t$ and $H(t) = t$ in Boas-Buck-type polynomials (4), then we obtain Appell polynomials. As a new devolopment, Njionou Sadjang [21] studied (p,q)-Appell polynomials in the year 2019.
Now, we present moments, central moments and important theorems for Szász-Baskakov operators including Appell polynomials.

Lemma 6.1. *For every $x \in [0,\infty)$, we have*

$$\begin{aligned}
D_s^R(1;x) &= 1,\\
D_s^R(e_1;x) &= \frac{s}{s-2}x + \frac{A(1)+A'(1)}{(s-2)A(1)}, s>2,\\
D_s^R(e_2;x) &= \frac{s^2}{(s-2)(s-3)}x^2 + \frac{s\,(4A(1)+2A'(1))}{(s-2)(s-3)A(1)}x + \frac{2A(1)+4A'(1)+A''(1)}{(s-2)(s-3)A(1)}, s>3,\\
D_s^R(e_3;x) &= \frac{s^3}{(s-2)(s-3)(s-4)}x^3 + \frac{s^2\,(9A(1)+3A'(1))}{(s-2)(s-3)(s-4)A(1)}x^2\\
&\quad + \frac{s\,(18A(1)+18A'(1)+3A''(1))}{(s-2)(s-3)(s-4)A(1)}x + \frac{6A(1)+18A'(1)+9A''(1)+A'''(1)}{(s-2)(s-3)(s-4)A(1)}, s>4,\\
D_s^R(e_4;x) &= \frac{s^4}{(s-2)(s-3)(s-4)(s-5)}x^4 + \frac{s^3\,(16A(1)+4A'(1))}{(s-2)(s-3)(s-4)(s-5)A(1)}x^3\\
&\quad + \frac{s^2(72A(1)+48A'(1)+6A''(1))}{(s-2)(s-3)(s-4)(s-5)A(1)}x^2 + \frac{s(96A(1)+144A'(1)+48A''(1)+4A'''(1))}{(s-2)(s-3)(s-4)(s-5)A(1)}x\\
&\quad + \frac{24A(1)+96A'(1)+72A''(1)+16A'''(1)+A^{(iv)}(1)}{(s-2)(s-3)(s-4)(s-5)A(1)}, s>5.
\end{aligned}$$

Lemma 6.2. *For every $x \in [0,\infty)$, the operators D_s^R satisfy*

$$\begin{aligned}
D_s^R(t-x;x) &= \frac{2}{s-2}x+\frac{A(1)+A'(1)}{(s-2)A(1)},\ s>2,\\
D_s^R((t-x)^2;x) &= \left(\frac{s^2}{(s-2)(s-3)}-\frac{2s}{(s-2)}+1\right)x^2+\left(\frac{s\,(4A(1)+2A'(1))}{(s-2)(s-3)A(1)}-\frac{2A(1)+2A'(1)}{(s-2)A(1)}\right)x\\
&\quad+\frac{2A(1)+4A'(1)+A''(1)}{(s-2)(s-3)A(1)},\ s>3,\\
D_s^R((t-x)^4;x) &= \left(\frac{s^4}{(s-2)(s-3)(s-4)(s-5)}-\frac{4s^3}{(s-2)(s-3)(s-4)}+\frac{6s^2}{(s-2)(s-3)}-\frac{4s}{(s-2)}+1\right)x^4\\
&\quad+\left(\frac{s^3\,(16A(1)+4A'(1))}{(s-2)(s-3)(s-4)(s-5)A(1)}-\frac{4s^2\,(9A(1)+3A'(1))}{(s-2)(s-3)(s-4)A(1)}\right.\\
&\quad\left.+\frac{6s\,(4A(1)+2A'(1))}{(s-2)(s-3)A(1)}-\frac{4A(1)+4A'(1)}{(s-2)A(1)}\right)x^3\\
&\quad+\left(\frac{s^2(72A(1)+48A'(1)+6A''(1))}{(s-2)(s-3)(s-4)(s-5)A(1)}-\frac{4s\,(18A(1)+18A'(1)+3A''(1))}{(s-2)(s-3)(s-4)A(1)}\right.\\
&\quad\left.+\frac{12A(1)+24A'(1)+6A''(1)}{(s-2)(s-3)A(1)}\right)x^2\\
&\quad+\left(\frac{s(96A(1)+144A'(1)+48A''(1)+4A'''(1))}{(s-2)(s-3)(s-4)(s-5)A(1)}-\frac{24A(1)+72A'(1)+36A''(1)+4A'''(1)}{(s-2)(s-3)(s-4)A(1)}\right)x\\
&\quad+\frac{24A(1)+96A'(1)+72A''(1)+16A'''(1)+A^{(iv)}(1)}{(s-2)(s-3)(s-4)(s-5)A(1)},\ s>5.
\end{aligned}$$

