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A Note on Scaling Properties of Hewitt Stromberg Measure

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Abstract. In this note, we investigate those Hewitt Stromberg measures which obey to a simple scaling law. Consider a dimension function h and let H^h be the corresponding Hewitt Stromberg measure. We say that H^h obeys an order α scaling law whenever taking $A \subset \mathbb{R}^m$ and $c > 0$, one has

$$\mathsf{H}^h(cA) = c^\alpha \mathsf{H}^h(A).$$

1. Introduction

It is well known that Lebsgue measure satisfies a scaling law, i.e., when magnified by a scalar λ, the length of a curve, the area of a plane region and the volume of a 3-dimensional object are multiplied respectively by λ, λ^2 and λ^3. Let L be a slowly varying function in the sense of [24], i.e., L is a real-valued, positive, measurable function on $[0, \infty)$ and

$$\lim_{r\to 0} \frac{L(rx)}{L(r)} = 1, \qquad \forall x > 0.$$

It's well known that Hausdorff measure of the form $\mathcal{H}^h$, where h is a dimension function, as stated in Definition 2.1, having the form $h(r) = r^\alpha L(r)$, obey a scaling law :

$$\mathcal{H}^h(cA) = c^\alpha \mathcal{H}^h(A), \quad \forall A \subset \mathbb{R}^m. \tag{1}$$

This is a fundamental property in the theory of fractals. In [20], it is proved that, for every continuous increasing concave function h and for every $0 \le \alpha \le 1$, $\mathcal{H}^h(cA) = c^\alpha \mathcal{H}^h(A)$ is satisfied for every $c > 0$ and $A \subset \mathbb{R}$, if and only if

$$\lim_{r\to 0} \frac{h(rc)}{h(r)} = c^\alpha, \qquad \forall c > 0.$$

In [8] the authors prove a general theorem characterizing the height dimensional functions for which the corresponding Hausdorff (or packing) measure scales (see [19] and [10] for further properties of packing measure).

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Hewitt-Stromberg measures were introduced in [16, Exercise (10.51)]. Since then, they have been investigated by several authors, highlighting their importance in the study of local properties of fractals and products of fractals. One can cite, for example [6, 7, 13–15, 17] and recently [1, 11–13, 18, 21] (see also [2–5] for a class of generalization of these measures). In particular, Edgar's textbook [9, pp. 32-36] provides an excellent and systematic introduction to these measures. Such measures appear also appears explicitly, for example, in Pesin's monograph [22, 5.3] and implicitly in Mattila's text [19].

The aim of this paper is to characterize these measures obeying an order α scaling law. First we determine a necessary and sufficing condition so that the Hewitt-Stromberg pre-measure $\overline{\mathsf{H}}^h$ (see (2) for the definition) obeys a scaling law. Then we prove that, if the dimension function h is the form $h(r) = c^\alpha L(r)$, where $c > 0$ and L is slowly varying, then the h-dimensional Hewitt-Stromberg measure H^h (see (3) for the definition) obeys a scaling law. As an application, we study the Hewitt-Stromberg dimension of sets under Lipschitz or similarity transformation and conclude thereby, that if the Hewitt-Stromberg dimension of a set A is strictly less then one, then A is totally disconnected.

2. Preliminary

Definition 2.1. *A function $h : \mathbb{R}_+ \to \mathbb{R}_+$ is called a dimension function if h is increasing, continuous and $h(0) = 0$. We denote by $\mathcal{F}$ the set of dimension functions and $\mathcal{F}_m$ the set of $h \in \mathcal{F}$ such that $h(r)/r^m$ is a decreasing function of r.*

The Hausdorff measure associated with a dimension function h is defined as follows. Let $E \subset \mathbb{R}^m$, $m \geq 1$, and $\varepsilon > 0$, we write

$$\mathcal{H}^h_\varepsilon(E) = \inf\left\{ \sum_i h\Big(\operatorname{diam}(E_i)\Big) \Big| \ E \subseteq \bigcup_i E_i, \ \operatorname{diam}(E_i) < \varepsilon \right\}.$$

Now, we define the $h-$dimensional Hausdorff measure $\mathcal{H}^h(E)$ of E by

$$\mathcal{H}^h(E) = \sup_{\varepsilon>0} \mathcal{H}^h_\varepsilon(E).$$

The reader can be referred to Rogers' classical text [25] for a systematic discussion of $\mathcal{H}^h$.

