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On Hyponormality of the Sum of Two Composition Operators

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Abstract. In this paper we study some properties of the sums of two composition operators on the Hardy space. In particular, we investigate hyponormality of the sums of two composition operators. We also provide some conditions for which the sums of composition operators with linear fractional symbols are hyponormal.

1. Introduction

Let $\mathbb{D}$ denote the open unit disk in the complex plane. The space $H^2(\mathbb{D})$, simply H^2, consists of all analytic functions on $\mathbb{D}$ having power series representations with square summable complex coefficients. The space $H^\infty(\mathbb{D})$, simply H^∞, consists of all the functions that are analytic and bounded on $\mathbb{D}$. If φ is an analytic mapping from $\mathbb{D}$ into itself, the *composition operator* C_φ is the operator on H^2 defined by $C_\varphi f = f \circ \varphi$ for any f in H^2. It is well known that the composition operator C_φ is always bounded on H^2 by the Littlewood subordination theorem (see [8] and [19]).

The Hardy space H^2 has reproducing kernels K_α for $\alpha \in \mathbb{D}$; if $f(\alpha) = \langle f, K_\alpha \rangle$ for any f in H^2. In fact, $K_\alpha(z) = \frac{1}{1-\overline{\alpha}z} = \sum_{n=0}^{\infty} \overline{\alpha}^n z^n$ and $\|K_\alpha\| = \frac{1}{\sqrt{1-|\alpha|^2}}$ for $\alpha \in \mathbb{D}$. The reproducing kernels have very useful properties. In particular, the span of reproducing kernels K_α for uncountably many α in $\mathbb{D}$ is dense in H^2 and the adjoint of C_φ satisfies the formula $C_\varphi^* K_\alpha = K_{\varphi(\alpha)}$ for any $\alpha \in \mathbb{D}$.

If φ is any analytic self-map of $\mathbb{D}$, we call $a \in \overline{\mathbb{D}}$ a *fixed point* of φ provided that $\lim_{r\to 1^-} \varphi(ra) = a$. For ζ on the unit circle and $\delta > 1$, a *nontangential approach region* at ζ is defined by $\Gamma(\zeta, \delta) = \{z \in \mathbb{D} : |z - \zeta| < \delta(1 - |z|)\}$. We say that a function f has a *nontangential limit* at ζ when $\lim_{z\to\zeta} f(z)$ exists in each nontangential region $\Gamma(\zeta, \delta)$. We also say φ has a *finite angular derivative* at $\zeta \in \partial\mathbb{D}$ if there exists η on $\partial\mathbb{D}$ so that $\frac{\varphi(z)-\eta}{z-\zeta}$ has a finite nontangential limit as $z \to \zeta$. If this limit exists, it is denoted by $\varphi'(\zeta)$. It is well known that if φ is an analytic self-map of $\mathbb{D}$, which is neither the identity map nor an elliptic automorphism of $\mathbb{D}$, then there is a point c of $\overline{\mathbb{D}}$ so that the iterates $\varphi_n := \varphi_{n-1} \circ \varphi$ of φ converges uniformly to a on compact subsets of $\mathbb{D}$.

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Moreover, c is the unique fixed point of φ in $\overline{\mathbb{D}}$ for which $|\varphi'(c)| \leq 1$. We say that the unique fixed point c is the *Denjoy-Wolff point* of φ. The Schwarz Lemma implies that φ has at most one fixed point in $\mathbb{D}$, and if c is a fixed point in $\mathbb{D}$, then it is the only one with $|\varphi'(c)| \leq 1$. If φ has a fixed point c inside $\mathbb{D}$, it is the Denjoy-Wolff point and $|\varphi'(c)| < 1$. There can be many fixed points on $\partial\mathbb{D}$ but at most one with $|\varphi'(c)| \leq 1$ and this c is the Denjoy-Wolff point; in this case, $0 < \varphi'(c) \leq 1$ (see [8] and [19] for more details).

Let $\mathcal{H}$ be a separable complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. If $T \in \mathcal{L}(\mathcal{H})$, then we shall use the notations $\sigma(T)$, $\sigma_p(T)$, and $\sigma_{ap}(T)$ for the spectrum, the point spectrum and the approximate point spectrum of T, respectively. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *normal* if T and T^* commute. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *subnormal* if there is a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ and a normal operator N on $\mathcal{K}$ such that N leaves $\mathcal{H}$ invariant and $T = N|_{\mathcal{H}}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is called *hyponormal* provided $T^*T \geq TT^*$. We say that $T \in \mathcal{L}(\mathcal{H})$ is *quasinormal* when $[T, T^*T] = 0$ where $[S, T] := ST - TS$ for operators S and T in $\mathcal{L}(\mathcal{H})$. It is well known that quasinormality implies subnormality and subnormality implies hyponormality. The famous Fuglede-Putnam theorem is as follows: for normal operators $S, T \in \mathcal{L}(\mathcal{H})$, if $SX = XT$ for $X \in \mathcal{L}(\mathcal{H})$, then $S^*X = XT^*$ (see [9], [16]).

In 1969, H. Radjavi and P. Rosenthal showed that if S and T are two normal operators such that their linear span consists of normal operators, then S and T commute (see [17]). In 1988, J. B. Conway and W. Szymanski had tried to generalize the results of [17] to the class of hyponormal and subnormal operators (see [4]). However, instead of giving the positive answer for these extension, they showed that the result of [17] does not extend to hyponormal and subnormal operators as finding two noncommuting hyponormal operators such that their linear span consists entirely of hyponormal. They also showed that if S and T are two hyponormal operators in $\mathcal{L}(\mathcal{H})$ and if $S^*T = TS^*$, then the linear span of S and T consists of hyponormal operators and both ST and TS are hyponormal. It means that there is a close relationship between hyponormality of the linear span of S and T and the equation $S^*T = TS^*$ for two hyponormal operators S and T in $\mathcal{L}(\mathcal{H})$. As an extension of the study for differences of composition operators, the linear combination of composition operators has received growing interest (see [2], [13], [14]).

From the above motivations, we focus on our work for hyponormality of the sum of two composition operators $\omega C_\varphi + C_\psi$ for any $\omega \in \mathbb{C}$.

2. Main results

In this section, we study some properties of the sum of two composition operators. Throughout this paper, we consider the linear pencils $\omega C_\varphi + C_\psi$ for nonzero complex number ω. We now investigate hyponormality of such sums.

Theorem 2.1. *Let φ and ψ be analytic maps from $\mathbb{D}$ into itself. If $\omega C_\varphi + C_\psi$ is hyponormal for any nonzero $\omega \in \mathbb{C}$, then both C_φ and C_ψ are hyponormal and*

$$|\langle [C_\varphi^*, C_\psi]h, h\rangle|^2 \leq \langle [C_\varphi^*, C_\varphi]h, h\rangle \langle [C_\psi^*, C_\psi]h, h\rangle \tag{1}$$

holds for all $h \in H^2$.

