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Linear Inequalities via Extension of Montgomery Identity and Weighted Hermite-Hadamard Inequalities with and without Green Functions

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Abstract. Weighted Hermite-Hadamard dual inequality in integral form is an important result as its left hand inequality is in fact Jensen inequality and right hand inequality is the Lah–Ribarić inequality. In this paper new linear inequalities are introduced via extension of Montgomery identity and weighted Hermite-Hadamard inequalities with and without Green functions in discrete and integral cases.

1. Introduction and Preliminaries

Here we recall weighted Hermite-Hadamard dual inequality for convex functions as under [12]:

Theorem 1.1. *Let $p : [a, b] \to \mathbb{R}$ be a nonnegative function. If f is a convex function given on an interval I, then we have*

$$f(\lambda) \leq \frac{1}{P}\int_a^b p(x)f(x)dx \leq \frac{b-\lambda}{b-a}f(a) + \frac{\lambda-a}{b-a}f(b) \tag{1}$$

or

$$Pf(\lambda) \leq \int_a^b p(x)f(x)dx \leq P\left[\frac{b-\lambda}{b-a}f(a) + \frac{\lambda-a}{b-a}f(b)\right] \tag{2}$$

where

$$P = \int_a^b p(x)dx \quad and \quad \lambda = \frac{1}{P}\int_a^b p(x)xdx.$$

Note that in this important inequality LH inequality is in fact Jensen's inequality and RH inequality is Lah-Ribarić inequality in integral form (see [11]).

The following result is due to Popoviciu [13, 14].

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Theorem 1.2. *The inequality*

$$\sum_{i=1}^{m} p_i f(x_i) \geq 0 \tag{3}$$

holds $\forall$ $n-$*convex functions* $f : [a,b] \to \mathbb{R}$, $n \in \mathbb{N}$, *iff the* $m-$*tuples* $\mathbf{x} = (x_1, x_2, \ldots, x_m) \in [a,b]^m$, $\mathbf{p} = (p_1, p_2, \ldots, p_m) \in \mathbb{R}^m$ *satisfy*

$$\sum_{i=1}^{m} p_i x_i^k = 0, \quad \forall\, k \in \{0, 1, \ldots, n-1\}, \tag{4}$$

$$\sum_{i=1}^{m} p_i (x_i - t)_+^{n-1} \geq 0, \quad \textit{for every } t \in [a,b], \tag{5}$$

where $y_+ = \max(y, 0)$.

In fact, Popoviciu proved a stronger result that it is enough to assume that the inequality in (5) holds for every $t \in [x_{(1)}, x_{(m-n+1)}]$, where $x_{(1)} \leq \cdots \leq x_{(m)}$ is the ordered $m-$tuple $\mathbf{x}$, since this, together with (4), implies that it holds for every $t \in [a,b]$ (see [15]). In the case of convex functions, i.e. $n = 2$, Pečarić [10] proved the result with the conditions (4) and (5) replaced with

$$\sum_{i=1}^{m} p_i = 0 \quad \text{and} \quad \sum_{i=1}^{m} p_i |x_i - x_k| \geq 0 \text{ for } k \in \{1, \ldots, m\}. \tag{6}$$

The integral analogue of Proposition 1.2 is given in the next proposition.

Theorem 1.3. *Let* $n \geq 2$, $p : [\alpha, \beta] \to \mathbb{R}$ *and* $g : [\alpha, \beta] \to [a,b]$. *The inequality*

$$\int_{\alpha}^{\beta} p(x) f(g(x))\, dx \geq 0 \tag{7}$$

holds forall $n-$*convex functions* $f : [a,b] \to \mathbb{R}$ *iff*

$$\int_{\alpha}^{\beta} p(x) g(x)^k\, dx = 0, \quad \forall\, k \in \{0, 1, \ldots, n-1\}, \tag{8}$$

$$\int_{\alpha}^{\beta} p(x) \left(g(x) - t\right)_+^{n-1}\, dx \geq 0, \quad \textit{for every } t \in [a,b]. \tag{9}$$

In [1] we can found following extension of Montgomery's identity via Taylor's formula (see also [2]).

Theorem 1.4. *Let* $n \in \mathbb{N}$, $f : I \to \mathbb{R}$ *be such that* $f^{(n-1)}$ *is absolutely continuous,* $I \subset \mathbb{R}$ *an open interval,* $a, b \in I$, $a < b$. *Then the following identity holds*

$$\begin{aligned} f(x) &= \frac{1}{b-a} \int_a^b f(t)\, dt + \frac{1}{b-a} \sum_{k=0}^{n-2} \frac{1}{k!\,(k+2)} \left[f^{(k+1)}(a)(x-a)^{k+2} - f^{(k+1)}(b)(x-b)^{k+2} \right] \\ &+ \frac{1}{(n-1)!} \int_a^b T_n(x,s) f^{(n)}(s)\, ds \end{aligned} \tag{10}$$

where

$$T_n(x,s) = \begin{cases} -\dfrac{(x-s)^n}{n(b-a)} + \dfrac{x-a}{b-a}(x-s)^{n-1}, & a \leq s \leq x, \\[2ex] -\dfrac{(x-s)^n}{n(b-a)} + \dfrac{x-b}{b-a}(x-s)^{n-1}, & x < s \leq b. \end{cases} \tag{11}$$

From this important identity we easily get Montgomery identity by putting $n = 1$ (see [4] and [9]).

Using this extension of Montgomery identity, Asif et. al in [3] stated and proved following results in discrete and integral form respectively.

Theorem 1.5. *Let $n \in \mathbb{N}$, $f : I \to \mathbb{R}$ be such that $f^{(n+1)}$ is absolutely continuous, $I \subset \mathbb{R}$ an open interval, $a, b \in I$, $a < b$ let T_n be given by* (11). *Furthermore, let $m \in \mathbb{N}$, $x_i \in [a, b]$ and $p_i \in \mathbb{R}$ for $i \in \{1, 2, \ldots, m\}$ be such that $\sum_{i=1}^{m} p_i = 0$. Then*

$$\sum_{i=1}^{m} p_i f(x_i) - \frac{1}{b-a} \sum_{k=0}^{n-2} \frac{1}{k!\,(k+2)} \sum_{i=1}^{m} p_i \left[f^{(k+1)}(a)\,(x_i - a)^{k+2} - f^{(k+1)}(b)\,(x_i - b)^{k+2} \right]$$
$$= \frac{1}{(n-1)!} \int_a^b \left(\sum_{i=1}^{m} p_i T_n(x_i, s) \right) f^{(n)}(s)\, ds \tag{12}$$

where T_n is as defined in (11).