The packing measure with dimension h is defined, for $\varepsilon > 0$, as follows

$$\overline{\mathcal{P}}^h_\varepsilon(E) = \sup\left\{ \sum_i h\Big(2r_i\Big)^t \right\},$$

where the supremum is taken over all closed balls $\Big(B(x_i, r_i)\Big)_i$ such that $r_i \leq \varepsilon$, $x_i \in E$ and $d(x_i, x_j) > r_i + r_j$ for $i \neq j$. The h-dimensional packing pre-measure $\overline{\mathcal{P}}^h(E)$ of E is now defined by

$$\overline{\mathcal{P}}^h(E) = \sup_{\varepsilon>0} \overline{\mathcal{P}}^h_\varepsilon(E).$$

Now, we define the h-dimensional packing measure $\mathcal{P}^h(E)$ of E as

$$\mathcal{P}^h(E) = \inf\left\{ \sum_i \overline{\mathcal{P}}^h(E_i) \ \Big| \ E \subseteq \bigcup_i E_i \right\}.$$

While Hausdorff and packing measures are defined using coverings and packings by families of sets with diameters less than a given positive number ε, the Hewitt-Stromberg measures are defined using covering of balls with the same diameter ε. The Hewitt-Stromberg pre-measures are defined as follows,

$$\overline{\mathsf{H}}^h(E) = \liminf_{r\to 0} \overline{\mathsf{H}}^h_r(E) \quad \text{where} \quad \overline{\mathsf{H}}^h_r(E) = N_r(E)\, h(2r) \tag{2}$$

where the covering number $N_r(E)$ of E is defined by

$$N_r(E) = \inf\Big\{\sharp\{I\} \quad \Big| \quad \Big(B(x_i,r)\Big)_{i\in I} \text{ is a family of closed balls with } E \subseteq \bigcup_i B(x_i,r)\Big\}$$

Now, we define the (lower) h-dimensional Hewitt-Stromberg measures, which we denote by $\mathsf{H}^h(E)$, as follows

$$\mathsf{H}^h(E) = \inf\left\{\sum_i \overline{\mathsf{H}}^h(E_i) \;\Big|\; E \subseteq \bigcup_i E_i\right\}. \tag{3}$$

We recall the basic inequalities satisfied by the Hewitt-Stromberg, the Hausdorff and the packing measures (see [6, Lemma 1.1])

$$\mathcal{H}^h(E) \le \mathsf{H}^h(E) \le h^*\mathcal{P}^h(E),$$

for any set $E \subset \mathbb{R}^m$ and $h^* := \limsup_{r\to 0} \frac{h(2r)}{h(r)}$.

We end this section by a useful lemma which will be used in the proof of Theorem 3.1.

Lemma 2.2. *Let μ be a probability measure on $\mathbb{R}^m$ with support K. Suppose that there exists a positive and finite number M and, for every $n \in \mathbb{N}$, a covering $\mathcal{A}_n$ of K by closed balls with radius x_n such that $x_n \to 0$, satisfying*

$$h(diam(A)) \ge M\mu(A), \quad \forall A \subset \mathbb{R}^m$$

and

$$\lim_{n\to+\infty} \sum_{\mathcal{A}_n} h(x_n) = M.$$

Then $\overline{\mathsf{H}}^h(K) = M$.

Proof. Let, for $\epsilon > 0$, $\mathcal{B}_\epsilon = \{B(y_i,\epsilon)\}_i$ be an ϵ−cover of K, then

$$\sum_{B\in\mathcal{B}_\epsilon} h(2\epsilon) \ge \sum_{B\in\mathcal{B}_\epsilon} M\mu(B) \ge M\mu(K).$$

Therefore, $N_\epsilon(K)h(2\epsilon) \ge M$ and then

$$\overline{\mathsf{H}}^h(K) \ge M.$$

On the other hand, let $n \in \mathbb{N}$ then

$$N_{x_n/2}(K)h(x_n) \le \sum_{\mathcal{A}_n} h(x_n)$$

and hence $\overline{\mathsf{H}}^h(K) \le M$. □

3. Main results

Our first result is the following.