Lemma 6.3. *We have following results*

$$\begin{aligned}
\lim_{s\to\infty} sD_s^R(t-x;x) &= 2x+\frac{A(1)+A'(1)}{A(1)},\\
\lim_{s\to\infty} sD_s^R((t-x)^2;x) &= x^2+2x,\\
\lim_{s\to\infty} s^2D_s^R((t-x)^4;x) &= 3x^4+12x^3+12x^2.
\end{aligned}$$

Theorem 6.4. *For $f, f', f'' \in C[0,\infty)\cap K$ and $x\in[0,\infty)$, we have*

$$\lim_{s\to\infty} s\left(D_s^R(f;x)-f(x)\right)=\left(2x+\frac{A(1)+A'(1)}{A(1)}\right)f'(x)+\frac{1}{2}\left(x^2+2x\right)f''(x)$$

uniformly in each compact subset of $[0,\infty)$.

Theorem 6.5. *If $f\in C_\rho^*[0,\infty)$, then*

$$\sup_{x\in[0,\infty)}\frac{|D_s^R(f;x)-f(x)|}{(1+x^2)^{\frac{5}{2}}} \le 2\left(2+M_0^R(s)+\sqrt{M_1^R(s)}\right)\Omega\left(f,\sqrt{M_0^R(s)}\right),$$

where

$$\begin{aligned}
M_0^R(s) &= \left(\frac{s^2}{(s-2)(s-3)}-\frac{2s}{s-2}+1\right)+\frac{1}{2}\left(\frac{s\,(4A(1)+2A'(1))}{(s-2)(s-3)A(1)}-\frac{2A(1)+2A'(1)}{(s-2)A(1)}\right)\\
&\quad+\frac{2A(1)+4A'(1)+A''(1)}{(s-2)(s-3)A(1)},\ s>3 \qquad (38)
\end{aligned}$$

and

$$
\begin{aligned}
M_1^R(s) &= \left(\frac{s^4}{(s-2)(s-3)(s-4)(s-5)} - \frac{4s^3}{(s-2)(s-3)(s-4)} + \frac{6s^2}{(s-2)(s-3)} - \frac{4s}{(s-2)} + 1\right) \\
&+ \frac{3\sqrt{3}}{16}\left(\frac{s^3\,(16A(1)+4A'(1))}{(s-2)(s-3)(s-4)(s-5)A(1)} - \frac{4s^2\,(9A(1)+3A'(1))}{(s-2)(s-3)(s-4)A(1)}\right. \\
&\left.+ \frac{6s\,(4A(1)+2A'(1))}{(s-2)(s-3)A(1)} - \frac{4A(1)+4A'(1)}{(s-2)A(1)}\right) + \frac{s^2(18A(1)+12A'(1)+3/2A''(1))}{(s-2)(s-3)(s-4)(s-5)A(1)} \\
&- \frac{s\,(18A(1)+18A'(1)+3A''(1))}{(s-2)(s-3)(s-4)A(1)} + \frac{3A(1)+6A'(1)+(3/2)A''(1)}{(s-2)(s-3)A(1)} \\
&+ \frac{3\sqrt{3}}{16}\left(\frac{s(96A(1)+144A'(1)+48A''(1)+4A'''(1))}{(s-2)(s-3)(s-4)(s-5)A(1)} - \frac{24A(1)+72A'(1)+36A''(1)+4A'''(1)}{(s-2)(s-3)(s-4)A(1)}\right) \\
&+ \frac{24A(1)+96A'(1)+72A''(1)+16A'''(1)+A^{(iv)}(1)}{(s-2)(s-3)(s-4)(s-5)A(1)}, \quad s>5. \qquad (39)
\end{aligned}
$$

Here, we obtain Hermite polynomials and Gould-Hopper polynomials under special choices of A.

