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Symmetric Operator Amenability of Operator Algebras

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Abstract. Using the notion of a symmetric virtual diagonal for a Banach algebra, we prove that a Banach algebra is symmetrically amenable if its second dual is symmetrically amenable. We introduce symmetric operator amenability in the category of completely contractive Banach algebras as an operator algebra analogue of symmetric amenability of Banach algebras. We give some equivalent formulations of symmetric operator amenability of completely contractive Banach algebras and investigate some hereditary properties of symmetric operator amenable algebras. We show that amenability of locally compact groups is equivalent to symmetric operator amenability of its Fourier algebra. Finally, we discuss about Jordan derivation on symmetrically operator amenable algebras.

1. Introduction

A Banach algebra $\mathcal{B}$ is called *amenable* if it has a *bounded approximate diagonal*, that is, there is a bounded net $\{\Delta_\alpha\}_\alpha$ in the algebraic tensor product $\mathcal{B}\otimes\mathcal{B}$ such that

$$\|\Delta_\alpha \cdot b - b\cdot\Delta_\alpha\| + \|m(\Delta_\alpha)b - b\| \to 0 \quad (b\in\mathcal{B})$$

where $m : \mathcal{B}\otimes\mathcal{B}\to\mathcal{B}$ is a natural product. Johnson initiated the theory of amenable Banach algebras and proved that a locally compact group G is amenable if and only if $L^1(G)$ is amenable as a Banach algebra. There are many alternative equivalent formulations of the notion of amenability, which many people have studied. For further details, we refer [11]. If a locally compact group G with its dual $\widehat{G}$ is abelian, the Fourier algebra $A(G)$ is isomorphic to $L^1(\widehat{G})$. Since the amenability of a locally compact group is that of $L^1(G)$ as a Banach algebra, we may naturally ask if the amenability of G is equivalent to that of $A(G)$. Johnson [9] gave the answer in the negative. Using this operator space structure, Ruan [12] introduced a weaker notion of amenability for completely contractive Banach algebras, which is called *operator amenability*. Since $A(G)$ has a natural operator space structure, he showed that the amenability of G is equivalent to the operator amenability of $A(G)$.

Johnson [10] introduced the notion of symmetric amenability of Banach algebras for understanding of the behavior of Jordan derivations on Banach algebras. A Banach algebra $\mathcal{B}$ is called *symmetrically amenable*

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if it has a bounded approximate diagonal consisting of symmetric tensors. Johnson [10] proved that for every amenable locally compact group G, $L^1(G)$ is symmetrically amenable. However, in general, the set of all symmetrically amenable Banach algebras is a proper subset of the set of all amenable Banach algebras since unital amenable Banach algebras without tracial states are not symmetrically amenable [10]. The purpose of this paper is to introduce the notion of symmetric operator amenability in the category of operator algebras as an operator analogue of symmetric amenability.

The contents of the paper are as follows. In section 2 we review the symmetric amenability in the category of Banach algebras and show that symmetric amenability of Banach algebras is implied by symmetric amenability of their second duals using a symmetric virtual diagonal. In section 3 we introduce the notion of symmetric operator amenability in the category of completely contractive Banach algebras and prove some equivalent formulations of symmetric operator amenability. In section 4 we investigate some hereditary properties of symmetric operator amenable algebras. In section 5 we prove that the amenability of locally compact groups is equivalent to the symmetric operator amenability of their Fourier algebras and introduce a symmetric operator diagonal in a completely contractive Banach algebra. Finally, we discuss about Jordan derivations on completely contractive Banach algebras which are symmetrically operator amenable.

2. symmetric virtual Diagonals on Banach Algebras

In this section, $\mathcal{B}$ and $\mathcal{B}^{**}$ denote a Banach algebra and its second dual equipped with the first Arens product, respectively, unless specified otherwise. This product can be characterized as the extension to $\mathcal{B}^{**} \times \mathcal{B}^{**}$ of the multiplication map m on $\mathcal{B}$ with the following continuity properties:

(i) for fixed $y \in \mathcal{B}^{**}$, the map $x \mapsto xy$ is weak*-continuous on $\mathcal{B}^{**}$,

(ii) for fixed $y \in \mathcal{B}$, the map $x \mapsto yx$ is weak*-continuous on $\mathcal{B}^{**}$.

We will identify $\mathcal{B}$ with its canonical image in $\mathcal{B}^{**}$. For basic definitions and properties, the reader is referred to [2].