Theorem 3.1. *Let $h \in \mathcal{F}_m$ and $f : \mathbb{R}_+ \to \mathbb{R}_+$. The following are equivalent*

1. $\overline{\mathsf{H}}^h(cA) \le f(c)\overline{\mathsf{H}}^h(A) \quad \forall c > 0, A \subset \mathbb{R}^m.$
2. $\displaystyle\limsup_{r\to 0} \frac{h(cr)}{h(r)} \le f(c) \quad \forall c > 0.$

For $h \in \mathcal{F}_m$ and $g : \mathbb{R}_+ \to \mathbb{R}_+$. The following are equivalent

3. $\overline{\mathsf{H}}^h(cA) \geq g(c)\overline{\mathsf{H}}^h(A) \quad \forall c > 0, A \subset \mathbb{R}^m.$
4. $\liminf_{r\to 0} \dfrac{h(cr)}{h(r)} \geq g(c) \quad \forall c > 0.$

Remark 3.2. *It's clear that for all functions f and g satisfying*

$$f(x)g(1/x) = 1, \tag{4}$$

we obtain that assumption (1) and (3) are equivalent. Indeed, (1) is equivalent to

$$\overline{\mathsf{H}}^h(A) \geq \frac{1}{f(c)}\overline{\mathsf{H}}^h(cA) \quad \forall c > 0, A \subset \mathbb{R}^m,$$

and by replacing A by $(1/c)A$ and c by $(1/c)$ we obtain (3).
Similarly (2) is equivalent to (4). *Since, for every f, there exists g such that* (4) *holds and for all g there exists f with this property, we only have to prove (2) implies (1) and (3) implies (4).*

Proof. (2) $\Longrightarrow$ (1). Suppose that
$$\limsup_{r\to 0} \frac{h(cr)}{h(r)} \leq f(c).$$
Then, for r small enough, we have
$$h(cr) \leq (f(c) + \widetilde{M}(r))h(r),$$
where $\widetilde{M}(r) \to 0$ as $r \to 0$. In particular we have, for ϵ small enough
$$h(2\epsilon) \leq (f(c) + M(\epsilon))h(2\epsilon/c),$$
where $M(\epsilon) \to 0$ as $\epsilon \to 0$. Let $\mathcal{A}$ be a (ϵ/c)-cover of a set $E \subseteq A$ by closed balls with radius ϵ/c, then $c\mathcal{A}$ is an ϵ-cover of cE by closed balls with radius ϵ. Therefore,
$$N_\epsilon(cE)h(2\epsilon) \leq N_{\epsilon/c}(E)(f(c) + M(\epsilon))h(2\epsilon/c).$$
Thus, letting ϵ tend to 0, we get
$$\overline{\mathsf{H}}^h(cE) \leq \overline{\mathsf{H}}^h(E)f(c).$$
Therefore, if $A \subseteq \bigcup_i E_i$, then
$$\mathsf{H}^h(cA) \leq \sum_i \overline{\mathsf{H}}^h(cE_i) \leq \sum_i \overline{\mathsf{H}}^h(E_i)f(c).$$
Since $\bigcup_i E_i$ is an arbitrarily cover of A we get
$$\mathsf{H}^h(cA) \leq \mathsf{H}^h(A)f(c). \tag{5}$$