Theorem 1.6. *Let $g : [\alpha, \beta] \to [a, b]$ and $p : [\alpha, \beta] \to \mathbb{R}$ be integrable functions such that $\int_\alpha^\beta p(x)dx = 0$. Let $n \in \mathbb{N}$, $I \subset \mathbb{R}$ be an open interval, $a, b \in I$, $a < b$ and $f : I \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous. Then*

$$\int_\alpha^\beta p(x)\, f(g(x))\, dx - \frac{1}{b-a} \sum_{k=0}^{n-2} \frac{1}{k!\,(k+2)} \int_\alpha^\beta p(x) \left[f^{(k+1)}(a)\,(g(x) - a)^{k+2} - f^{(k+1)}(b)\,(g(x) - b)^{k+2} \right] dx$$
$$= \frac{1}{(n-1)!} \int_a^b \left(\int_\alpha^\beta p(x)\, T_n(g(x), s)\, dx \right) f^{(n)}(s)\, ds \tag{13}$$

where T_n is as defined in (11).

2. Linear inequalities via extension of Montgomery identity and weighted Hermite-Hadamard inequalities

Theorem 2.1. *Let all the assumptions of Theorem* 1.5 *hold.*

If

$$\sum_{i=1}^{m} p_i T_n(x_i, s) \geq 0, \quad \text{for all } s \in [a, b], \tag{14}$$

then:

1. *for every $(n+2)$–convex function $f : I \to \mathbb{R}$ the following inequalities hold*

$$\begin{aligned} \frac{P_1(n)}{(n-1)!} f^{(n)}(\lambda_1(n)) &\leq \sum_{i=1}^{m} p_i f(x_i) - \frac{1}{b-a} \sum_{k=0}^{n-2} \frac{1}{k!\,(k+2)} \sum_{i=1}^{m} p_i \left[f^{(k+1)}(a)\,(x_i - a)^{k+2} - f^{(k+1)}(b)\,(x_i - b)^{k+2} \right] \\ &\leq \frac{P_1(n)}{(n-1)!} \left[\frac{b - \lambda_1(n)}{b-a} f^{(n)}(a) + \frac{\lambda_1(n) - a}{b-a} f^{(n)}(b) \right] \end{aligned} \tag{15}$$

where

$$\begin{aligned} P_1(n) &= \sum_{i=1}^{m} p_i \int_a^b T_n(x_i, s)\, ds \\ &= \sum_{i=1}^{m} p_i \frac{(x_i - a)^{n+1} - (x_i - b)^{n+1}}{(n+1)\,(b-a)} \end{aligned}$$

and

$$\begin{aligned}\lambda_1(n) &= \frac{\sum_{i=1}^{m} p_i \int_a^b sT_n(x_i,s)\,ds}{\sum_{i=1}^{m} p_i \int_a^b T_n(x_i,s)\,ds}\\ &= \frac{1}{(n+1)(b-a)P_1(n)}\sum_{i=1}^{m} p_i\left[a(x_i-a)^n - b(x_i-b)^n\right]-1\end{aligned}$$

or

$$\lambda_1(n) = \frac{1}{n(n+1)P_1(n)}\sum_{i=1}^{m} p_i x_i^{n+1} - \frac{1}{P_1(n)(b-a)}\sum_{k=0}^{n-2}\frac{1}{k(k+2)}\binom{n-1}{k-1}$$
$$\times\sum_{i=1}^{m} p_i\left[a^{n-k}(x_i-a)^{k+2} - b^{n-k}(x_i-b)^{k+2}\right],$$

2. *for every* $(n+2)-$*concave functions* $f : I \to \mathbb{R}$, (15) *holds with the reversed sign of inequalities.*

Proof. 1. Since f is $(n+2)$ convex, then $f^{(n)}$ is convex. Applying weighted Hermite-Hadamard inequalities (2) on a convex function $f^{(n)}$ with weight $\sum p_i T_n(x_i,s)$, we get

$$\begin{aligned}\frac{P_1(n)}{(n-1)!}f^{(n)}(\lambda_1(n)) &\le \frac{1}{(n-1)!}\int_a^b\left(\sum_{i=1}^{m} p_i T_n(x_i,s)\right)f^{(n)}(s)ds\\ &\le \frac{P_1(n)}{(n-1)!}\left[\frac{b-\lambda_1(n)}{b-a}f^{(n)}(a)+\frac{\lambda_1(n)-a}{b-a}f^{(n)}(b)\right].\end{aligned}$$

Now by substituting value of

$$\frac{1}{(n-1)!}\int_a^b\left(\sum_{i=1}^{m} p_i T_n(x_i,s)\right)f^{(n)}(s)ds$$

from identity (12), we get our required result.
Now we find value of $P_1(n)$ as follows. First we consider

$$T_n(x,s) = \begin{cases} -\dfrac{(x-s)^n}{n(b-a)} + \dfrac{x-a}{b-a}(x-s)^{n-1}, & a \le s \le x,\\[2ex] -\dfrac{(x-s)^n}{n(b-a)} + \dfrac{x-b}{b-a}(x-s)^{n-1}, & x < s \le b.\end{cases}$$

We replace x by x_i,

$$T_n(x_i,s) = \begin{cases} -\dfrac{(x_i-s)^n}{n(b-a)} + \dfrac{x_i-a}{b-a}(x_i-s)^{n-1}, & a \le s \le x_i,\\[2ex] -\dfrac{(x_i-s)^n}{n(b-a)} + \dfrac{x_i-b}{b-a}(x_i-s)^{n-1}, & x_i < s \le b.\end{cases}$$

Now we calculate $\int_a^b T_n(x_i, s)\, ds$ as under:

$$\begin{aligned}
\int_a^b T_n(x_i, s)\, ds &= \int_a^{x_i} T_n(x_i, s)\, ds + \int_{x_i}^b T_n(x_i, s)\, ds \\
&= \int_a^{x_i} -\frac{(x_i - s)^n}{n(b-a)} + \frac{x_i - a}{b-a}(x_i - s)^{n-1}\, ds + \int_{x_i}^b -\frac{(x_i - s)^n}{n(b-a)} + \frac{x_i - b}{b-a}(x_i - s)^{n-1}\, ds \\
&= \left.\frac{(x_i - s)^{n+1}}{n(n+1)(b-a)} - \frac{x_i - a}{n(b-a)}(x_i - s)^n\right|_a^{x_i} + \left.\frac{(x_i - s)^{n+1}}{n(n+1)(b-a)} - \frac{x_i - b}{n(b-a)}(x_i - s)^n\right|_{x_i}^b \\
&= 0 - \frac{(x_i - a)^{n+1}}{n(n+1)(b-a)} - 0 + \frac{x_i - a}{n(b-a)}(x_i - a)^n \\
&\quad + \frac{(x_i - b)^{n+1}}{n(n+1)(b-a)} - 0 - \frac{x_i - b}{n(b-a)}(x_i - b)^n + 0 \\
&= -\frac{(x_i - a)^{n+1}}{n(n+1)(b-a)} + \frac{(x_i - a)^{n+1}}{n(b-a)} + \frac{(x_i - b)^{n+1}}{n(n+1)(b-a)} - \frac{(x_i - b)^{n+1}}{n(b-a)} \\
&= \frac{(x_i - a)^{n+1} - (x_i - b)^{n+1}}{(n+1)(b-a)}
\end{aligned}$$

Finally we get

$$\begin{aligned}
P_1(n) &= \sum_{i=1}^m p_i \int_a^b T_n(x_i, s)\, ds \\
&= \sum_{i=1}^m p_i \frac{(x_i - a)^{n+1} - (x_i - b)^{n+1}}{(n+1)(b-a)}
\end{aligned}$$