We denote by $\mathcal{B} \otimes^{\gamma} \mathcal{B}$ the Banach space projective tensor product of $\mathcal{B}$ with itself. An element x of $\mathcal{B} \otimes^{\gamma} \mathcal{B}$ is called *symmetric* if $x^{\bullet} = x$ where the flip map on $\mathcal{B} \otimes^{\gamma} \mathcal{B}$ is defined by $(a \otimes b)^{\bullet} = b \otimes a$. The opposite algebra $\mathcal{B}^{\bullet}$ is the Banach space $\mathcal{B}$ with product $a \bullet b = ba$. We can easily see that $\{\Delta_\alpha\}_\alpha$ is an approximate diagonal for $\mathcal{B}$ if and only if $\{\Delta_\alpha^{\bullet}\}_\alpha$ is an approximate diagonal for $\mathcal{B}^{\bullet}$, so that $\mathcal{B}$ is amenable if and only if $\mathcal{B}^{\bullet}$ is amenable. We see that any symmetric element $x = \sum a_i \otimes b_i$ can be written as a linear combination of symmetric elements

$$x = \frac{1}{4} \sum \Big[(a_i + b_i) \otimes (a_i + b_i) - (a_i - b_i) \otimes (a_i - b_i) \Big]. \tag{1}$$

Definition 2.1. *A symmetric virtual diagonal for $\mathcal{B}$ is an element M in $(\mathcal{B} \otimes^{\gamma}_{sym} \mathcal{B})^{**}$ such that for all $a \in \mathcal{B}$,*

$$a \cdot M = M \cdot a \quad \textit{and} \quad a \cdot m^{**}(M) = a$$

where $\mathcal{B} \otimes^{\gamma}_{sym} \mathcal{B}$ is the Banach space of all symmetric elements in $\mathcal{B} \otimes^{\gamma} \mathcal{B}$.

If M is a symmetric virtual diagonal for $\mathcal{B}$, then we have that

$$a \bullet M = M \bullet a, \quad a \cdot (m^{\bullet})^{**}(M) = a \quad \text{and} \quad m^{**}(M) = (m^{\bullet})^{**}(M). \tag{2}$$

where $a \bullet (b \otimes c) = b \otimes ac$, $(b \otimes c) \bullet a = ba \otimes c$ and $m^{\bullet}(a \otimes b) = ba$. The following proposition is the symmetric analogue of [8, Lemma 1.2].

Proposition 2.2. *A Banach algebra is symmetrically amenable if and only if it has a symmetric virtual diagonal.*

Proof. If a Banach algebra $\mathcal{B}$ is symmetrically amenable, then there is a bounded approximate diagonal $\{x_\alpha\}$ in $\mathcal{B}\otimes^\gamma_{\rm sym}\mathcal{B}$. Since $\{x_\alpha\}$ is bounded, it has a weak*-cluster point M in $(\mathcal{B}\otimes^\gamma_{\rm sym}\mathcal{B})^{**}$. It is easy to see that M is a symmetric virtual diagonal.

Conversely, suppose that $\mathcal{B}$ has a symmetric virtual diagonal M. There is a bounded net $\{x_\alpha\}$ in $\mathcal{B}\otimes^\gamma_{\rm sym}\mathcal{B}$ such that x_α converges to M in the weak*-topology. The remaining of proof is the same as that in [8, Theorem 1.3]. □

Lemma 2.3. *([5, Lemma 1.7]) There is a continuous linear map $\Psi : \mathcal{B}^{**}\otimes^\gamma\mathcal{B}^{**}\to(\mathcal{B}\otimes^\gamma\mathcal{B})^{**}$ such that for $a, b, c\in\mathcal{B}$ and $x\in\mathcal{B}^{**}\otimes^\gamma\mathcal{B}^{**}$, the following holds;*

(i) $\Psi(a\otimes b) = a\otimes b$,

(ii) $\Psi(x)\cdot c = \Psi(x\cdot c)$ *and* $c\cdot\Psi(x) = \Psi(c\cdot x)$,

(iii) $m^{**}(\Psi(x)) = \overline{m}(x)$ *where* $\overline{m} : \mathcal{B}^{**}\otimes^\gamma\mathcal{B}^{**}\to\mathcal{B}^{**}$ *is a natural product.*