Conversely, if C_φ and C_ψ are hyponormal and $Re\{\overline{\omega}[C_\varphi^*, C_\psi]\} \geq 0$ for any nonzero $\omega \in \mathbb{C}$, then $\omega C_\varphi + C_\psi$ is hyponormal. In particular, if C_φ and C_ψ are normal, then $\omega C_\varphi + C_\psi$ is normal.

Proof. We see that $\omega C_\varphi + C_\psi$ is hyponormal if and only if

$$[(\omega C_\varphi + C_\psi)^*, \omega C_\varphi + C_\psi] \geq 0,$$

equivalently,

$$(\omega C_\varphi + C_\psi)^*(\omega C_\varphi + C_\psi) \geq (\omega C_\varphi + C_\psi)(\omega C_\varphi + C_\psi)^*,$$

which means that

$$\begin{aligned}
&\Leftrightarrow \quad |\omega|^2(C_\varphi^*C_\varphi - C_\varphi C_\varphi^*) + (C_\psi^*C_\psi - C_\psi C_\psi^*)\\
&\qquad +\overline{\omega}(C_\varphi^*C_\psi - C_\psi C_\varphi^*) + \omega(C_\psi^*C_\varphi - C_\varphi C_\psi^*) \geq 0\\
&\Leftrightarrow \quad |\omega|^2[C_\varphi^*, C_\varphi] + [C_\psi^*, C_\psi] + \overline{\omega}[C_\varphi^*, C_\psi] + \omega[C_\psi^*, C_\varphi] \geq 0. \qquad (2)
\end{aligned}$$

Since $\omega \neq 0$, we can set $\omega = re^{i\theta}$ for any $r > 0$ and an arbitrary real θ. Then we obtain from (2) that

$$r^2[C_\varphi^*, C_\varphi] + [C_\psi^*, C_\psi] + re^{-i\theta}[C_\varphi^*, C_\psi] + re^{i\theta}[C_\psi^*, C_\varphi] \geq 0.$$

Hence it follows that

$$[C_\varphi^*, C_\varphi] + \frac{1}{r^2}[C_\psi^*, C_\psi] + \frac{e^{-i\theta}}{r}[C_\varphi^*, C_\psi] + \frac{e^{i\theta}}{r}[C_\psi^*, C_\varphi] \geq 0.$$

Letting $r \to \infty$, we have $[C_\varphi^*, C_\varphi] \geq 0$. Hence C_φ is hyponormal. Furthermore, since $[C_\psi^*, C_\varphi] = [C_\varphi^*, C_\psi]^*$, it follows that (2)

$$\begin{aligned}
&\Leftrightarrow \quad |\omega|^2[C_\varphi^*, C_\varphi] + [C_\psi^*, C_\psi] + 2\mathrm{Re}\{\overline{\omega}[C_\varphi^*, C_\psi]\} \geq 0\\
&\Leftrightarrow \quad |\omega|^2\langle[C_\varphi^*, C_\varphi]h, h\rangle + \langle[C_\psi^*, C_\psi]h, h\rangle + 2\mathrm{Re}\{\overline{\omega}\langle[C_\varphi^*, C_\psi]h, h\rangle\} \geq 0\\
&\Rightarrow \quad |\omega|^2\langle[C_\varphi^*, C_\varphi]h, h\rangle + \langle[C_\psi^*, C_\psi]h, h\rangle + 2|\omega||\langle[C_\varphi^*, C_\psi]h, h\rangle| \geq 0
\end{aligned}$$

for all $h \in H^2$. Since both $[C_\varphi^*, C_\varphi]$ and $[C_\psi^*, C_\psi]$ are self-adjoint, both $\langle[C_\varphi^*, C_\varphi]h, h\rangle$ and $\langle[C_\psi^*, C_\psi]h, h\rangle$ are real and hence

$$|\langle[C_\varphi^*, C_\psi]h, h\rangle|^2 \leq \langle[C_\varphi^*, C_\varphi]h, h\rangle\langle[C_\psi^*, C_\psi]h, h\rangle. \qquad (3)$$

Since C_φ is hyponormal, $\langle[C_\varphi^*, C_\varphi]h, h\rangle \geq 0$ for all $h \in H^2$. Hence, C_ψ is hyponormal from (3).

Conversely, we get that

$$\begin{aligned}
& [(\omega C_\varphi + C_\psi)^*, \omega C_\varphi + C_\psi]\\
= \;& (\omega C_\varphi + C_\psi)^*(\omega C_\varphi + C_\psi) - (\omega C_\varphi + C_\psi)(\omega C_\varphi + C_\psi)^*\\
= \;& |\omega|^2[C_\varphi^*, C_\varphi] + [C_\psi^*, C_\psi] + \overline{\omega}[C_\varphi^*, C_\psi] + \omega[C_\psi^*, C_\varphi]\\
= \;& |\omega|^2[C_\varphi^*, C_\varphi] + [C_\psi^*, C_\psi] + 2\mathrm{Re}\{\overline{\omega}[C_\varphi^*, C_\psi]\} \qquad (4)
\end{aligned}$$

due to $[C_\psi^*, C_\varphi] = [C_\varphi^*, C_\psi]^*$. Since C_φ and C_ψ are hyponormal, $[C_\varphi^*, C_\varphi] \geq 0$ and $[C_\psi^*, C_\psi] \geq 0$. Thus, if $\mathrm{Re}\{\overline{\omega}[C_\varphi^*, C_\psi]\} \geq 0$, then we obtain from (4) that

$$[(\omega C_\varphi + C_\psi)^*, \omega C_\varphi + C_\psi] \geq 0.$$

Therefore, $\omega C_\varphi + C_\psi$ is hyponormal. In particular, if C_φ and C_ψ are normal, then

$$[C_\varphi^*, C_\varphi] = 0 = [C_\psi^*, C_\psi]. \qquad (5)$$

Since C_φ and C_ψ are normal, we can set φ and ψ as

$$\begin{cases} \varphi(z) = \gamma z \text{ for some } \gamma \text{ with } |\gamma| \leq 1, \text{ and}\\ \psi(z) = \delta z \text{ for some } \delta \text{ with } |\delta| \leq 1. \end{cases}$$

Thus, $(\varphi \circ \psi)(z) = (\psi \circ \varphi)(z) = \gamma\delta z$ and so

$$C_\varphi C_\psi = C_{\psi\circ\varphi} = C_{\varphi\circ\psi} = C_\psi C_\varphi.$$

By Fuglede-Putnam theorem, we ensure that $C_\varphi^*C_\psi = C_\psi C_\varphi^*$ and hence

$$[C_\varphi^*, C_\psi] = 0. \qquad (6)$$

Therefore, we obtain from (4) with (5) and (6) that

$$[(\omega C_\varphi + C_\psi)^*, \omega C_\varphi + C_\psi] = 0.$$

Thus, in this case, $\omega C_\varphi + C_\psi$ is normal. □

Corollary 2.2. *Let φ and ψ be analytic maps from $\mathbb{D}$ into itself and let $\omega C_\varphi + C_\psi$ is hyponormal for any nonzero $\omega \in \mathbb{C}$. Then the following statements hold.*
(i) $\varphi(0) = \psi(0) = 0$.
(ii) If C_ψ is invertible, then C_ψ is normal and $[C_\varphi^*, C_\psi] = 0$.