Method 2 for $P_1(n)$
Starting from following identity

$$\begin{aligned}
&\sum_{i=1}^m p_i f(x_i) - \frac{1}{b-a}\sum_{k=0}^{n-2}\frac{1}{k!(k+2)}\sum_{i=1}^m p_i \left[f^{(k+1)}(a)(x_i - a)^{k+2} - f^{(k+1)}(b)(x_i - b)^{k+2}\right] \\
&= \frac{1}{(n-1)!}\int_a^b \left(\sum_{i=1}^m p_i T_n(x_i, s)\right) f^{(n)}(s)\, ds
\end{aligned} \tag{16}$$

If we choose $f(x) = \frac{x^n}{n!}$ in (16), then we obtain

$$\begin{aligned}
&\sum_{i=1}^m p_i \frac{x_i^n}{n!} - \frac{1}{n!(b-a)}\sum_{k=0}^{n-2}\frac{1}{k!(k+2)}\sum_{i=1}^m p_i n(n-1)\cdots(n-k)\left[a^{n-k-1}(x_i - a)^{k+2} - b^{n-k-1}(x_i - b)^{k+2}\right] \\
&= \frac{1}{(n-1)!}\int_a^b \sum_{i=1}^m p_i T_n(x_i, s)\, ds
\end{aligned}$$

We know that $P_1(n) = \int_a^b \sum_{i=1}^m p_i T_n(x_i, s)\, ds$, so we can write

$$\frac{P_1(n)}{(n-1)!} = \sum_{i=1}^m p_i \frac{x_i^n}{n!} - \frac{1}{n!(b-a)}\sum_{k=0}^{n-2}\frac{1}{k!(k+2)}\sum_{i=1}^m p_i n(n-1)\cdots(n-k)\left[a^{n-k-1}(x_i - a)^{k+2} - b^{n-k-1}(x_i - b)^{k+2}\right]$$

after some simplification we obtain

$$P_1(n) = \frac{1}{n}\sum_{i=1}^{m} p_i x_i^n - \frac{1}{b-a}\sum_{k=0}^{n-2}\frac{1}{k+2}\binom{n-1}{k}\sum_{i=1}^{m} p_i \left[a^{n-k-1}(x_i-a)^{k+2} - b^{n-k-1}(x_i-b)^{k+2}\right],$$

where we used the fact that

$$\begin{aligned}\frac{(n-1)\cdots(n-k)}{(k+2)k!} &= \frac{(n-1)\cdots(n-k)(n-k-1)!}{(k+2)\,k!\,(n-k-1)!}\\ &= \frac{1}{k+2}\frac{(n-1)!}{k!\,(n-k-1)!}\\ &= \frac{1}{k+2}\binom{n-1}{k}.\end{aligned}$$

In order to calculate value of $\lambda_1(n)$, we calculate $\int_a^b sT_n(x_i,s)\,ds$ as under by using integration by parts:

$$\begin{aligned}\int_a^b sT_n(x_i,s)\,ds &= \int_a^{x_i} sT_n(x_i,s)\,ds + \int_{x_i}^b sT_n(x_i,s)\,ds\\ &= s\int T_n(x_i,s)\,ds\Big|_a^{x_i} + s\int T_n(x_i,s)\,ds\Big|_{x_i}^b - \int_a^b T_n(x_i,s)\,ds\\ &= s\frac{(x_i-s)^{n+1}}{n(n+1)(b-a)} - s\frac{x_i-a}{n(b-a)}(x_i-s)^n\Big|_a^{x_i}\\ &\quad + s\frac{(x_i-s)^{n+1}}{n(n+1)(b-a)} - s\frac{x_i-b}{n(b-a)}(x_i-s)^n\Big|_{x_i}^b - \int_a^b T_n(x_i,s)\,ds\\ &= 0 - \frac{a(x_i-a)^{n+1}}{n(n+1)(b-a)} - 0 + \frac{a(x_i-a)}{n(b-a)}(x_i-a)^n\\ &\quad + \frac{b(x_i-b)^{n+1}}{n(n+1)(b-a)} - 0 - \frac{b(x_i-b)}{n(b-a)}(x_i-b)^n + 0 - \int_a^b T_n(x_i,s)\,ds\\ &= -a\frac{(x_i-a)^{n+1}}{n(n+1)(b-a)} + a\frac{(x_i-a)^{n+1}}{n(b-a)}\\ &\quad + b\frac{(x_i-b)^{n+1}}{n(n+1)(b-a)} - b\frac{(x_i-b)^{n+1}}{n(b-a)} - \int_a^b T_n(x_i,s)\,ds\\ &= \frac{a(x_i-a)^{n+1} - b(x_i-b)^{n+1}}{(n+1)(b-a)} - \int_a^b T_n(x_i,s)\,ds\end{aligned}$$

Now multiplying by p_i and taking sum over i from 1 to m, we get:

$$\sum_{i=1}^{m} p_i\int_a^b sT_n(x_i,s)\,ds = \sum_{i=1}^{m} p_i\frac{a(x_i-a)^{n+1} - b(x_i-b)^{n+1}}{(n+1)(b-a)} - \sum_{i=1}^{m} p_i\int_a^b T_n(x_i,s)\,ds$$

But we know that

$$P_1(n) = \sum_{i=1}^{m} p_i\int_a^b T_n(x_i,s)\,ds$$

So we have

$$\sum_{i=1}^{m} p_i\int_a^b sT_n(x_i,s)\,ds = \sum_{i=1}^{m} p_i\frac{a(x_i-a)^{n+1} - b(x_i-b)^{n+1}}{(n+1)(b-a)} - P_1(n)$$

If we divide by $P_1(n)$ we finally get

$$\begin{aligned}\lambda_1(n) &= \frac{\sum_{i=1}^{m} p_i \int_a^b sT_n(x_i, s)\,ds}{\sum_{i=1}^{m} p_i \int_a^b T_n(x_i, s)\,ds} \\ &= \frac{1}{(n+1)(b-a)P_1(n)} \sum_{i=1}^{m} p_i \left[a(x_i-a)^n - b(x_i-b)^n\right] - 1.\end{aligned}$$

Method 2 for $\lambda_1(n)$

Starting from following identity

$$\begin{aligned}&\sum_{i=1}^{m} p_i f(x_i) - \frac{1}{b-a}\sum_{k=0}^{n-2}\frac{1}{k!(k+2)}\sum_{i=1}^{m} p_i \left[f^{(k+1)}(a)(x_i-a)^{k+2} - f^{(k+1)}(b)(x_i-b)^{k+2}\right] \\ &= \frac{1}{(n-1)!}\int_a^b \left(\sum_{i=1}^{m} p_i T_n(x_i, s)\right) f^{(n)}(s)\, ds \qquad (17)\end{aligned}$$