6.1.1. Hermite polynomials

If $A(u) = e^{-\frac{\xi u^2}{2}}$ then $R_s(x) = H_s^{(\xi)}(x)$ is the Hermite polynomials of variance ξ, which is

$$
H_v^{(\xi)}(x) = \sum_{m=0}^{[\frac{v}{2}]} \left(-\frac{\xi}{2}\right)^m \frac{v!}{m!(v-2m)!} x^{v-2m}. \qquad (40)
$$

Here, [.] denotes the integer part. The generating function of the Hermite polynomials is given by

$$
\sum_{v=0}^{\infty} \frac{u^v}{v!} H_v^{(\xi)}(x) = e^{ux-\frac{\xi}{2}u^2}. \qquad (41)
$$

Then the Szász-Baskakov operators including Hermite polynomials of variance ξ is given by

$$
D_s^H(x) = (s-1)e^{\xi/2-sx} \sum_{v=0}^{\infty} \frac{H_v^{(\xi)}(sx)}{v!} \int_0^{\infty} \binom{s+v-1}{v} \frac{t^v}{(1+t)^{s+v}} f(t)\,dt. \qquad (42)
$$

Under the assumption $\xi \le 0$, restrictions i)–iii) and assumptions (10) for the operators D_s^H are verified.

Lemma 6.6. *We have following results*

$$
\begin{aligned}
\lim_{s\to\infty} sD_s^H(t-x;x) &= 2x+1-\xi, \\
\lim_{s\to\infty} sD_s^H((t-x)^2;x) &= x^2+2x, \\
\lim_{s\to\infty} s^2D_s^H((t-x)^4;x) &= 3x^4+12x^3+12x^2.
\end{aligned}
$$

Theorem 6.7. *For $f, f', f'' \in C[0,\infty) \cap K$ and $x \in [0,\infty)$, we get*

$$
\lim_{s\to\infty} s\left(D_s^H(f;x) - f(x)\right) = (2x+1-\xi)\,f'(x) + \frac{1}{2}\left(x^2+2x\right) f''(x)
$$

uniformly in each compact subset of $[0,\infty)$.

Theorem 6.8. *If* $f \in C_p^*[0,\infty)$*, then*

$$\sup_{x\in[0,\infty)} \frac{|D_s^H(f;x) - f(x)|}{(1+x^2)^{\frac{5}{2}}} \leq 2\left(2 + M_0^H(s) + \sqrt{M_1^H(s)}\right)\Omega\left(f, \sqrt{M_0^H(s)}\right),$$

where

$$M_0^H(s) = \frac{2s + 11 - 8\xi + \xi^2}{(s-2)(s-3)},$$

$$M_1^H(s) = \frac{3(8+3\sqrt{3})s^2 + s(596 + 273\sqrt{3} - 6(25+17\sqrt{3})\xi + (6+9\sqrt{3})\xi^2)}{4(s-2)(s-3)(s-4)(s-5)} + \frac{816 + 270\sqrt{3} - 3(424+195\sqrt{3})\xi + 36(17+5\sqrt{3})\xi^2 - (88+15\sqrt{3})\xi^3 + 4\xi^4}{4(s-2)(s-3)(s-4)(s-5)}.$$

By taking $f(x) = \frac{x^2}{1+x^3}$, we can see the error estimation of the Szász-Baskakov operators including Hermite polynomials by the help of weighted modulus of continuity in Table 1.