Proof. (i) Since both C_φ and C_ψ are hyponormal from Theorem 2.1, it follows from [7, Theorem 2] that $\varphi(0) = \psi(0) = 0$.

(ii) If C_ψ is invertible, ψ is an automorphism of $\mathbb{D}$ such that $\psi(z) = \lambda \frac{z-c}{\bar{c}z-1}$ where $|\lambda| = 1$ and $|c| < 1$ from [11, Corollary 2.0.2]. Since C_ψ is hyponormal by Theorem 2.1, $\psi(0) = 0$. Hence $\psi(z) = -\lambda z$ and so C_ψ is normal. Thus $[C_\psi^*, C_\psi] = 0$ which implies $[C_\varphi^*, C_\psi] = 0$ by Theorem 2.1. □

Example 2.3. *Let $\varphi(z) = \frac{1}{2}z + \frac{1}{2}$ and $\psi(z) = \frac{1}{4}z + \frac{3}{4}$ be analytic maps from $\mathbb{D}$ into itself. Since $\varphi(0) \neq 0$ and $\psi(0) \neq 0$, $\omega C_\varphi + C_\psi$ is not hyponormal for some nonzero $\omega \in \mathbb{C}$ from Corollary 2.2 (i). On the other hand, we now let $\varphi(z) = \frac{z}{z+2}$ and $\psi(z) = z$. Then*

$$C_\varphi^* C_\psi K_\alpha = C_\varphi^* \frac{1}{1 - \overline{\alpha} z} = C_\varphi^* K_\alpha = K_{\varphi(\alpha)}$$

and

$$C_\psi C_\varphi^* K_\alpha = C_\psi K_{\varphi(\alpha)} = \frac{1}{1 - \overline{\varphi(\alpha)} z} = K_{\varphi(\alpha)}.$$

Thus $[C_\varphi^*, C_\psi] = 0$. Since C_φ and C_ψ are hyponormal, $\omega C_\varphi + C_\psi$ is hyponormal for any $\omega \neq 0$ in $\mathbb{C}$ from Theorem 2.1.

Corollary 2.4. *Let φ be analytic map from $\mathbb{D}$ into itself. If $\omega C_\varphi + C_\varphi^*$ is hyponormal for any nonzero $\omega \in \mathbb{C}$, then $\varphi(z) = \gamma z$ for some γ with $|\gamma| \leq 1$.*

Proof. If we replace C_ψ by C_φ^* in Theorem 2.1, then C_φ and C_φ^* are hyponormal. Thus C_φ is normal and so $\varphi(z) = \gamma z$ for some γ with $|\gamma| \leq 1$. □

Theorem 2.5. *Let φ and ψ be analytic maps from $\mathbb{D}$ into itself. If $\omega C_\varphi + C_\psi$ is hyponormal for any nonzero $\omega \in \mathbb{C}$, then*

$$|\langle [C_\varphi^*, C_\psi] K_\alpha, K_\alpha \rangle|^2 \leq (\|K_\alpha\|^2 - \|K_{\varphi(\alpha)}\|^2)(\|K_\alpha\|^2 - \|K_{\psi(\alpha)}\|^2)$$

holds for any $\alpha \in \mathbb{D}$. Furthermore, in this case, either $|\alpha| = |\varphi(\alpha)|, |\psi(\alpha)|$ or $|\alpha| \geq |\varphi(\alpha)|, |\psi(\alpha)|$ holds for any $\alpha \in \mathbb{D}$.

Proof. Suppose $\omega C_\varphi + C_\psi$ is hyponormal for nonzero $\omega \in \mathbb{C}$. Then $\varphi(0) = \psi(0) = 0$ from Corollary 2.2 (i). Thus using [5, Theorem 2.1] we have $\|C_\varphi\| = 1$ and $\|C_\psi\| = 1$. Hence we observe that for any $\alpha \in \mathbb{D}$

$$\begin{aligned}
\langle [C_\varphi^*, C_\varphi] K_\alpha, K_\alpha \rangle &= \langle C_\varphi K_\alpha, C_\varphi K_\alpha \rangle - \langle C_\varphi^* K_\alpha, C_\varphi^* K_\alpha \rangle \\
&= \|C_\varphi K_\alpha\|^2 - \langle K_{\varphi(\alpha)}, K_{\varphi(\alpha)} \rangle \\
&= \|C_\varphi K_\alpha\|^2 - \|K_{\varphi(\alpha)}\|^2 \\
&\leq \|C_\varphi\|^2 \|K_\alpha\|^2 - \|K_{\varphi(\alpha)}\|^2
\end{aligned}$$

$$\leq \|K_\alpha\|^2 - \|K_{\varphi(\alpha)}\|^2. \tag{7}$$

Similarly, we get that for any $\alpha \in \mathbb{D}$

$$\begin{aligned}\langle [C_\psi^*, C_\psi]K_\alpha, K_\alpha\rangle &= \|C_\psi K_\alpha\|^2 - \|K_{\psi(\alpha)}\|^2 \\ &\leq \|C_\psi\|^2\|K_\alpha\|^2 - \|K_{\psi(\alpha)}\|^2 \\ &\leq \|K_\alpha\|^2 - \|K_{\psi(\alpha)}\|^2. \end{aligned}\tag{8}$$

Thus we obtain from (1) in Theorem 2.1 with (7) and (8) that

$$|\langle [C_\varphi^*, C_\psi]K_\alpha, K_\alpha\rangle|^2 \leq (\|K_\alpha\|^2 - \|K_{\varphi(\alpha)}\|^2)(\|K_\alpha\|^2 - \|K_{\psi(\alpha)}\|^2)$$

for any $\alpha \in \mathbb{D}$. In addition, we have

$$\begin{aligned} 0 &\leq (\|K_\alpha\|^2 - \|K_{\varphi(\alpha)}\|^2)(\|K_\alpha\|^2 - \|K_{\psi(\alpha)}\|^2) \\ &= (\frac{1}{1-|\alpha|^2} - \frac{1}{1-|\varphi(\alpha)|^2})(\frac{1}{1-|\alpha|^2} - \frac{1}{1-|\psi(\alpha)|^2}) \end{aligned}$$

since $|\langle [C_\varphi^*, C_\psi]K_\alpha, K_\alpha\rangle|^2 \geq 0$. Thus it holds that

$$|\alpha| \leq |\varphi(\alpha)|, |\psi(\alpha)| \text{ or } |\alpha| \geq |\varphi(\alpha)|, |\psi(\alpha)|$$

holds for any $\alpha \in \mathbb{D}$. Moreover, since φ and ψ are analytic maps from $\mathbb{D}$ into itself and $\varphi(0) = 0$ and $\psi(0) = 0$, Schwartz lemma implies that

$$|\varphi(\alpha)| \leq |\alpha| \text{ and } |\psi(\alpha)| \leq |\alpha|$$

for all $\alpha \in \mathbb{D}$. Thus, we obtain that

$$|\alpha| = |\varphi(\alpha)|, |\psi(\alpha)| \text{ or } |\alpha| \geq |\varphi(\alpha)|, |\psi(\alpha)|$$

holds for any $\alpha \in \mathbb{D}$. □

Corollary 2.6. *If $|\varphi(\alpha)| < |\alpha| < |\psi(\alpha)|$ or $|\psi(\alpha)| < |\alpha| < |\varphi(\alpha)|$ for some $\alpha \in \mathbb{D}$, then $\omega C_\varphi + C_\psi$ is not hyponormal for some nonzero $\omega \in \mathbb{C}$.*