If we choose $f(x) = \dfrac{x^{n+1}}{(n+1)!}$ in (17), then we obtain

$$\begin{aligned}&\sum_{i=1}^{m} p_i \frac{x_i^{n+1}}{(n+1)!} - \frac{1}{(n+1)!(b-a)}\sum_{k=0}^{n-2}\frac{1}{k!(k+2)} \\ &\times \sum_{i=1}^{m} p_i (n+1)n\cdots(n-k+1)\left[a^{n-k}(x_i-a)^{k+2} - b^{n-k}(x_i-b)^{k+2}\right] \\ &= \frac{1}{(n-1)!}\int_a^b \sum_{i=1}^{m} p_i s T_n(x_i, s)\, ds.\end{aligned}$$

We know that $\lambda_1(n) = \dfrac{\int_a^b \sum_{i=1}^{m} p_i s T_n(x_i, s)\,ds}{\int_a^b \sum_{i=1}^{m} p_i T_n(x_i, s)\,ds}$,

so we can write

$$\begin{aligned}&\frac{P_1(n)}{(n-1)!}\lambda_1(n) = \sum_{i=1}^{m} p_i \frac{x_i^{n+1}}{(n+1)!} - \frac{1}{(n+1)!(b-a)}\sum_{k=0}^{n-2}\frac{1}{k!(k+2)} \\ &\times \sum_{i=1}^{m} p_i (n+1)n\cdots(n-k+1)\left[a^{n-k}(x_i-a)^{k+2} - b^{n-k}(x_i-b)^{k+2}\right]\end{aligned}$$

after some simplification we obtain

$$\begin{aligned}&\lambda_1(n) = \frac{1}{n(n+1)P_1(n)}\sum_{i=1}^{m} p_i x_i^{n+1} - \frac{1}{P_1(n)(b-a)}\sum_{k=0}^{n-2}\frac{1}{k(k+2)}\binom{n-1}{k-1} \\ &\times \sum_{i=1}^{m} p_i \left[a^{n-k}(x_i-a)^{k+2} - b^{n-k}(x_i-b)^{k+2}\right],\end{aligned}$$

where we used the fact that

$$\begin{aligned}\frac{(n-1)\cdots(n-k+1)}{(k+2)k!} &= \frac{(n-1)\cdots(n-k+1)(n-k)!}{(k+2)\,k!\,(n-k)!}\\ &= \frac{1}{k(k+2)}\frac{(n-1)!}{(k-1)!\,(n-k)!}\\ &= \frac{1}{k(k+2)}\binom{n-1}{k-1}.\end{aligned}$$

2. Since f is $(n+2)-$concave, i.e., $-f^{(n+2)} \geq 0$, then clearly $-f^{(n)}$ is a convex function we get inequalities (15) in reverse direction by using weighted Hermite-Hadamard inequalities for convex function $-f^{(n)}$ and condition (14).

□

Corollary 2.2. *Let the $m-$tuples $\mathbf{x} = (x_1, x_2, \ldots, x_m) \in [a,b]^m$ and $\mathbf{p} = (p_1, p_2, \ldots, p_m) \in \mathbb{R}^m$ satisfy (4) and (5). Furthermore, let $\lambda_1(n)$ and $P_1(n)$ be as in Theorem 2.1 and let T_n be given by (11). Then, for a function $f : I \to \mathbb{R}$ which is $(n+2)-$convex inequalities in (15) hold, while the reverse inequalities in (15) hold if f is $(n+2)-$concave.*

Proof. In [3] it was proved that $T_n(x,s)$ is an $n-$convex function with respect to x. Therefore for each $s \in [a,b]$ by Theorem 1.2 we have $\sum_{i=1}^m p_i T_n(x_i, s) \geq 0$, so assumption (14) of Theorem 2.1 holds and hence we get our required result. □

Corollary 2.3. *Let the $m-$tuples $\mathbf{x} = (x_1, x_2, \ldots, x_m) \in [a,b]^m$ and $\mathbf{p} = (p_1, p_2, \ldots, p_m) \in \mathbb{R}^m$ satisfy*

$$\sum_{i=1}^{m} p_i = 0 \quad \text{and} \quad \sum_{i=1}^{m} p_i |x_i - x_k| \geq 0 \text{ for } k \in \{1, \ldots, m\}. \tag{18}$$

Furthermore, let $\lambda_1(n)$ and $P_1(n)$ be as in Theorem 2.1 and let T_n be given by (11). Then, for a 4$-$convex function $f : I \to \mathbb{R}$ following inequality holds,

$$P_1(2) f^{(2)}(\lambda_1(2)) \leq \sum_{i=1}^{m} p_i f(x_i) \leq P_1(2)\left[\frac{b-\lambda_1(2)}{b-a} f^{(2)}(a) + \frac{\lambda_1(2)-a}{b-a} f^{(2)}(b)\right] \tag{19}$$

while the reverse inequality (19) holds if f is 4$-$concave.

Proof. Since $T_2(x,s)$ is a convex function with respect to x for each $s \in [a,b]$. Therefore by using (6) we have that $\sum_{i=1}^m p_i T_2(x_i, s) \geq 0$, so assumption (14) of Theorem 2.1 holds for $n = 2$ and hence we get our required result. □

Now we state integral version of Theorem 2.1 as under. Since proving techniques are of similar nature so we omit the details.

Theorem 2.4. *Let all the assumptions of Theorem 1.6 hold. If*

$$\int_{\alpha}^{\beta} p(x)\, T_n(g(x), s)\, dx \geq 0, \quad \text{for all } s \in [a,b]., \tag{20}$$

then:

1. *for every $(n+2)-$convex function $f : I \to \mathbb{R}$ the following inequalities hold*

$$\begin{aligned}\frac{P_2(n)}{(n-1)!} f^{(n)}(\lambda_2(n)) &\leq \int_{\alpha}^{\beta} p(x) f(g(x))\, dx - \frac{1}{b-a}\sum_{k=0}^{n-2}\frac{1}{k!\,(k+2)}\\ &\times \int_{\alpha}^{\beta} p(x)\left[f^{(k+1)}(a)(g(x)-a)^{k+2} - f^{(k+1)}(b)(g(x)-b)^{k+2}\right] dx\\ &\leq \frac{P_2(n)}{(n-1)!}\left[\frac{b-\lambda_2(n)}{b-a} f^{(n)}(a) + \frac{\lambda_2(n)-a}{b-a} f^{(n)}(b)\right]\end{aligned} \tag{21}$$

where

$$\begin{aligned} P_2(n) &= \int_\alpha^\beta p(x)\left(\int_a^b T_n(g(x),s)\,ds\right)dx \\ &= \frac{1}{(n+1)(b-a)}\int_\alpha^\beta p(x)\left[(g(x)-a)^{n+1}-(g(x)-b)^{n+1}\right]dx. \end{aligned}$$

or

$$P_2(n) = \frac{1}{n}\int_\alpha^\beta p(x)\,[g(x)]^n d(x) - \frac{1}{b-a}\sum_{k=0}^{n-2}\frac{1}{k+2}\binom{n-1}{k}$$
$$\times\int_\alpha^\beta p(x)\left[a^{n-k-1}(g(x)-a)^{k+2} - b^{n-k-1}(g(x)-b)^{k+2}\right]dx,$$

and

$$\begin{aligned} \lambda_2(n) &= \frac{\int_\alpha^\beta p(x)\left(\int_a^b sT_n(g(x),s)\,ds\right)dx}{\int_\alpha^\beta p(x)\left(\int_a^b T_n(g(x),s)\,ds\right)dx} \\ &= \frac{1}{(n+1)(b-a)P_2(n)}\int_\alpha^\beta p(x)\left[a(g(x)-a)^n - b(g(x)-b)^n\right]dx - 1, \end{aligned}$$

or

$$\lambda_2(n) = \frac{1}{n(n+1)P_2(n)}\int_\alpha^\beta p(x)\,[g(x)]^{n+1} - \frac{1}{P_2(n)(b-a)}\sum_{k=0}^{n-2}\frac{1}{k(k+2)}\binom{n-1}{k-1}$$
$$\times\int_\alpha^\beta p(x)\left[a^{n-k}(g(x)-a)^{k+2} - b^{n-k}(g(x)-b)^{k+2}\right],$$