(3) $\Longrightarrow$ (4). Let $c > 0$ and choose a strictly decreasing sequence $(z_n)_{n\geq 1}$ converging to 0 for which
$$\lim_{n\to\infty} \frac{h(cz_n)}{h(z_n)} = \liminf_{r\to 0} \frac{h(cr)}{h(r)}. \tag{6}$$
If $h \in \mathcal{F}_m$ we can construct, (see the proof of Theorem 2 in [8]), a set K, a probability measure μ with $\mu(K) = 1$, a finite and positive number M and, for $n \in \mathbb{N}$, a covering $\mathcal{A}_n$ of K by closed balls with diameter x_n such that $x_n \in \{z_1, z_2 \ldots\}$, $x_n \to 0$,
$$h\big(diam(A)\big) \geq M\mu(A), \quad \forall A \subset \mathbb{R}^m$$

and

$$\lim_{n\to 0}\sum_{\mathcal{A}_n} h(x_n) \to M.$$

using Lemma 2.2 we obtain $\overline{\mathsf{H}}^h(K) = M > 0$. Therefore, under our assumption,

$$\overline{\mathsf{H}}^h(cK) \geq g(c)\overline{\mathsf{H}}^h(K) = g(c)M$$

and since $c\mathcal{A}_n$ is a covering of cK, then

$$\begin{aligned} g(c) &\leq \frac{\overline{\mathsf{H}}^h(cK)}{M} \leq \liminf_{n\to+\infty}\frac{\sum_{B\in\mathcal{A}_n} h(c\ diam(B))}{\sum_{B\in\mathcal{A}_n} h(diamB)} \\ &= \liminf_{n\to+\infty}\frac{\sum_{B\in\mathcal{A}_n} h(cx_n)}{\sum_{B\in\mathcal{A}_n} h(x_n)} = \liminf_{n\to\infty}\frac{h(cx_n)}{h(x_n)} \\ &= \liminf_{r\to 0}\frac{h(cr)}{h(r)}, \end{aligned}$$

where we have used (6). □

Let $h \in \mathcal{F}_m$ and $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\lim_{r\to 0}\frac{h(xr)}{h(r)} \leq f(x)$. Then, a similar proof to that of Theorem 3.1 $\Big($ (2) $\Longrightarrow$ (1) $\Big)$ allows us to establish the following :

$$\mathsf{H}^h(S(A)) \leq f(c)\mathsf{H}^h(A), \quad \forall A \subset \mathbb{R}^m, \tag{7}$$

where S is a lipschitz function, i.e.,

$$|S(x) - S(y)| \leq c|x - y|, \qquad \forall x, y \in \mathbb{R}^m,$$

for some $c \geq 0$. In particular, for any set A such that $\mathsf{H}^h(A) = 0$ then $\mathsf{H}^h(S(A)) = 0$. As an immediate consequence of Theorem 3.1 we have the following result.

Corollary 3.3. *Let $h \in \mathcal{F}_m$. Then, the following are equivalent*

1. $\lim_{r\to 0}\frac{h(cr)}{h(r)} = c^\alpha, \quad \forall c > 0.$
2. $\overline{\mathsf{H}}^h(cA) = c^\alpha\overline{\mathsf{H}}^h(A) \quad \forall c > 0, A \subset \mathbb{R}^m.$

Clearly assumption (1) of Corollary 3.3 means that h is of the form

$$h(r) = r^\alpha L(r)$$

where L is slowly varying. These are the types of dimension functions which are so common in dynamics and stochastic processes. It follows from Remark 3.2 and (5), we have the following result.

Corollary 3.4. *Take $h \in \mathcal{F}_m$ and assume that*

$$\lim_{r\to 0}\frac{h(cr)}{h(r)} = c^\alpha, \quad c > 0$$

then

$$\mathsf{H}^h(cA) = c^\alpha\mathsf{H}^h(A) \quad \forall c > 0, A \subset \mathbb{R}^m.$$

Proof. Take $f(x) = x^\alpha$ in Theorem 3.3, we get

$$\overline{\mathsf{H}}^h(cE) \leq c^\alpha \overline{\mathsf{H}}^h(E)$$

for any $E \subset \mathbb{R}^m$ and $c > 0$. Now, let $A \subset \mathbb{R}^m$ and $\{E_i\}$ be any cover of A. Then, for any $c > 0$, w have

$$\mathsf{H}^h(cA) \leq \sum_i \overline{\mathsf{H}}^h(cE_i) \leq c^\alpha \sum_i \overline{\mathsf{H}}^h(E_i).$$