Proof. By Theorem 2.5, we observe that if

$$\|K_{\varphi(\alpha)}\|^2 < \|K_\alpha\|^2 < \|K_{\psi(\alpha)}\|^2 \text{ or } \|K_{\psi(\alpha)}\|^2 < \|K_\alpha\|^2 < \|K_{\varphi(\alpha)}\|^2$$

for some $\alpha \in \mathbb{D}$, then $\omega C_\varphi + C_\psi$ is not hyponormal for some nonzero $\omega \in \mathbb{C}$. Since $\|K_{\varphi(\alpha)}\|^2 = \frac{1}{1-|\varphi(\alpha)|^2}$, $\|K_{\psi(\alpha)}\|^2 = \frac{1}{1-|\psi(\alpha)|^2}$, and $\|K_\alpha\|^2 = \frac{1}{1-|\alpha|^2}$, we obtain the result. □

Example 2.7. *Let $\varphi(z) = \frac{1}{2}z + \frac{1}{2}$ and $\psi(z) = \frac{1}{2}z$ be analytic maps from $\mathbb{D}$ into itself. Take $\alpha = \frac{1}{2}$. Then $\varphi(\frac{1}{2}) = \frac{3}{4}$ and $\psi(\frac{1}{2}) = \frac{1}{4}$. Thus, $|\psi(\alpha)| < |\alpha| < |\varphi(\alpha)|$ holds at $\alpha = \frac{1}{2}$ and so $C_\varphi + C_\psi$ is not hyponormal.*

Recall that a closed subspace $\mathcal{M}$ of a Hilbert space $\mathcal{H}$ is said to be an *invariant subspace* for an operator $T \in \mathcal{L}(\mathcal{H})$ if $Th \in \mathcal{M}$ whenever $h \in \mathcal{M}$. In other words, if $T\mathcal{M} \subseteq \mathcal{M}$. We call that $\mathcal{M}$ is a *reducing subspace* for $T \in \mathcal{L}(\mathcal{H})$ if $T\mathcal{M} \subseteq \mathcal{M}$ and $T\mathcal{M}^\perp \subseteq \mathcal{M}^\perp$. For a positive integer i and α in $\mathbb{D}$, *the ith derivative evaluation kernel* at α, denoted as $K_\alpha^{[i]}$, is the function in H^2 such that $\langle f, K_\alpha^{[i]}\rangle = f^{(i)}(\alpha)$ for any function f on H^2. In particular, it is easy to see that $K_0^{[i]} = i!z^i$ for a positive integer i.

Theorem 2.8. *Let φ and ψ be analytic maps from $\mathbb{D}$ into itself and let c be the Denjoy-Wolff point of φ and ψ in $\mathbb{D}$. If m is a positive integer, then $\mathcal{M}_m(c) := span\{K_c, K_c^{[1]}, \cdots, K_c^{[m]}\}$ is an invariant subspace of $\overline{\omega}C_\varphi^* + C_\psi^*$ for any nonzero $\omega \in \mathbb{C}$. In particular, if $\omega C_\varphi + C_\psi$ is hyponormal for nonzero $\omega \in \mathbb{C}$, then $(\omega C_\varphi + C_\psi)\big|_{\mathcal{M}_n(0)^\perp}$ is hyponormal and if $\omega C_\varphi + C_\psi$ is cohyponormal, then $(\omega C_\varphi + C_\psi)\big|_{\mathcal{M}_n(0)}$ is normal.*

Proof. We first show that

$$\begin{aligned}(\omega C_\varphi + C_\psi)^* K_c &= \overline{\omega} C_\varphi^* K_c + C_\psi^* K_c \\ &= \overline{\omega} K_{\varphi(c)} + K_{\psi(c)} = (\overline{\omega}+1)K_c. \end{aligned} \tag{9}$$

For any function $f \in H^2$ and any positive integer n, we obtain that

$$\begin{aligned}\langle f, (\omega C_\varphi + C_\psi)^* K_c^{[n]}\rangle &= \langle (\omega C_\varphi + C_\psi) f, K_c^{[n]}\rangle \\ &= \langle \omega(f\circ\varphi) + (f\circ\psi), K_c^{[n]}\rangle \\ &= \frac{d^n}{dz^n}\Big[\omega f(\varphi(z)) + f(\psi(z))\Big]\Big|_{z=c} \\ &= \sum_{i=1}^{n-1}[g_i(c) + h_i(c)] f^{(i)}(c) + (\omega(\varphi'(c))^n + (\psi'(c))^n) f^{(n)}(c) \\ &= \langle f, \sum_{i=1}^{n-1}[\overline{g_i(c) + h_i(c)}]K_c^{[i]} + (\overline{\omega(\varphi'(c))^n + (\psi'(c))^n})K_c^{[n]}\rangle \end{aligned}$$

where $g_i(z)$ and $h_i(z)$ are appropriate sums for various products of derivatives of φ and ψ, respectively. Hence

$$(\omega C_\varphi + C_\psi)^* K_c^{[n]} = \sum_{i=1}^{n-1}[\overline{g_i(c) + h_i(c)}]K_c^{[i]} + (\overline{\omega(\varphi'(c))^n + (\psi'(c))^n})K_c^{[n]}. \tag{10}$$

Therefore $\mathcal{M}_n(c)$ is an invariant subspace of $(\omega C_\varphi + C_\psi)^*$ and so $\mathcal{M}_n(c)^\perp$ is an invariant subapce of $\omega C_\varphi + C_\psi$. In particular, if $\omega C_\varphi + C_\psi$ is hyponormal for nonzero $\omega \in \mathbb{C}$, then zero is the Denjoy-Wolff point of φ and ψ from Corollary 2.2 (i) and so $(\omega C_\varphi + C_\psi)\big|_{\mathcal{M}_n(0)^\perp}$ is hyponormal. In addition, since $\mathcal{M}_n(0)$ is an invariant subapce for $(\omega C_\varphi + C_\psi)^*$ and it is the finite dimensional subspace, if $(\omega C_\varphi + C_\psi)^*$ is hyponormal, then $(\omega C_\varphi + C_\psi)^*$ is normal on $\mathcal{M}_n(0)$. Thus, $\omega C_\varphi + C_\psi$ is normal on $\mathcal{M}_n(0)$. Since every normal operator on a finite dimensional space is reductive, $\mathcal{M}_n(0)$ is a reducing subspace for $\omega C_\varphi + C_\psi$ and so $(\omega C_\varphi + C_\psi)\big|_{\mathcal{M}_n(0)}$ is normal. □