s	$\xi = -0.005$	$\xi = -0.6$	$\xi = -0.85$	$\xi = -1.5$
10	2.09861	2.2582	2.32598	2.50571
10^2	0.344189	0.348233	0.35008	0.355279
10^3	0.106257	0.10639	0.106452	0.106626
10^4	0.0337866	0.0337909	0.0337929	0.0337985
10^5	0.0107167	0.0107169	0.0107169	0.0107171
10^6	0.00339262	0.00339262	0.00339263	0.00339263
10^7	0.00107322	0.00107322	0.00107322	0.00107322

Table 1: Error estimation of D_s^H by using weighted modulus of continuity

6.1.2. Gould-Hopper polynomials

By taking $A(u) = e^{hu^m}$, then $R_s(x) = G_s^m(x,h)$ is the Gould–Hopper polynomials. The explicit representation of these polynomials is given as:

$$G_v^m(x,h) = \sum_{j=0}^{[\frac{v}{m}]} \frac{v!}{j!(v-mj)!} h^j x^{v-mj}. \tag{43}$$

Here, [.] denotes the integer part. The generating function of the Gould–Hopper polynomials [14] is given by

$$\sum_{v=0}^{\infty} \frac{u^v}{v!} G_v^m(x,h) = e^{ux+hu^m}. \tag{44}$$

Then the explicit form of the Szász-Baskakov operators including Gould-Hopper polynomials are given by

$$D_s^G(f;x) = (s-1)e^{-h-sx} \sum_{v=0}^{\infty} \frac{G_v^m(sx,h)}{v!} \int_0^{\infty} \binom{s+v-1}{v} \frac{t^v}{(1+t)^{s+v}} f(t)\,dt. \tag{45}$$

It is worhty to note that for $h = 0$, we have $G_v^m(sx, 0) = (sx)^v$ and the operators (45) lead to the Szász-Baskakov operators. Under the assumption $h \leq 0$, restrictions i)–iii) and assumptions (10) for the operators D_s^G are verified.

Theorem 6.9. *For $f, f', f'' \in C[0,\infty) \cap K$ and $x \in [0,\infty)$, we get*

$$\lim_{s\to\infty} s\left(D_s^G(f;x) - f(x)\right) = (2x+1+hm)f'(x) + \frac{1}{2}\left(x^2+2x\right)f''(x)$$

uniformly in each compact subset of $[0,\infty)$.

Theorem 6.10. *If $f \in C_\rho^*[0,\infty)$, then*

$$\sup_{x\in[0,\infty)} \frac{|D_s^G(f;x) - f(x)|}{(1+x^2)^{\frac{5}{2}}} \leq 2\left(2 + M_0^G(s) + \sqrt{M_1^G(s)}\right)\Omega\left(f, \sqrt{M_0^G(s)}\right),$$

where

$$M_0^G(s) = \frac{h^2m^2 + hm^2 + 6hm + 2s + 11}{(s-2)(s-3)},$$

$$\begin{aligned} M_1^G(s) &= \frac{816 + 270\sqrt{3} + 4h^4m^4 + h^3m^3(40 + 15\sqrt{3} + 24m) + (596 + 273\sqrt{3})s + 3(8 + 3\sqrt{3})s^2}{4(s-2)(s-3)(s-4)(s-5)} \\ &+ \frac{h^2m^2(260 + 90\sqrt{3} + 15(8 + 3\sqrt{3})m + 28m^2 + 6s + 9\sqrt{3}s)}{4(s-2)(s-3)(s-4)(s-5)} \\ &+ \frac{hm(560 + 345\sqrt{3} + 5(8 + 3\sqrt{3})m^2 + 4m^3 + 138s + 84\sqrt{3}s + m(260 + 90\sqrt{3} + 6s + 9\sqrt{3}s))}{4(s-2)(s-3)(s-4)(s-5)}. \end{aligned}$$

By taking $f(x) = \frac{x^2}{1+x^3}$ and $m = 0.5$, we obtain the error approximation of the Szász-Baskakov operators including Gould-Hopper polynomials by using weighted modulus of continuity as we see in Table 2.