2. *for every* $(n+2)-$*concave functions* $f : I \to \mathbb{R}$, (21) *holds with the reversed sign of inequalities.*

Corollary 2.5. *Let* $g : [\alpha,\beta] \to [a,b]$ *and* $p : [\alpha,\beta] \to \mathbb{R}$ *be integrable functions such that* $\int_\alpha^\beta p(x)dx = 0$ *satisfy* (8) *and* (9). *Furthermore, let* $\lambda_2(n)$ *and* $P_2(n)$ *be as in Theorem 2.4 and let* T_n *be given by* (11). *Then, for a function* $f : I \to \mathbb{R}$ *which is* $(n+2)-$*convex inequalities in* (21) *hold, while the reverse inequalities in* (21) *hold if* f *is* $(n+2)-$*concave.*

3. Linear inequalities via extension of Montgomery identity and weighted Hermite-Hadamard inequalities with Green functions

From [5] and [16] (see also [6]), we recall the definitions of different Green functions $G_l : [a,b]\times[a,b]$ for $l \in \{0,1,2,3,4\}$ respectively

$$G_0(s,t) = \begin{cases} \dfrac{(s-b)(t-a)}{b-a}, & a \le t \le s, \\ \dfrac{(t-b)(s-a)}{b-a}, & s \le t \le b. \end{cases} \tag{22}$$

$$G_1(s,t) = \begin{cases} a-t, & a \le t \le s, \\ a-s, & s \le t \le b, \end{cases} \tag{23}$$

$$G_2(s,t) = \begin{cases} s-b, & a \le t \le s, \\ t-b, & s \le t \le b. \end{cases} \tag{24}$$

$$G_3(s,t)=\begin{cases} s-a, & a\le t\le s,\\ t-a, & s\le t\le b.\end{cases} \tag{25}$$

$$G_4(s,t)=\begin{cases} b-t, & a\le t\le s,\\ b-s, & s\le t\le b.\end{cases} \tag{26}$$

The functions G_l for $l\in\{0,1,2,3,4\}$ are continuous, symmetric and convex with respect to both variables s and t.

Before we proceed further we need here results related to extension of Montgomery identity involving Green functions from [7] and [8].

Theorem 3.1. *Fix $l\in\{0,1,2,3,4\}$. Let $\mathbf{x}=(x_1,\dots,x_m)\in[a,b]^m$, $\mathbf{p}=(p_1,\dots,p_m)\in\mathbb{R}^m$ satisfy conditions*

$$\sum_{i=1}^{m} p_i=0,\qquad \sum_{i=1}^{m} p_i x_i=0.$$

Also let $f:I\to\mathbb{R}$ be a function such that $f^{(n-1)}$ is absolutely continuous for $n\in\mathbb{N}$ $n\ge 3$, $I\subset\mathbb{R}$ an open interval, $a,b\in I$, $a<b$, then for all $s\in[a,b]$ we have the following identity

$$\begin{aligned}&\sum_{i=1}^{m} p_i f(x_i)-\frac{f'(a)-f'(b)}{b-a}\int_a^b\sum_{i=1}^{m} p_iG_l(x_i,s)ds\\ &-\sum_{k=2}^{n-1}\frac{k}{(k-1)!}\int_a^b\sum_{i=1}^{m} p_iG_l(x_i,s)\frac{f^{(k)}(a)(s-a)^{k-1}-f^{(k)}(b)(s-b)^{k-1}}{b-a}ds\\ &=\frac{1}{(n-3)!}\int_a^b f^{(n)}(t)\left(\int_a^b\sum_{i=1}^{m} p_iG_l(x_i,s)\tilde{T}_{n-2}(s,t)ds\right)dt\end{aligned} \tag{27}$$

where

$$\tilde{T}_{n-2}(s,t)=\begin{cases}\dfrac{1}{b-a}\left[\dfrac{(s-t)^{n-2}}{(n-2)}+(s-a)(s-t)^{n-3}\right], & a\le t\le s\le b,\\[2ex] \dfrac{1}{b-a}\left[\dfrac{(s-t)^{n-2}}{(n-2)}+(s-b)(s-t)^{n-3}\right], & a\le s<t\le b.\end{cases} \tag{28}$$

and G_l are as defined in (22) − (26)*. Moreover, we also have the following identity*

$$\begin{aligned}&\sum_{i=1}^{m} p_i f(x_i)-\frac{f'(a)-f'(b)}{b-a}\int_a^b\sum_{i=1}^{m} p_iG_l(x_i,s)ds\\ &-\sum_{k=3}^{n-1}\frac{k-2}{(k-1)!}\int_a^b\sum_{i=1}^{m} p_iG_l(x_i,s)\frac{f^{(k)}(a)(s-a)^{k-1}-f^{(k)}(b)(s-b)^{k-1}}{b-a}ds\\ &=\frac{1}{(n-3)!}\int_a^b f^{(n)}(t)\left(\int_a^b\sum_{i=1}^{m} p_iG_l(x_i,s)T_{n-2}(s,t)ds\right)dt\end{aligned} \tag{29}$$

where T_n is as defined in Proposition 1.4.

The integral version of the above results may be stated as follows.

Theorem 3.2. *Fix $l\in\{0,1,2,3,4\}$. Let $g:[\alpha,\beta]\to[a,b]$ be a function and let $p:[\alpha,\beta]\to\mathbb{R}$ be a continuous integrable function such that $\int_\alpha^\beta p(x)dx=0$ and $\int_\alpha^\beta p(x)g(x)dx=0$. Let $f:I\to\mathbb{R}$ be a function such that $f^{(n-1)}$ is* *absolutely continuous, $I \subset \mathbb{R}$ an open interval, $a, b \in I$, $a < b$, then for all $s \in [a, b]$ we have the following identity*

$$\int_\alpha^\beta p(x) f(g(x))\,dx - \frac{f'(a)-f'(b)}{b-a}\int_a^b \int_\alpha^\beta p(x) G_l(g(x),s)dxds$$
$$-\sum_{k=2}^{n-1}\frac{k}{(k-1)!}\int_a^b\left(\int_\alpha^\beta p(x) G_l(g(x),s)\,dx\right)\frac{f^{(k)}(a)(s-a)^{k-1}-f^{(k)}(b)(s-b)^{k-1}}{b-a}ds$$
$$=\frac{1}{(n-3)!}\int_a^b f^{(n)}(t)\left(\int_a^b\left(\int_\alpha^\beta p(x) G_l(g(x),s)\,dx\right)\tilde{T}_{n-2}(s,t)ds\right)dt. \tag{30}$$