Corollary 2.9. *Let φ and ψ be analytic maps from $\mathbb{D}$ into itself and let c be the Denjoy-wolff point of φ and ψ in $\mathbb{D}$. If $\omega C_\varphi + C_\psi$ is cohyponormal for any nonzero $\omega \in \mathbb{C}$, then $span\{K_c\}$ is a reducing subspace for $\omega C_\varphi + C_\psi$.*

Proof. From (9) in Theorem 2.8,

$$(\omega C_\varphi + C_\psi)^* K_c = (\overline{\omega}+1)K_c. \tag{11}$$

Thus, we see that

$$K_c \in \ker((\omega C_\varphi + C_\psi)^* - (\overline{\omega}+1)).$$

Since $\omega C_\varphi + C_\psi$ is cohyponormal,

$$\ker((\omega C_\varphi + C_\psi)^* - (\overline{\omega}+1)) \subseteq \ker(\omega C_\varphi + C_\psi - (\omega+1)).$$

Hence, it holds that $K_c \in \ker(\omega C_\varphi + C_\psi - (\omega+1))$ and so

$$(\omega C_\varphi + C_\psi)K_c = (\omega+1)K_c. \tag{12}$$

Thus, the conclusion follows from (11) and (12). □

Theorem 2.10. *Let φ and ψ be analytic maps from $\mathbb{D}$ into itself and let c be the Denjoy-Wolff point of φ and ψ in $\mathbb{D}$. Then*

$$\sigma_p(\omega C_\varphi + C_\psi) \subseteq \{0, \omega + 1, \omega\varphi'(c) + \psi'(c), \omega(\varphi'(c))^2 + (\psi'(c))^2, \cdots\} \subseteq \sigma(\omega C_\varphi + C_\psi) \cup \{0\}.$$

Proof. Let $\gamma \in \sigma_p(\omega C_\varphi + C_\psi)$. Then there exists a nonzero function $f \in H^2$ such that $(\omega C_\varphi + C_\psi)f = \gamma f$. If $\gamma = 0$, it is trivial. If $\gamma \neq 0$, then

$$\omega f(\varphi(z)) + f(\psi(z)) = \gamma f(z) \tag{13}$$

for $z \in \mathbb{D}$. Let f have a zero of order n at c. If $n = 0$, put $z = c$ in (13). Then

$$\omega f(\varphi(c)) + f(\psi(c)) = \gamma f(c).$$

Thus, we get that

$$\omega f(c) + f(c) = \gamma f(c)$$

and so $\gamma = \omega + 1$. For $n = 1, 2, 3, \cdots$, if we differentiate (13) n times, then we obtain that

$$\sum_{i=1}^{n-1}[g_i(z)f^{(i)}(\varphi(z)) + h_i(z)f^{(i)}(\psi(z))] + \omega f^{(n)}(\varphi(z))(\varphi'(z))^n + f^{(n)}(\psi(z))(\psi'(z))^n = \gamma f^{(n)}(z) \tag{14}$$

where $g_i(z)$ and $h_i(z)$ are appropriate sums for various products of derivatives of φ and ψ, respectively. Put $z = c$ in (14). Then we obtain that

$$\omega f^{(n)}(c)(\varphi'(c))^n + f^{(n)}(c)(\psi'(c))^n = \gamma f^{(n)}(c)$$

and so $\gamma = \omega(\varphi'(c))^n + (\psi'(c))^n$ which gives the first inclusion is true.

We now show that the second inclusion holds. For an arbitrary positive integer m, set $\mathcal{M}_m(c) = \operatorname{span}\{K_c, K_c^{[1]}, \cdots, K_c^{[m]}\}$ and $\mathcal{M}_0(c) = \operatorname{span}\{K_c\}$. Then this set is linearly independent. Indeed, we assume that there exist $t_j \in \mathbb{C}$ for $j = 0, 1, 2, \cdots, m$ such that $\sum_{n=0}^m t_n K_c^{[n]} = 0$. If we set $g_j(z) = \frac{1}{j!}(z-c)^j$ for $j = 0, 1, 2, \cdots, m$, then

$$0 = \langle g_j, \sum_{n=0}^m t_n K_c^{[n]}\rangle = \sum_{n=0}^m \overline{t_n} g_j^{(n)}(c) = \overline{t_j}$$

for $j = 0, 1, 2, \cdots, m$. In addition, $\mathcal{M}_m(c)$ is invariant for $(\omega C_\varphi + C_\psi)^*$ from Theorem 2.8. Thus, the adjoint of $\omega C_\varphi + C_\psi$ can be written as

$$(\omega C_\varphi + C_\psi)^* = \begin{pmatrix} (\omega C_\varphi + C_\psi)^*\big|_{\mathcal{M}_m(c)} & A \\ 0 & B_m \end{pmatrix}$$

on $\mathcal{M}_m(c) \oplus \mathcal{M}_m(c)^\perp$. In particular, using (9) and (10) in Theorem 2.8, we can write $(\omega C_\varphi + C_\psi)^*\big|_{\mathcal{M}_m(c)}$ as the upper triangular matrix whose diagonal elements are $\overline{\omega(\varphi'(c))^j + (\psi'(c))^j}$ for $j = 0, 1, 2, \cdots, m$. Since $\mathcal{M}_m(c)$ is also finite dimensional, $\overline{\omega(\varphi'(c))^j + (\psi'(c))^j}$ with $j = 0, 1, 2, \cdots, m$ are eigenvalues for $(\omega C_\varphi + C_\psi)^*$. Taking m sufficiently large, we thus obtain that

$$\{\overline{\omega + 1}, \overline{\omega\varphi'(c) + \psi'(c)}, \overline{\omega(\varphi'(c))^2 + (\psi'(c))^2}, \cdots\} \subseteq \sigma((\omega C_\varphi + C_\psi)^*).$$

Thus, it holds that

$$\{\omega + 1, \omega\varphi'(c) + \psi'(c), \omega(\varphi'(c))^2 + (\psi'(c))^2, \cdots\} \subseteq \sigma(\omega C_\varphi + C_\psi)$$

as we desired. □

Corollary 2.11. *Let φ and ψ be analytic maps from $\mathbb{D}$ into itself. If $\omega C_\varphi + C_\psi$ is hyponormal for nonzero $\omega \in \mathbb{C}$, then*

$$iso\sigma(\omega C_\varphi + C_\psi) \subseteq \{0, \omega+1, \omega\varphi'(0)+\psi'(0), \omega(\varphi'(0))^2 + (\psi'(0))^2, \cdots\} \subseteq \sigma_{ap}(\overline{\omega}C_\varphi^* + C_\psi^*)^* \cup \{0\}$$

for any subset Δ of $\mathbb{C}$, $\Delta^* = \{\overline{z} : z \in \Delta\}$.