s	$h = 0.005$	$h = 1.5$	$h = 2$	$h = 4$
10	2.09782	2.2599	2.31555	2.55013
10^2	0.34417	0.348459	0.350073	0.357392
10^3	0.106256	0.106397	0.106451	0.106698
10^4	0.0337866	0.0337911	0.0337929	0.0338008
10^5	0.0107167	0.0107169	0.0107169	0.0107172
10^6	0.00339262	0.00339262	0.00339263	0.00339263
10^7	0.00107322	0.00107322	0.00107322	0.00107322

Table 2: Error of D_s^G by using weighted modulus of continuity for $m = 0.5$

6.2. *Laguerre polynomials*

The Laguerre polynomials are Boas–Buck-type polynomials with $A(u) = \frac{1}{(1-u)^{\alpha+1}}$, $H(u) = -\frac{u}{(1-u)}$, $B(u) = e^u$, where $0 \leq u < 1$, $x < 0$ and α is a nonnegative integer. The Laguerre polynomial of degree v is defined as

$$L_v^{(\alpha)}(x) = \sum_{j=0}^{v} \binom{v+\alpha}{v-j} \frac{(-x)^j}{j!}, \ \alpha > -1.$$

It can be seen that for $x \leq 0$ Laguerre polynomials are positive. Generating functions of $L_v^{(\alpha)}$ are

$$\frac{1}{(1-u)^{\alpha+1}} e^{\frac{-xu}{1-u}} = \sum_{v=0}^{\infty} L_v^{(\alpha)}(x) u^v, \ 0 \leq u < 1. \tag{46}$$

In order to ensure the restrictions i)-iii) ans assumption (10), the generating function (46) should be modified when $u \to u/2$ and $x \to -x/2$. The new form of the generating function is given as follows

$$\frac{1}{\left(1-\frac{u}{2}\right)^{\alpha+1}} e^{\frac{xu}{2(2-u)}} = \sum_{v=0}^{\infty} \frac{L_v^{(\alpha)}(\frac{-x}{2})}{2^v} u^v, \ 0 \le u < 2.$$

Thus, D_s^L the Szász-Mirakyan-Baskakov operators including Laguerre polynomials are given by

$$D_s^L(f;x) = \frac{(s-1)e^{-sx/2}}{2^{\alpha+1}} \sum_{v=0}^{\infty} \frac{L_v^{(\alpha)}(\frac{-sx}{2})}{2^v} \int_0^{\infty} \binom{s+v-1}{v} \frac{t^v}{(1+t)^{s+v}} f(t)\, dt.$$

Theorem 6.11. *For $f, f', f'' \in C[0,\infty) \cap K$ and $x \in [0,\infty)$, we get*

$$\lim_{s\to\infty} s\left(D_s^L(f;x) - f(x)\right) = (2x+\alpha+2) f'(x) + \frac{1}{2}\left(x^2+2x\right) f''(x)$$

uniformly in each compact subset of $[0,\infty)$.

Theorem 6.12. *If $f \in C_\rho^*[0,\infty)$, then*

$$\sup_{x\in[0,\infty)} \frac{|D_s^L(f;x) - f(x)|}{(1+x^2)^{\frac{5}{2}}} \le 2\left(2 + M_0^L(s) + \sqrt{M_1^L(s)}\right)\Omega\left(f, \sqrt{M_0^L(s)}\right),$$

where

$$\begin{aligned} M_0^L(s) &= \frac{20+3s+10\alpha+\alpha^2}{(s-2)(s-3)}, \\ M_1^L(s) &= \frac{24(124+45\sqrt{3}) + 6(10+3\sqrt{3})s^2 + 68(44+15\sqrt{3})\alpha + (932+225\sqrt{3})\alpha^2}{4(s-2)(s-3)(s-4)(s-5)} \\ &\quad + \frac{(104+15\sqrt{3})\alpha^3 + 4\alpha^4 + s(1352+870\sqrt{3}+3(94+79\sqrt{3})\alpha + 6(1+3\sqrt{3})\alpha^2)}{4(s-2)(s-3)(s-4)(s-5)}. \end{aligned}$$