Moreover, we also have the following identity

$$\int_\alpha^\beta p(x) f(g(x))dx - \frac{f'(a)-f'(b)}{b-a}\int_a^b \int_\alpha^\beta p(x) G_l(g(x),s)\,dx\,ds$$
$$-\sum_{k=3}^{n-1}\frac{k-2}{(k-1)!}\int_a^b\left(\int_\alpha^\beta p(x) G_l(g(x),s)\,dx\right)\frac{f^{(k)}(a)(s-a)^{k-1}-f^{(k)}(b)(s-b)^{k-1}}{b-a}ds$$
$$=\frac{1}{(n-3)!}\int_a^b f^{(n)}(t)\left(\int_a^b\left(\int_\alpha^\beta p(x) G_l(g(x),s)\,dx\right)T_{n-2}(s,t)ds\right)dt \tag{31}$$

where $\tilde{T}_n$, T_n and G_l are as in Theorem 3.1.

Now we obtain our main results of this section by using the previously defined Green functions together with the weighted Hermite-Hadamard inequalities and extension of Montgomery identity both in discrete and integral forms.

Theorem 3.3. *Let all the assumptions of Theorem* 3.1 *hold with the additional condition*

$$\int_a^b \sum_{i=1}^m p_i G_l(x_i,s)\tilde{T}_{n-2}(s,t)ds \geq 0, \quad \forall\, t \in [a,b]. \tag{32}$$

Then:

1. *for every $(n+2)$−convex function $f : I \to \mathbb{R}$ the following inequalities hold*

$$\frac{P_3(n)}{(n-3)!}f^{(n)}\left(\lambda_3(n)\right) \leq \sum_{i=1}^m p_i f(x_i) - \frac{f'(a)-f'(b)}{b-a}\int_a^b \sum_{i=1}^m p_i G_l(x_i,s)ds$$
$$-\sum_{k=2}^{n-1}\frac{k}{(k-1)!}\int_a^b \sum_{i=1}^m p_i G_l(x_i,s)\frac{f^{(k)}(a)(s-a)^{k-1}-f^{(k)}(b)(s-b)^{k-1}}{b-a}ds$$
$$\leq \frac{P_3(n)}{(n-3)!}\left[\frac{b-\lambda_3(n)}{b-a}f^{(n)}(a)+\frac{\lambda_3(n)-a}{b-a}f^{(n)}(b)\right] \tag{33}$$

where

$$\begin{aligned} P_3(n) &= \int_a^b\left(\int_a^b \sum_{i=1}^m p_i G_l(x_i,s)\tilde{T}_{n-2}(s,t)ds\right)dt \\ &= \frac{1}{n(n-1)(n-2)}\sum_{i=1}^m p_i x_i^n - \frac{a^{n-1}-b^{n-1}}{(n-1)(n-2)(b-a)}\int_a^b \sum_{i=1}^m p_i G_l(x_i,s)ds \\ &- \sum_{k=2}^{n-1}\frac{k(n-3)!}{(k-1)!\,(n-k)!(b-a)}\int_a^b \sum_{i=1}^m p_i G_l(x_i,s)\left[a^{n-k}(s-a)^{k-1}-b^{n-k}(s-b)^{k-1}\right]ds, \end{aligned}$$

and

$$\lambda_3(n) = \frac{\int_a^b t\left(\int_a^b \sum_{i=1}^m p_i G_l(x_i,s)\tilde{T}_{n-2}(s,t)ds\right)dt}{\int_a^b \left(\int_a^b \sum_{i=1}^m p_i G_l(x_i,s)\tilde{T}_{n-2}(s,t)ds\right)dt}$$

$$= \frac{1}{(n+1)n(n-1)(n-2)P_3(n)}\sum_{i=1}^m p_i x_i^{n+1} - \frac{a^n-b^n}{n(n-1)(n-2)(b-a)P_3(n)}\int_a^b \sum_{i=1}^m p_i G_l(x_i,s)ds$$

$$- \sum_{k=2}^{n-1}\frac{k(n-3)!}{(k-1)!(n-k+1)!(b-a)P_3(n)}\int_a^b \sum_{i=1}^m p_i G_l(x_i,s)\left[a^{n-k+1}(s-a)^{k-1} - b^{n-k+1}(s-b)^{k-1}\right]ds,$$

2. *for every* $(n+2)-$*concave functions* $f: I \to \mathbb{R}$, (15) *holds with the reversed sign of inequalities.*

Proof. 1. By using convexity of $f^{(n)}$ and

$$\int_a^b \sum_{i=1}^m p_i G_l(x_i,s)\tilde{T}_{n-2}(s,t)ds \geq 0, \quad \forall\, t\in[a,b], \quad \text{for} \quad l\in\{0,1,2,3,4\}$$

in weighted Hermite-Hadamard inequality (2) and dividing by $(n-3)!$ we get

$$\frac{P_3(n)}{(n-3)!}f^{(n)}(\lambda_3(n)) \leq \frac{1}{(n-3)!}\int_a^b \left(\int_a^b \sum_{i=1}^m p_i G_l(x_i,s)\tilde{T}_{n-2}(s,t)ds\right)f^{(n)}(t)dt$$

$$\leq \frac{P_3(n)}{(n-3)!}\left[\frac{b-\lambda_3(n)}{b-a}f^{(n)}(a) + \frac{\lambda_3(n)-a}{b-a}f^{(n)}(b)\right]$$

Now by substituting value of

$$\frac{1}{(n-3)!}\int_a^b \left(\int_a^b \sum_{i=1}^m p_i G_l(x_i,s)\tilde{T}_{n-2}(s,t)ds\right)f^{(n)}(t)dt$$

from identity (27) we get our required result.
Now we find value of $P_3(n)$ as follows. First we consider the identity

$$\sum_{i=1}^m p_i f(x_i) - \frac{f'(a)-f'(b)}{b-a}\int_a^b \sum_{i=1}^m p_i G_l(x_i,s)ds$$
$$-\sum_{k=2}^{n-1}\frac{k}{(k-1)!}\int_a^b \sum_{i=1}^m p_i G_l(x_i,s)\frac{f^{(k)}(a)(s-a)^{k-1} - f^{(k)}(b)(s-b)^{k-1}}{b-a}ds$$
$$= \frac{1}{(n-3)!}\int_a^b f^{(n)}(t)\left(\int_a^b \sum_{i=1}^m p_i G_l(x_i,s)\tilde{T}_{n-2}(s,t)ds\right)dt \tag{34}$$