Proof. Since $\omega C_\varphi + C_\psi$ is hyponormal, it holds that

$$\text{iso}\sigma(\omega C_\varphi + C_\psi) \subseteq \sigma_p(\omega C_\varphi + C_\psi).$$

From Corollary 2.2 (i), we know zero is the Denjoy-Wolff point of φ and ψ. Thus, we get from Theorem 2.10 that

$$\text{iso}\sigma(\omega C_\varphi + C_\psi) \subseteq \{0, \omega+1, \omega\varphi'(0)+\psi'(0), \omega(\varphi'(0))^2 + (\psi'(0))^2, \cdots\}.$$

In addition, it is known that $\sigma(T) = \sigma_{ap}(T^*)^*$ for any hyponormal operator $T \in \mathcal{L}(\mathcal{H})$. Thus the result follows from Theorem 2.10. □

We next consider the sums of composition operators with linear fractional symbols.

Lemma 2.12. *Let $\varphi(z) = \frac{z}{uz+v}$ with $|v| \geq 1 + |u|$ and $\psi(z) = \frac{z}{sz+t}$ with $|t| \geq 1+|s|$. Then $[C_\varphi^*, C_\psi] = 0$ if and only if one of the following cases occurs.*
(i) $\varphi(z) = \frac{z}{v}$ and $\psi(z) = \frac{z}{t}$.
(ii) $\varphi(z) = z$ and $\psi(z) = \frac{z}{sz+t}$.
(iii) $\varphi(z) = \frac{z}{uz+v}$ and $\psi(z) = z$.

Proof. Suppose that $[C_\varphi^*, C_\psi] = 0$. We obtain that for any $\alpha \in \mathbb{D}$

$$C_\varphi^* C_\psi K_\alpha = C_\varphi^* \frac{1}{1-\overline{\alpha}\psi(z)} = C_\varphi^* \frac{1}{1-\overline{\alpha}\frac{z}{sz+t}} = C_\varphi^* \frac{sz+t}{(s-\overline{\alpha})z+t}.$$

Note that for $s \neq \overline{\alpha}$, we can write

$$\begin{aligned} \frac{sz+t}{(s-\overline{\alpha})z+t} &= \frac{s}{s-\overline{\alpha}} + \frac{t - \frac{st}{s-\overline{\alpha}}}{t+(s-\overline{\alpha})z} = \frac{s}{s-\overline{\alpha}} + \frac{1-\frac{s}{s-\overline{\alpha}}}{1+\frac{s-\overline{\alpha}}{t}z} \\ &= \frac{s}{s-\overline{\alpha}}K_0 + \left(1 - \frac{s}{s-\overline{\alpha}}\right)K_{\frac{\alpha-\overline{s}}{\overline{t}}}. \end{aligned}$$

Thus, we induce that for any $\alpha \in \mathbb{D}$ with $s \neq \overline{\alpha}$

$$\begin{aligned} C_\varphi^* C_\psi K_\alpha &= C_\varphi^*\left[\frac{s}{s-\overline{\alpha}}K_0 + \left(1-\frac{s}{s-\overline{\alpha}}\right)K_{\frac{\alpha-\overline{s}}{\overline{t}}}\right] \\ &= \frac{s}{s-\overline{\alpha}}K_{\varphi(0)} + \left(1-\frac{s}{s-\overline{\alpha}}\right)K_{\varphi(\frac{\alpha-\overline{s}}{\overline{t}})} \\ &= \frac{s}{s-\overline{\alpha}} - \frac{\overline{\alpha}}{s-\overline{\alpha}}\frac{1}{1-\frac{\frac{\overline{\alpha}-s}{t}}{\overline{u}(\frac{\overline{\alpha}-s}{t})+\overline{v}}z} \\ &= \frac{s}{s-\overline{\alpha}} - \frac{\overline{\alpha}}{s-\overline{\alpha}-\frac{(s-\overline{\alpha})^2}{\overline{u}(s-\overline{\alpha})-\overline{v}t}z} \\ &= \frac{s}{s-\overline{\alpha}} - \frac{\overline{\alpha}[\overline{u}(s-\overline{\alpha})-\overline{v}t]}{(s-\overline{\alpha})[\overline{u}(s-\overline{\alpha})-\overline{v}t-(s-\overline{\alpha})z]} \\ &= \frac{s[\overline{u}(s-\overline{\alpha})-\overline{v}t-(s-\overline{\alpha})z] - \overline{\alpha}[\overline{u}(s-\overline{\alpha})-\overline{v}t]}{(s-\overline{\alpha})[\overline{u}(s-\overline{\alpha})-\overline{v}t-(s-\overline{\alpha})z]}. \end{aligned} \tag{15}$$

On the other hand, we assert that for any $\alpha \in \mathbb{D}$

$$\begin{aligned} C_\psi C_\varphi^* K_\alpha &= C_\psi K_{\varphi(\alpha)} = K_{\varphi(\alpha)} \circ \psi = \frac{1}{1-\overline{\varphi(\alpha)}\psi(z)} \\ &= \frac{1}{1-\left(\frac{\overline{\alpha}}{\overline{u\alpha}+\overline{v}}\right)\left(\frac{z}{sz+t}\right)} = \frac{(\overline{u\alpha}+\overline{v})(sz+t)}{(\overline{u\alpha}+\overline{v})(sz+t)-\overline{\alpha}z}. \end{aligned} \tag{16}$$

Since $[C_\varphi^*, C_\psi] = 0$, we get from (15) and (16) that

$$\frac{s[\overline{u}(s-\overline{\alpha})-\overline{v}t-(s-\overline{\alpha})z]-\overline{\alpha}[\overline{u}(s-\overline{\alpha})-\overline{v}t]}{(s-\overline{\alpha})[\overline{u}(s-\overline{\alpha})-\overline{v}t-(s-\overline{\alpha})z]} = \frac{(\overline{u\alpha}+\overline{v})(sz+t)}{(\overline{u\alpha}+\overline{v})(sz+t)-\overline{\alpha}z}$$

for any $\alpha \in \mathbb{D}$ with $s \neq \overline{\alpha}$. This implies that

$$\begin{aligned} &\{s[\overline{u}(s-\overline{\alpha})-\overline{v}t-(s-\overline{\alpha})z]-\overline{\alpha}[\overline{u}(s-\overline{\alpha})-\overline{v}t]\}\{(\overline{u\alpha}+\overline{v})(sz+t)-\overline{\alpha}z\} \\ = \ &(s-\overline{\alpha})[\overline{u}(s-\overline{\alpha})-\overline{v}t-(s-\overline{\alpha})z](\overline{u\alpha}+\overline{v})(sz+t) \end{aligned} \tag{17}$$

for any $\alpha \in \mathbb{D}$ with $s \neq \overline{\alpha}$. A computation gives from (17) that

$$0 = z^2[\overline{u}s\overline{\alpha}^3 - s(1+\overline{u}s-\overline{v})\overline{\alpha}^2 + s^2(1-\overline{v})\overline{\alpha}] + z\{[\overline{u}(t-1)]\overline{\alpha}^3 + [(2-t)\overline{u}s]\overline{\alpha}^2 - \overline{u}s^2\overline{\alpha}\} \tag{18}$$

for any $\alpha \in \mathbb{D}$ with $s \neq \overline{\alpha}$. Since (18) holds for any $z \in \mathbb{D}$, both the coefficient of z^2 and the coefficient of z in (18) must be zero. This means that

$$\begin{cases} \overline{u}s\overline{\alpha}^3 - s(1+\overline{u}s-\overline{v})\overline{\alpha}^2 + s^2(1-\overline{v})\overline{\alpha} = 0 \text{ and} \\ [\overline{u}(t-1)]\overline{\alpha}^3 + [(2-t)\overline{u}s]\overline{\alpha}^2 - \overline{u}s^2\overline{\alpha} = 0. \end{cases} \tag{19}$$