If we take $f(x) = \frac{x^2}{1+x^3}$, then we achieve the error approximation of the Szász-Baskakov operators including Laguerre polynomials with the help of weighted modulus of continuity in Table 3.

s	$\alpha = 1.0$	$\alpha = 2.0$	$\alpha = 3.0$	$\alpha = 4.0$
10	3.10694	3.44874	3.83697	4.27149
10^2	0.429377	0.437536	0.446734	0.456903
10^3	0.130051	0.130325	0.13064	0.130997
10^4	0.0413435	0.0413524	0.0413627	0.0413743
10^5	0.0131207	0.013121	0.0131213	0.0131217
10^6	0.00415461	0.00415462	0.00415463	0.00415464
10^7	0.00131438	0.00131438	0.00131438	0.00131438

Table 3: Error of D_s^L by using weighted modulus of continuity

6.3. Charlier polynomials

The Charlier polynomial of degree v is defined as

$$L_v^{(b)}(x) = \sum_{j=0}^{v} \binom{v}{j} \frac{(-x)_j}{v!b^j},$$

where $b > 1$, $x \in [0, \infty)$, $(x)_0 = 1$, $(x)_j = x(x+1)\ldots(x+j-1)$, $j \geq 1$ Generating functions of teh Charlier polynomials are

$$e^u \left(1 - \frac{u}{b}\right)^x = \sum_{v=0}^{\infty} C_v^{(b)}(x) u^v, \ |u| < b. \tag{47}$$

The Charlier polynomials are Boas–Buck-type polynomials with $A(u) = e^u$, $H(u) = \ln\left(1 - \frac{u}{b}\right)$, $B(u) = e^u$, where $|u| < b$. In order to ensure the restrictions i)-iii) ans assumption (10), the generating function (47) should be modified. The new form of the generating function is given as follows

$$e^u e^{-(b-1)x \ln(1-u/b)} = \sum_{v=0}^{\infty} C_v^{(b)}(x) u^v, \ |u| < b.$$

Thus, D_s^C the Szász-Mirakyan-Baskakov operators including Charlier polynomials are given by

$$D_s^C(f;x) = \frac{(s-1)\left(1-\frac{1}{b}\right)^{(b-1)sx}}{e} \sum_{v=0}^{\infty} C_v^{(b)}(-(b-1)x) \int_0^{\infty} \binom{s+v-1}{v} \frac{t^v}{(1+t)^{s+v}} f(t)\, dt.$$

Theorem 6.13. *For $f, f', f'' \in C[0,\infty) \cap K$ and $x \in [0,\infty)$, we get*

$$\lim_{s\to\infty} s\left(D_s^C(f;x) - f(x)\right) = (2x+2) f'(x) + \frac{1}{2}\left(x^2 + \left(2 - \frac{1}{(1-b)^2}\right)x\right) f''(x)$$

uniformly in each compact subset of $[0,\infty)$.

Theorem 6.14. *If $f \in C_\rho^*[0,\infty)$, then*

$$\sup_{x\in[0,\infty)} \frac{|D_s^C(f;x) - f(x)|}{(1+x^2)^{\frac{5}{2}}} \leq 2\left(2 + M_0^C(s) + \sqrt{M_1^C(s)}\right)\Omega\left(f, \sqrt{M_0^C(s)}\right),$$