If we choose $f(x) = \frac{x^n}{n!}$ in (34), then we obtain

$$\sum_{i=1}^m p_i \frac{x_i^n}{n!} - \frac{a^{n-1}-b^{n-1}}{(n-1)!(b-a)}\int_a^b \sum_{i=1}^m p_i G_l(x_i,s)ds$$
$$-\sum_{k=2}^{n-1}\frac{kn(n-1)\cdots(n-k+1)}{(k-1)!n!(b-a)}\int_a^b \sum_{i=1}^m p_i G_l(x_i,s)\left[a^{n-k}(s-a)^{k-1} - b^{n-k}(s-b)^{k-1}\right]ds$$
$$= \frac{1}{(n-3)!}\int_a^b \left(\int_a^b \sum_{i=1}^m p_i G_l(x_i,s)\tilde{T}_{n-2}(s,t)ds\right)dt$$

or we can write

$$\frac{P_3(n)}{(n-3)!} = \sum_{i=1}^{m} p_i \frac{x_i^n}{n!} - \frac{a^{n-1} - b^{n-1}}{(n-1)!(b-a)} \int_a^b \sum_{i=1}^{m} p_i G_l(x_i, s) ds$$
$$- \sum_{k=2}^{n-1} \frac{kn(n-1)\cdots(n-k+1)}{(k-1)!n!(b-a)} \int_a^b \sum_{i=1}^{m} p_i G_l(x_i, s) \left[a^{n-k}(s-a)^{k-1} - b^{n-k}(s-b)^{k-1}\right] ds$$

after some simplification we obtain

$$P_3(n) = \frac{1}{n(n-1)(n-2)} \sum_{i=1}^{m} p_i x_i^n - \frac{a^{n-1} - b^{n-1}}{(n-1)(n-2)(b-a)} \int_a^b \sum_{i=1}^{m} p_i G_l(x_i, s) ds$$
$$- \sum_{k=2}^{n-1} \frac{k(n-3)!}{(k-1)!\,(n-k)!(b-a)} \int_a^b \sum_{i=1}^{m} p_i G_l(x_i, s) \left[a^{n-k}(s-a)^{k-1} - b^{n-k}(s-b)^{k-1}\right] ds$$

where we used the fact that

$$\begin{aligned} \frac{k(n-3)!n(n-1)\cdots(n-k+1)}{(k-1)!n!} &= \frac{k(n-3)!n(n-1)\cdots(n-k+1)(n-k)!}{(k-1)!n!(n-k)!} \\ &= \frac{k(n-3)!}{(k-1)!\,(n-k)!} \end{aligned}$$

Now we find value of $\lambda_3(n)$ by choosing $f(x) = \frac{x^{n+1}}{(n+1)!}$ in (34), we obtain

$$\sum_{i=1}^{m} p_i \frac{x_i^{n+1}}{(n+1)!} - \frac{a^n - b^n}{n!(b-a)} \int_a^b \sum_{i=1}^{m} p_i G_l(x_i, s) ds$$
$$- \sum_{k=2}^{n-1} \frac{k(n+1)n\cdots(n-k+2)}{(k-1)!(n+1)!(b-a)} \int_a^b \sum_{i=1}^{m} p_i G_l(x_i, s) \left[a^{n-k+1}(s-a)^{k-1} - b^{n-k+1}(s-b)^{k-1}\right] ds$$
$$= \frac{1}{(n-3)!} \int_a^b t \left(\int_a^b \sum_{i=1}^{m} p_i G_l(x_i, s) \tilde{T}_{n-2}(s, t) ds \right) dt$$

or we can write

$$\frac{P_3(n)}{(n-3)!} \lambda_3(n) = \sum_{i=1}^{m} p_i \frac{x_i^{n+1}}{(n+1)!} - \frac{a^n - b^n}{n!(b-a)} \int_a^b \sum_{i=1}^{m} p_i G_l(x_i, s) ds$$
$$- \sum_{k=2}^{n-1} \frac{k(n+1)n\cdots(n-k+2)}{(k-1)!(n+1)!(b-a)} \int_a^b \sum_{i=1}^{m} p_i G_l(x_i, s) \left[a^{n-k+1}(s-a)^{k-1} - b^{n-k+1}(s-b)^{k-1}\right] ds$$

after some simplification we obtain

$$\lambda_3(n) = \frac{1}{(n+1)n(n-1)(n-2)P_3(n)} \sum_{i=1}^{m} p_i x_i^{n+1} - \frac{a^n - b^n}{n(n-1)(n-2)(b-a)P_3(n)} \int_a^b \sum_{i=1}^{m} p_i G_l(x_i, s) ds$$
$$- \sum_{k=2}^{n-1} \frac{k(n-3)!}{(k-1)!(n-k+1)!(b-a)} \int_a^b \sum_{i=1}^{m} p_i G_l(x_i, s) \left[a^{n-k+1}(s-a)^{k-1} - b^{n-k+1}(s-b)^{k-1}\right] ds,$$

where we used the fact that

$$\begin{aligned} \frac{k(n-3)!(n+1)n\cdots(n-k+2)}{(k-1)!(n+1)!} &= \frac{k(n-3)!(n+1)n\cdots(n-k+2)(n-k+1)!}{(k-1)!(n+1)!(n-k+1)!} \\ &= \frac{k(n-3)!}{(k-1)!\,(n-k+1)!}. \end{aligned}$$

2. For idea of the proof see proof of part (2) of Theorem 2.1.
□

Here we have another results similar to Theorem 3.3, since proving techniques are same so we omit the details.

Theorem 3.4. *Let all the assumptions of Theorem* 3.1 *hold with the additional condition*

$$\int_a^b \sum_{i=1}^m p_i G_l(x_i, s) T_{n-2}(s,t) ds \geq 0, \quad \forall\, t \in [a,b]. \tag{35}$$

Then:

1. *for every* $(n+2)-$*convex function* $f : I \to \mathbb{R}$ *the following inequalities hold*

$$\frac{P_4(n)}{(n-3)!} f^{(n)}(\lambda_4(n)) \leq \sum_{i=1}^m p_i f(x_i) - \frac{f'(a)-f'(b)}{b-a}\int_a^b \sum_{i=1}^m p_i G_l(x_i,s) ds$$
$$-\sum_{k=3}^{n-1} \frac{k-2}{(k-1)!}\int_a^b \sum_{i=1}^m p_i G_l(x_i,s) \frac{f^{(k)}(a)(s-a)^{k-1} - f^{(k)}(b)(s-b)^{k-1}}{b-a} ds$$
$$\leq \frac{P_4(n)}{(n-3)!}\left[\frac{b-\lambda_4(n)}{b-a} f^{(n)}(a) + \frac{\lambda_4(n)-a}{b-a} f^{(n)}(b)\right] \tag{36}$$

where

$$P_4(n) = \frac{1}{n(n-1)(n-2)}\sum_{i=1}^m p_i x_i^n - \frac{a^{n-1}-b^{n-1}}{(n-1)(n-2)(b-a)}\int_a^b \sum_{i=1}^m p_i G_l(x_i,s) ds$$
$$-\sum_{k=3}^{n-1} \frac{(k-2)(n-3)!}{(k-1)!\,(n-k)!(b-a)}\int_a^b \sum_{i=1}^m p_i G_l(x_i,s)\left[a^{n-k}(s-a)^{k-1} - b^{n-k}(s-b)^{k-1}\right] ds,$$

and

$$\lambda_4(n) = \frac{1}{(n+1)n(n-1)(n-2)P_4(n)}\sum_{i=1}^m p_i x_i^{n+1} - \frac{a^n-b^n}{n(n-1)(n-2)(b-a)P_4(n)}\int_a^b \sum_{i=1}^m p_i G_l(x_i,s) ds$$
$$-\sum_{k=3}^{n-1} \frac{(k-2)(n-3)!}{(k-1)!(n-k+1)!(b-a)P_4(n)}\int_a^b \sum_{i=1}^m p_i G_l(x_i,s)\left[a^{n-k+1}(s-a)^{k-1} - b^{n-k+1}(s-b)^{k-1}\right] ds,$$

2. *for every* $(n+2)-$*concave functions* $f : I \to \mathbb{R}$, (36) *holds with the reversed sign of inequalities.*

Now we state integral version of Theorem 3.3 as under. Since proof techniques are of similar nature so we omit the details.