Since $\bigcup_i E_i$ is an arbitrarily cover of A we get

$$\mathsf{H}^h(cA) \leq c^\alpha \mathsf{H}^h(A).$$

Now, by replacing A by $(1/c)A$ and c by $(1/c)$ we obtain the other inequality. □

Let $t \in (0, 1)$ and h_t is the dimension function defined by $h_t(r) = r^t$, then H^h is the usual H^t measure. In this case, one has $\lim\limits_{r\to 0} \dfrac{h_t(rx)}{h_t(r)} = x^t$.

Example 3.5. *Let K be the middle third Cantor set. Then,*

$$\mathsf{H}^t(cK) = c^t, \qquad \forall c > 0,$$

where $t = \dfrac{\log 2}{\log 3}$. *Indeed it's enough, by Corollary 3.4, to prove that* $\mathsf{H}^t(K) = 1$. *We call* E_k *the intervals that make up the sets in the construction of level-k intervals. Thus,* E_k *consists of* 2^k *level−k intervals each of length* 3^{-k}. *Letting* $\delta_k = \frac{1}{2}3^{-k}$ *and taking the intervals of* E_k *as a* δ_k*−cover of K gives that,*

$$\overline{\mathsf{H}}^t_{\delta_k}(K) \leq \sum_{E_k} 3^{-kt} = 1.$$

Thus, $\overline{\mathsf{H}}^t(K) \leq 1$. *On the other hand, we have (see [23])*

$$\mathsf{H}^t(K) \geq \mathcal{H}^t(K) = 1.$$

Example 3.6. *We consider again the dimension function* $h_t(r) = r^\alpha$. *Let* $S : [0, 1] \to \mathbb{R}^2$ *be a Lispschitz continuous function with ratio c and define the graph of S by*

$$G = \Big\{(x, f(x)) : x \in [0, 1]\Big\} \subset \mathbb{R}^2.$$

By Corollary 3.4, we have

$$\mathsf{H}^1(G) \leq c^\alpha \mathsf{H}^1([0, 1]) = c^\alpha.$$

On the other hand the function $f : G \to [0, 1]$ *given by* $f(x, y) = x$ *is the inverse of S. From*

$$|f(x, y) - f(x_1, y_1)| \leq |x - x_1| \leq |(x, y) - (x_1, y_1)|,$$

we see that f is Lipschitz continuous with ratio 1. *Therefore*

$$1 = \mathsf{H}^1([0, 1]) \leq \mathsf{H}^1(G).$$

We conclude that

$$1 \leq \mathsf{H}^1(G) \leq c^\alpha.$$

In particular, if $c = 1$, *then* $\mathsf{H}^1(G) = 1$.

In Fractal geometry, it is very interesting to consider a geometric transformations, such as similarity transformation S, that is,

$$|S(x) - S(y)| = c|x - y|, \qquad \forall x, y \in \mathbb{R}^m,$$

with the ratio $c > 0$. Indeed, We can construct a self-similar fractal by transforming a geometric figure using a combination of similarity transformations, see for example the construction of Cantor set. Hence, using a similarity transformation S gives us a new way to create many new self-similar fractal designs. Therefore, it is interesting to compare the size of a set A by $S(A)$.

Corollary 3.7. *Let $h \in \mathcal{F}_m$ such that $\lim_{r \to 0} \frac{h(xr)}{h(r)} = x^\alpha$, for all $x > 0$. Then,*

$$\mathsf{H}^h(S(A)) = c^\alpha \mathsf{H}^h(A),$$

where $S : \mathbb{R}^m \to \mathbb{R}^m$ is a similarity of ration or scale $c > 0$.

In the following we give a new characterization of the dimension functions h for which the associated Hewitt-Stromberg measure obeys a scaling law.