where

$$M_0^C(s) = \frac{38 + 3s + 2(b^2 - 2b)(19 + 2s)}{2(s-2)(s-3)(b-1)^2},$$

$$\begin{aligned} M_1^C(s) = {} & \frac{4312 + 1740\sqrt{3} - 17248b - 6960\sqrt{3}b + 25872b^2 + 10440\sqrt{3}b^2 - 17248b^3 - 6960\sqrt{3}b^3}{8(s-2)(s-3)(s-4)(s-5)(b-1)^4} \\ & + \frac{4312b^4 + 1740\sqrt{3}b^4 + 1092s + 432\sqrt{3}s - 5088bs - 2322\sqrt{3}bs + 8472b^2s + 4131\sqrt{3}b^2s}{8(s-2)(s-3)(s-4)(s-5)(b-1)^4} \\ & + \frac{-5968b^3s - 3000\sqrt{3}b^3s + 1492b^4s750\sqrt{3}b^4s + 30s^2 + 9\sqrt{3}s^2 - 144bs^2 - 54\sqrt{3}bs^2}{8(s-2)(s-3)(s-4)(s-5)(b-1)^4} \\ & + \frac{264b^2s^2 + 99\sqrt{3}b^2s^2 - 192b^3s^2 - 72\sqrt{3}b^3s^2 + 48b^4s^2 + 18\sqrt{3}b^4s^2}{8(s-2)(s-3)(s-4)(s-5)(b-1)^4}. \end{aligned}$$

By taking $f(x) = \frac{x^2}{1+x^3}$, we get the error analysis of the Szász-Baskakov operators including Charlier polynomials by the help of weighted modulus of continuity which is shown in Table 4.

s	$b = 2.5$	$b = 4.5$	$b = 5.5$	$b = 6.5$
10	2.22605	2.31353	2.3208	3.32444
10^2	0.332545	0.347242	0.348516	0.34916
10^3	0.100514	0.0105397	0.10582	0.106033
10^4	0.0318717	0.0334487	0.0335853	0.0336542
10^5	0.010105	0.0106072	0.0106507	0.0106726
10^6	0.0031987	0.00335785	0.00337163	0.00337858
10^7	0.00101186	0.00106222	0.00106658	0.00106878

Table 4: Error of D_s^C by using weighted modulus of continuity

7. Conclusion

In this work, the Szász-Baskakov operators including Boas-Buck-type polynomials are introduced. Uniform convergence of these operators is shown in each compact subset of $[0, \infty)$. The rate of convergence of the boundless function defined on $[0, \infty)$ is computed in weighted spaces and by using weighted modulus of continuity. Then a Voronovskaya-type theorem is given for quantitative asymptotic estimation. In the last part, some special polynomials are obtained under particular choices of analytic functions A, B and H. Finally, the error approximations of the Szász-Baskakov operators including special polynomials are given by using weighted modulus of continuity.