In addition, (19) holds for any $\alpha \in \mathbb{D}$ with $s \neq \overline{\alpha}$. Thus, we just find the solutions for which satisfy the following equations:

$$\begin{cases} \overline{u}s = 0, s(1+\overline{u}s-\overline{v}) = 0, s^2(1-\overline{v}) = 0 \\ \overline{u}(t-1) = 0, (2-t)\overline{u}s = 0, \overline{u}s^2 = 0. \end{cases}$$

This ensures that

$$\begin{cases} \varphi(z) = \frac{z}{v} \text{ and } \psi(z) = \frac{z}{t} \text{ or} \\ \varphi(z) = z \text{ and } \psi(z) = \frac{z}{sz+t} \text{ or} \\ \varphi(z) = \frac{z}{uz+v} \text{ and } \psi(z) = z. \end{cases}$$

We now show the converse. If $\varphi(z) = \frac{z}{v}$ and $\psi(z) = \frac{z}{t}$, then $(\varphi \circ \psi)(z) = \frac{z}{vt} = (\psi \circ \varphi)(z)$. Thus, C_φ and C_ψ commute. In addition, in this case, C_φ and C_ψ is normal. Thus, we ensures that C_φ^* and C_ψ commute by Fuglede-Putnam Theorem. Since every composition operator induced by the identity map is the identity operator, the other cases are also trivial. □

Theorem 2.13. *Let φ and ψ be linear fractional maps from $\mathbb{D}$ into itself. If C_φ and C_ψ are hyponormal and $\langle [C_\varphi^*, C_\psi]K_\alpha, K_\alpha\rangle = 0$ for all $\alpha \in \mathbb{D}$, then at least one of C_φ and C_ψ is normal. Furthermore, in this case, $\omega C_\varphi + C_\psi$ is hyponormal.*

Proof. If C_φ and C_ψ are hyponormal, then $\varphi(0) = 0$ and $\psi(0) = 0$ from [7, Theorem 2]. Thus, we can write

$$\varphi(z) = \frac{z}{uz+v} \text{ with } |v| \geq 1 + |u| \text{ and } \psi(z) = \frac{z}{sz+t} \text{ with } |t| \geq 1 + |s|$$

since φ and ψ are linear fractional maps from $\mathbb{D}$ into itself. Define e_k by $e_k(z) = z^k$. Since $\langle [C_\varphi^*, C_\psi]K_\alpha, K_\alpha\rangle = 0$ for all $\alpha \in \mathbb{D}$,

$$\begin{aligned} 0 &= \langle [C_\varphi^*, C_\psi] \sum_{k=0}^{\infty} \overline{\alpha}^k e_k, \sum_{j=0}^{\infty} \overline{\alpha}^j e_j\rangle \\ &= \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \overline{\alpha}^k \alpha^j \langle [C_\varphi^*, C_\psi] e_k, e_j\rangle. \end{aligned}$$

Set $\alpha = re^{i\theta}$. Then, for all $\alpha \in \mathbb{D}$

$$\begin{aligned} 0 &= \langle [C_\varphi^*, C_\psi]K_\alpha, K_\alpha\rangle e^{-in\theta} \\ &= \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} r^{j+k} e^{i(j-k-n)\theta} \langle [C_\varphi^*, C_\psi] e_k, e_j\rangle. \end{aligned}$$

Thus, we have

$$\begin{aligned} 0 &= \frac{1}{2\pi}\int_0^{2\pi} \langle [C_\varphi^*, C_\psi]K_\alpha, K_\alpha\rangle e^{-in\theta} d\theta \\ &= \sum_{k=0}^{\infty} r^{2k+n} \langle [C_\varphi^*, C_\psi] e_k, e_{k+n}\rangle \end{aligned}$$

for every $0 < r < 1$. This implies that $\langle [C_\varphi^*, C_\psi] e_k, e_{k+n}\rangle = 0$ and so

$$\langle [C_\varphi^*, C_\psi] e_k, e_m\rangle = 0$$

for all integers k and m where $m \geq k \geq 0$. Therefore

$$\langle [C_\varphi^*, C_\psi]^* K_\alpha, K_\alpha\rangle = \overline{\langle [C_\varphi^*, C_\psi] K_\alpha, K_\alpha\rangle} = 0$$

for all $\alpha \in \mathbb{D}$. Thus $\langle e_k, [C_\varphi^*, C_\psi] e_m\rangle = 0$ for all integers k and m where $m \geq k \geq 0$. Hence

$$\langle [C_\varphi^*, C_\psi] e_k, e_m\rangle = 0$$

for all non-negative integers k and m and so $[C_\varphi^*, C_\psi] = 0$. Thus, we know from Lemma 2.12 that $[C_\varphi^*, C_\psi] = 0$ implies that φ and ψ satisfy the one of (i), (ii) and (iii) in Lemma 2.12. Here, we note that if $\varphi(z) = \frac{z}{v}$ with $|v| \geq 1$ and $\psi(z) = \frac{z}{t}$ with $|t| \geq 1$, then C_φ and C_ψ are normal. Every composition operator with the identity map is the identity operator. In addition, if we take $|u| = v - 1$ with $u > 1$ and $|s| = t - 1$ with $t > 1$, then both C_φ and C_ψ are subnormal or hyponormal from [6, Theorem 5]. Furthermore, since C_φ and C_ψ are hyponormal and $[C_\varphi^*, C_\psi] = 0$, $\omega C_\varphi + C_\psi$ is hyponormal from Theorem 2.1. □

Proposition 2.14. *Let $\varphi(z) = \frac{z}{uz+v}$ with $|u| = v - 1$ and $v > 1$ and $\psi(z) = \gamma z$ with $|\gamma| \leq 1$. Then*

$$\langle [C_\varphi^*, C_\psi]K_\alpha, K_\alpha\rangle = \frac{-\overline{u\alpha}|\alpha|^2(\gamma^2 - \gamma)}{(\overline{u\alpha}\gamma + \overline{v} - \gamma|\alpha|^2)(\overline{u\alpha} + \overline{v} - \gamma|\alpha|^2)}$$

for any α in $\mathbb{D}$.