Theorem 3.8. *Let $h \in \mathcal{F}_1$ and suppose that $\lim_{r \to 0} \frac{h(rx)}{h(r)}$ exists and is positive for all x in a set of positive Lebesgue measure, then there exists $\alpha > 0$, such that*

$$\mathsf{H}^h(cA) = c^\alpha \mathsf{H}^h(A), \qquad \forall A \subset \mathbb{R}.$$

Proof. Clearly, by Corollary 3.7, it's enough to prove that

$$\lim_{r \to 0} \frac{h(rx)}{h(r)} = x^\alpha, \qquad \forall x > 0, \tag{8}$$

for some $\alpha \in \mathbb{R}$. Note that, under the hypothesis of Theorem 3.8, there exists a set K such that

$$\forall x \in K, \qquad \lim_{r \to 0} \frac{h(re^x)}{h(r)},$$

exists and is positive. Thus, one can define the function $\phi : K \to \mathbb{R}$ by

$$\phi(x) = \lim_{r \to 0} \Big\{ \ln h(re^x) - \ln h(r) \Big\}.$$

Since K is an additive subgroup of $\mathbb{R}$, we have that $K = \mathbb{R}$ and

$$\phi(x + y) = \phi(x) + \phi(y).$$

Finally we have, by continuity of ϕ, that

$$\phi(x) = \phi(1)x, \qquad \forall x \in \mathbb{R}.$$

Now (8) follows. □

Remark 3.9. *Let us mention that if we replace, in the preview theorem, H^h by $\mathcal{H}^h$, then our result gives a new characterization for Hausdorff measure.*

4. Application

In this section we consider, for $t > 0$, the dimension function $h_t(r) = r^t$ so that we can define the Hewitt-Stromberg dimension by

$$\dim_{MB}(E) = \sup\left\{t \geq 0 \;\middle|\; \mathsf{H}^t(E) = +\infty\right\} = \inf\left\{t \geq 0 \;\middle|\; \mathsf{H}^t(E) = 0\right\}.$$

We prove that Lipschitz transformation does not increase Hewitt-Stromberg dimension. In addition it is preserved by any similarity or bi-Lipschitz transformation. The following result is a direct consequence of (7), with $f(x) = x^\alpha$, $\alpha > 0$.

Theorem 4.1. *If $S : \mathbb{R}^m \to \mathbb{R}^m$ is a Lipschitz transformation, then*

$$\dim_{MB}(S(A)) \leq \dim_{MB}(A).$$

In addition, if $S : \mathbb{R}^m \to \mathbb{R}^m$ is a bi-Lipschitz transformation, i.e.

$$c_1|x - y| \leq |S(x) - S(y)| \leq c_2|x - y| \quad \forall x, y \in \mathbb{R}^m,$$

for $0 < c_1 \leq c_2 < \infty$, then

$$\dim_{MB}(S(A)) = \dim_{MB}(A).$$

Indeed, applying The previous Theorem to $S^{-1} : S(A) \to A$ gives the other inequality. In particular, if S is a similarity transformation, then

$$\dim_{MB}(S(A)) = \dim_{MB}(A).$$

Remark 4.2. *Hewitt Stromberg dimension is invariant under bi-Lipschitz transformations. Thus, if two sets have different dimensions, there cannot be a bi-Lipschitz mapping from one onto the other.*

Corollary 4.3. *Let $A \subset \mathbb{R}^m$ be such that $\dim_{MB}(A) < 1$. Then A is totally disconnected.*

Proof. Let x and y be distinct points of A. Define a mapping $S : \mathbb{R}^m \to [0, +\infty)$ by

$$S(z) = |z - x|.$$

Since $|S(z) - S(w)| \leq |z - w|$, then we have

$$\dim_{MB}(S(A)) \leq \dim_{MB}(A) < 1.$$

Thus, $S(A)$ is a subset of $\mathbb{R}$ of H^1 measure or length zero, and so it has a dense complement. Choosing $r \notin S(A)$ with $0 < r < S(y)$ gives that

$$A = \left\{z \in A : \; S(z) < r\right\} \cup \left\{z \in A : \; S(z) > r\right\}.$$

Therefore, A is contained in two disjoint open sets with x in one set and y in the other one, so that x and y lie in different connected components of A. □

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