References

[1] P.N. Agrawal, H. Karsli, M. Goyal, Szász-Baskakov type operators based on q-integers. J Inequal Appl (2014) 441.
[2] F. Altomare, M. Campiti, Korovkin-type Approximation Theory and its Applications, De Gruyter Studies in Mathematics 17 (1994) W. De Gruyter, Berlin-New York.
[3] K.J. Ansari, M. Mursaleen, S. Rahman, Approximation by Jakimovski–Leviatan operators of Durrmeyer type involving multiple Appell polynomials, Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat, 113(2) (2019) 1007–1024.
[4] K.J. Ansari, S. Rahman, M. Mursaleen, Approximation and error estimation by modified Păltănea operators associating Gould–Hopper polynomials, Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. 113(3) (2019) 2827–2851.
[5] K.J. Ansari, M. Mursaleen, K.P.M. Shareef, M. Ghouse, Approximation by modified Kantorovich-Szász type operators involving Charlier polynomials, Advances in Difference Equations, vol. 2020 no. 1 (2020).
[6] K.J. Ansari, M.A. Salman, M. Mursaleen, A.H.H Al-Abied, On Jakimovski-Leviatan-Păltănea approximating operators involving Boas-Buck-type polynomials, Journal of King Saud University-Science, 32(7) (2020) 3018-3025.
[7] P.E. Appell, Sur une classe de polynômes, Ann. Sci. École Norm. Sup., 9 (1880) pp. 119–144.
[8] M. Ayman Mursaleen, S. Serra-Capizzano, Statistical Convergence via q-Calculus and a Korovkin's Type Approximation Theorem, Axioms 11 (2022) 70 (doi: https://doi.org/10.3390/axioms11020070)
[9] V.A. Baskakov, An instance of a sequence of linear positive operators in the space of continuous functions, Dokl. Akad. Nauk SSSR, 113(2), 249-251 (1957).
[10] Q.-B. Cai, A. Kilicman, M. A. Mursaleen et al. Approximation Properties and q-Statistical Convergence of Stancu-Type Generalized Baskakov-Szász Operators, Journal of Function Spaces, Vol. (2022) 1-9. DOI: 10.1155/2022/2286500
[11] M.-Y. Chen, Md Nasiruzzaman, M. A. Mursaleen, N. Rao, A. Kilicman, On Shape Parameter α-Based Approximation Properties and q-Statistical Convergence of Baskakov-Gamma Operators, Journal of Mathematics, Volume (2022), Article ID 4190732.
[12] B. Çekim, R. Aktaş, G. İçöz, Kantorovich-Stancu type operators including Boas-Buck type polynomials, Hacet. J. Math. Stat. 48(2) (2019) 460-471.
[13] E. Gadjieva, O. Doğru, Ağırlıklı Uzaylarda Szazs Tipinde Operatörler Dizisinin Sürekli fonksiyonlara Yaklaşımı. II. Kızılırmak Uluslararası Fen Bilimleri Kongresi. 20-22 Mayıs (1998), KIRIKKALE.
[14] H.W. Gould, A.T. Hopper, Operational formulas connected with two generalizations of Hermite polynomials, Duke Mathematical Journal 29 (1962) 51-63.
[15] M. Ismail, On a generalization of Szász operators, Mathematica 16(39)(2) (1974) 259-267.
[16] M.E.H. Ismail, Classical and Quantum Orthogonal Polynomials in One Variable, Cambridge, UK: Cambridge University Press, 2005.
[17] A. Jakimovski, D. Leviatan, Generalized Szász operators for the approximation in the infinite interval, Mathematica (Cluj) 34 (1969) 97-103.
[18] A. Kilicman, M.A. Mursaleen, A.A.H.A. Al-Abied, Stancu Type Baskakov-Durrmeyer Operators and Approximation Properties, Mathematics (2020) 8(7) 1164. https://doi.org/10.3390/math8071164

[19] M. Mursaleen, A.H. Al-Abied, A.M. Acu, Approximation by Cholodowsky type of Szász operators based on Boas-Buck-type polynomials, Turkish Journal of Mathematics 42 (2018) 2243-2259.
[20] M. Mursaleen, S. Rahman, K.J. Ansari, On the approximation by Bézier–Păltănea operators based on Gould–Hopper polynomials, Math. Commun. 24 (2019) 147–164.
[21] P. Njionou Sadjang, On (p, q)-Appell Polynomials, Anal Math 45 (2019) 583–598.
[22] G. Prasad, P.N. Agrawal, H.S. Kasana, Approximation of functions on $[0, \infty)$ by a new sequence of modified Szász operators, Math. Forum 6(2) (1983) 1–11.
[23] O. Szász, Generalization of S. Bernstein's polynomials to the infinite interval, J. Research Nat. Bur. Standards 45 (1950) 239-245.
[24] S. Sucu, G. Içöz, S. Varma, On some extensions of Szász operators including Boas-Buck-type polynomials, Abstract and Applied Analysis (2012) Article ID:680340.
[25] S. Varma, S. Sucu, G. İçöz, Generalization of Szász operators involving Brenke type polynomials, Comput. Math. Appl. 64(2) (2012) 121-127.
[26] S. Varma, F. Taşdelen, Szász type operators involving Charlier polynomials, Math. Comput. Model. 56(5-6) (2012) 118-122.
[27] A.D. Gadjiev, The convergence problem for a sequence of positive linear operators on bounded sets and theorems analogous to that of P.P. Korovkin, Soviet Dokl. Akad. Nauk SSSR, 218 (1974) 1001-1004 (in Russian).
[28] A.D. Gadjiev, Theorems of Korovkin type, Mat Zametki 20(5) (1976) 781-786 (in Russian).
[29] I. Yüksel, N. Ispir, Weighted approximation by a certain family of summation integral-type operators, Comput. Math. Appl. 52 (2006) 1463-1470.