Proof. We note that

$$\begin{aligned} \langle [C_\varphi^*, C_\psi]K_\alpha, K_\alpha\rangle &= \langle C_\varphi^* C_\psi K_\alpha, K_\alpha\rangle - \langle C_\psi C_\varphi^* K_\alpha, K_\alpha\rangle \\ &= \langle C_\varphi^* C_\psi K_\alpha, K_\alpha\rangle - \langle C_\varphi^* K_\alpha, C_\psi^* K_\alpha\rangle \end{aligned} \tag{20}$$

for any $\alpha \in \mathbb{D}$. Since

$$C_\varphi^* C_\psi K_\alpha = C_\varphi^* K_\alpha(\psi(z)) = C_\varphi^* \left(\frac{1}{1-\overline{\alpha}\gamma z} \right) = C_\varphi^* K_{\alpha\overline{\gamma}}(z) = K_{\varphi(\alpha\overline{\gamma})}(z),$$

we get that

$$\begin{aligned} \langle C_\varphi^* C_\psi K_\alpha, K_\alpha \rangle &= \langle K_{\varphi(\alpha\overline{\gamma})}(z), K_\alpha(z) \rangle \\ &= K_{\varphi(\alpha\overline{\gamma})}(\alpha) = \frac{1}{1-\frac{|\alpha|^2\gamma}{\overline{u\alpha}\gamma+\overline{v}}}. \end{aligned} \tag{21}$$

On the other hand,

$$\begin{aligned} \langle C_\varphi^* K_\alpha, C_\psi^* K_\alpha \rangle &= \langle K_{\varphi(\alpha)}(z), K_{\psi(\alpha)}(z) \rangle \\ &= K_{\varphi(\alpha)}(\psi(\alpha)) = \frac{1}{1-\frac{|\alpha|^2\gamma}{\overline{u\alpha}+\overline{v}}}. \end{aligned} \tag{22}$$

Hence, a computation gives from (21) and (22) that

$$\langle C_\varphi^* C_\psi K_\alpha, K_\alpha \rangle - \langle C_\varphi^* K_\alpha, C_\psi^* K_\alpha \rangle = \frac{-\overline{u\alpha}|\alpha|^2(\gamma^2-\gamma)}{(\overline{u\alpha}\gamma+\overline{v}-\gamma|\alpha|^2)(\overline{u\alpha}+\overline{v}-\gamma|\alpha|^2)},$$

which gives the result from (20). □

The following example explains that $C_\varphi + C_\psi$ may not be hyponormal even if both C_φ and C_ψ are hyponormal.

Example 2.15. *Let $\varphi(z) = \frac{z}{z+2}$ and $\psi(z) = \frac{1}{2}z$ be analytic maps from $\mathbb{D}$ into itself. Then $C_\varphi + C_\psi$ is not hyponormal and $Re\{[C_\varphi^*, C_\psi]\} < 0$.*

Proof. In Proposition 2.14, take $u = 1$, $v = 2$, $\gamma = \frac{1}{2}$, and $\alpha = -\frac{1}{2}$. Then a direct computation gives that

$$\langle [C_\varphi^*, C_\psi] K_{-\frac{1}{2}}, K_{-\frac{1}{2}} \rangle \quad = \quad \frac{\frac{1}{2}\cdot\frac{1}{4}(\frac{1}{4}-\frac{1}{2})}{(-\frac{1}{4}+2-\frac{1}{2}\cdot\frac{1}{4})(-\frac{1}{2}+2-\frac{1}{2}\cdot\frac{1}{4})} = -\frac{2}{143}.$$

Thus $|\langle [C_\varphi^*, C_\psi] K_{-\frac{1}{2}}, K_{-\frac{1}{2}} \rangle|^2 = \frac{4}{20449}$. However, $[C_\psi^*, C_\psi] = 0$ since C_ψ is normal. Since (1) in Theorem 2.1 does not hold, $C_\varphi + C_\psi$ is not hyponormal. Moreover, since C_φ and C_ψ are hyponormal, $\mathrm{Re}\{[C_\varphi^*, C_\psi]\} < 0$ from Theorem 2.1. □

Corollary 2.16. *Let $\varphi(z) = \frac{z}{uz+v}$ with $|u| = v-1$ and $v > 1$ and $\psi(z) = \gamma z$ with $|\gamma| \le 1$. If $\gamma = 1$, then $\omega C_\varphi + C_\psi$ is hyponormal.*

Proof. By Proposition 2.14,

$$\langle [C_\varphi^*, C_\psi] K_\alpha, K_\alpha \rangle = \frac{-\overline{u\alpha}|\alpha|^2(\gamma^2-\gamma)}{(\overline{u\alpha}\gamma+\overline{v}-\gamma|\alpha|^2)(\overline{u\alpha}+\overline{v}-\gamma|\alpha|^2)}$$

for any α in $\mathbb{D}$. If $\gamma = 1$, then $\langle [C_\varphi^*, C_\psi] K_\alpha, K_\alpha \rangle = 0$ for any α in $\mathbb{D}$. Hence the proof follows from Theorem 2.13. □

Recall that for an operator $T \in \mathcal{L}(\mathcal{H})$, the *spectral radius* $r(T)$ and *numerical radius* $w(T)$ of T are defined by $r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}$ and $w(T) = \sup\{|\langle Tx, x\rangle| : \|x\| = 1\}$, respectively. If $r(T) = \|T\|$, then T is said to be *normaloid* and if $w(T) = r(T)$, then T is said to be *spectraloid*.

Proposition 2.17. *Let φ and ψ be analytic maps from $\mathbb{D}$ into itself. If $\omega C_\varphi + C_\psi$ is hyponormal for any nonzero $\omega \in \mathbb{D}$, then*

$$||\omega| - 1| \le r(\omega C_\varphi + C_\psi) = w(\omega C_\varphi + C_\psi) \le |\omega| + 1.$$

Proof. We know from [5, Theorem 2.1] that

$$\begin{cases} \frac{1}{\sqrt{1-|\varphi(0)|^2}} \le \|C_\varphi\| \le \frac{1+|\varphi(0)|}{\sqrt{1-|\varphi(0)|^2}} \text{ and} \\ \frac{1}{\sqrt{1-|\psi(0)|^2}} \le \|C_\psi\| \le \frac{1+|\psi(0)|}{\sqrt{1-|\psi(0)|^2}}. \end{cases} \tag{23}$$

If $\omega C_\varphi + C_\psi$ is hyponormal for any nonzero $\omega \in \mathbb{C}$, then $\varphi(0) = \psi(0) = 0$ from Corollary 2.2 (i). Thus we see $\|C_\varphi\| = \|C_\psi\| = 1$. Hence we have

$$\|\omega C_\varphi + C_\psi\| \le |\omega|\|C_\varphi\| + \|C_\psi\| = |\omega| + 1$$

and

$$\|\omega C_\varphi + C_\psi\| \ge \Big||\omega|\|C_\varphi\| - \|C_\psi\|\Big| = ||\omega| - 1|.$$

Since every hyponormal operator is normaloid and every normaloid operator is spectraloid from [1], we obtain the result. □

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