Theorem 3.5. *Let all the assumptions of Theorem* 1.6 *hold with the additional condition*

$$\int_a^b \int_\alpha^\beta p(x)\, G_l(g(x), s)\, \tilde{T}_{n-2}(s,t)\, dx\, ds \geq 0, \quad \forall\, t \in [a,b]. \tag{37}$$

Then:

1. *for every* $(n+2)-$*convex function* $f : I \to \mathbb{R}$ *the following inequalities hold*

$$\frac{P_5(n)}{(n-3)!} f^{(n)}(\lambda_5(n)) \leq \int_\alpha^\beta p(x) f(g(x)) dx - \frac{f'(a)-f'(b)}{b-a}\int_a^b \int_\alpha^\beta p(x)\, G_l(g(x),s) dx ds$$
$$-\sum_{k=2}^{n-1} \frac{k}{(k-1)!}\int_a^b \left(\int_\alpha^\beta p(x)\, G_l(g(x),s) dx\right) \frac{f^{(k)}(a)(s-a)^{k-1} - f^{(k)}(b)(s-b)^{k-1}}{b-a} ds$$
$$\leq \frac{P_5(n)}{(n-3)!}\left[\frac{b-\lambda_5(n)}{b-a} f^{(n)}(a) + \frac{\lambda_5(n)-a}{b-a} f^{(n)}(b)\right] \tag{38}$$

where

$$P_5(n) = \frac{1}{n(n-1)(n-2)} \int_\alpha^\beta p(x)(g(x))^n\,dx - \frac{a^{n-1}-b^{n-1}}{(n-1)(n-2)(b-a)}$$
$$\times \int_a^b \int_\alpha^\beta p(x)\,G_l(g(x),s)dxds - \sum_{k=2}^{n-1} \frac{k(n-3)!}{(k-1)!\,(n-k)!(b-a)P_5(n)}$$
$$\times \int_a^b \left(\int_\alpha^\beta p(x)\,G_l(g(x),s)dx \right) \int_\alpha^\beta p(x) \left[a^{n-k}(s-a)^{k-1} - b^{n-k}(s-b)^{k-1} \right] ds,$$

and

$$\lambda_5(n) = \frac{1}{(n+1)n(n-1)(n-2)P_5(n)} \int_\alpha^\beta p(x)(g(x))^{n+1}\,dx$$
$$- \frac{a^n-b^n}{n(n-1)(n-2)(b-a)P_5(n)} \int_a^b \int_\alpha^\beta p(x)\,G_l(g(x),s)dxds$$
$$- \sum_{k=2}^{n-1} \frac{k(n-3)!}{(k-1)!(n-k+1)!(b-a)P_5(n)} \int_a^b \left(\int_\alpha^\beta p(x)\,G_l(g(x),s)dx \right)$$
$$\times \left[a^{n-k+1}(s-a)^{k-1} - b^{n-k+1}(s-b)^{k-1} \right] ds,$$

2. *For every* $(n+2)-$*concave functions* $f : I \to \mathbb{R}$, (38) *holds with the reversed sign of inequalities.*

Theorem 3.6. *Let all the assumptions of Theorem* 1.6 *hold with the additional condition*

$$\int_a^b \int_\alpha^\beta p(x)\,G_l(g(x),s) T_{n-2}(s,t)\,dx\,ds \geq 0, \quad \forall\, t \in [a,b]. \tag{39}$$

Then:

1. *for every* $(n+2)-$*convex function* $f : I \to \mathbb{R}$ *the following inequalities hold*

$$\frac{P_6(n)}{(n-3)!} f^{(n)}(\lambda_6(n)) \leq \int_\alpha^\beta p(x) f(g(x))dx - \frac{f'(a)-f'(b)}{b-a} \int_a^b \int_\alpha^\beta p(x)\,G_l(g(x),s)\,dx\,ds$$
$$- \sum_{k=3}^{n-1} \frac{k-2}{(k-1)!} \int_a^b \left(\int_\alpha^\beta p(x)\,G_l(g(x),s)\,dx \right) \frac{f^{(k)}(a)(s-a)^{k-1} - f^{(k)}(s-b)^{k-1}}{b-a} ds$$
$$\leq \frac{P_6(n)}{(n-3)!} \left[\frac{b-\lambda_6(n)}{b-a} f^{(n)}(a) + \frac{\lambda_6(n)-a}{b-a} f^{(n)}(b) \right] \tag{40}$$

where

$$P_6(n) = \frac{1}{n(n-1)(n-2)} \int_\alpha^\beta p(x)(g(x))^n\,dx - \frac{a^{n-1}-b^{n-1}}{(n-1)(n-2)(b-a)} \int_a^b \int_\alpha^\beta p(x)\,G_l(g(x),s)\,dx\,ds$$
$$- \sum_{k=3}^{n-1} \frac{(k-2)(n-3)!}{(k-1)!\,(n-k)!(b-a)} \int_a^b \left(\int_\alpha^\beta p(x)\,G_l(g(x),s)\,dx \right)$$
$$\times \int_\alpha^\beta p(x) \left[a^{n-k}(s-a)^{k-1} - b^{n-k}(s-b)^{k-1} \right] ds,$$

and

$$\lambda_6(n) = \frac{1}{(n+1)n(n-1)(n-2)P_6(n)} \int_\alpha^\beta p\,(x)\,(g(x))^{n+1}\,dx$$
$$-\frac{a^n - b^n}{n(n-1)(n-2)(b-a)P_6(n)} \int_a^b \int_\alpha^\beta p\,(x)\,G_l(g(x),s)\,dx\,ds$$
$$-\sum_{k=3}^{n-1} \frac{(k-2)(n-3)!}{(k-1)!(n-k+1)!(b-a)P_6(n)} \int_a^b \left(\int_\alpha^\beta p\,(x)\,G_l(g(x),s)\,dx \right)$$
$$\times \int_\alpha^\beta p\,(x) \left[a^{n-k+1}\,(s-a)^{k-1} - b^{n-k+1}\,(s-b)^{k-1} \right] ds,$$

2. *for every* $(n+2)-$*concave functions* $f : I \to \mathbb{R}$, (40) *holds with the reversed sign of inequalities.*

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