Recall that there is an isometric isomorphism between the space of bounded bilinear functionals on $\mathcal{B}\times\mathcal{B}$ and the space of continuous linear functionals on $\mathcal{B}\otimes^\gamma\mathcal{B}$, that is, $T\mapsto\varphi_T$ where $\varphi_T(a\otimes b) = T(a, b)$. Let $T : \mathcal{B}\times\mathcal{B}\to\mathbb{C}$ be a continuous bilinear form. If $\overline{T} : \mathcal{B}^{**}\times\mathcal{B}^{**}\to\mathbb{C}$ is its continuous extension, then $\overline{T}(A, B) = \lim_{\alpha,\beta} T(a_\alpha, b_\beta)$ for any $A, B\in\mathcal{B}^{**}$ where $\{a_\alpha\}$ and $\{b_\beta\}$ are bounded nets in $\mathcal{B}$ with $a_\alpha\to A$ and $b_\beta\to B$ in the weak*-sense (see [1] for the details).

Let Ψ be the continuous linear map in Lemma 2.3 and let $X = \sum A_i\otimes B_i$ be a symmetric tensor in $\mathcal{B}^{**}\otimes^\gamma\mathcal{B}^{**}$. We claim that $\Psi(X)\in(\mathcal{B}\otimes^\gamma_{\rm sym}\mathcal{B})^{**}$. To show this, it suffices to consider elements of form $A\otimes A$ with $A\in\mathcal{B}^{**}$. Indeed, let A be any element in $\mathcal{B}^{**}$ and take a bounded net (a_α) such that $a_\alpha\to A$ in the weak*-topology. For any $T\in(\mathcal{B}\times\mathcal{B})^*$, we have that

$$\left(\Psi(A\otimes A), \varphi_T\right) = \overline{T}(A, A) = \lim_\alpha\varphi_T(a_\alpha\otimes a_\alpha), \tag{3}$$

which implies that $\Psi(\mathcal{B}^{**}\otimes^\gamma_{\rm sym}\mathcal{B}^{**})\subset(\mathcal{B}\otimes^\gamma_{\rm sym}\mathcal{B})^{**}$.

Theorem 2.4. *If $\mathcal{B}$ is a Banach algebra whose the second dual $\mathcal{B}^{**}$ is symmetrically amenable, then $\mathcal{B}$ is symmetrically amenable.*

Proof. Suppose that $\mathcal{B}^{**}$ is symmetrically amenable. Then there is a bounded approximate diagonal $\{X_\alpha\}$ for $\mathcal{B}^{**}$ consisting of symmetric tensors. By Lemma 2.3, we have that

$$b\cdot\Psi(X_\alpha) - \Psi(X_\alpha)\cdot b\to 0 \quad\text{and}\quad \overline{m}(\Psi(X_\alpha)b)\to b$$

for all $b\in\mathcal{B}$. Moreover, it follows from the equation (3) that $\Psi(X_\alpha)\in(\mathcal{B}\otimes^\gamma_{\rm sym}\mathcal{B})^{**}$. If M is a weak*-cluster point of $\{\Psi(X_\alpha)\}$ in $(\mathcal{B}\otimes^\gamma_{\rm sym}\mathcal{B})^{**}$, then we have that $b\cdot M = M\cdot b$ and $m^{**}(M)b = b$ for all $b\in\mathcal{B}$. This implies that M is a symmetric virtual diagonal for $\mathcal{B}$. By Proposition 2.2, $\mathcal{B}$ is symmetrically amenable. □

Corollary 2.5. *([5, Corollary 1.9]) Let G be a discrete left (or right) cancellative semigroup with an identity. Then $L^1(G)^{**}$ is symmetrically amenable if and only if G is a finite group.*

Proof. If $L^1(G)^{**}$ is symmetrically amenable, then $L^1(G)^{**}$ is also amenable. By [5, Corollary 1.9], G is a finite group. Conversely, suppose that G is a finite group. It follows from [10, Theorem 4.1] that $L^1(G)$ is symmetrically amenable, so that $L^1(G)^{**}$ is symmetrically amenable. □

3. Symmetric Operator Amenability of Operator Algebras

In this section, we introduce a notion of symmetric operator amenability in the category of completely contractive Banach algebras, which is the operator analogue of symmetric amenability of Banach algebras. We first recall that operator spaces are (norm closed) subspaces of $\mathcal{B}(\mathcal{H})$ together with the operator matrix norms obtained from $\mathcal{B}(\mathcal{H})$. An associative algebra $\mathcal{A}$ is a *completely contractive Banach algebra* if it is an operator space and the multiplication map $m : \mathcal{A}\widehat{\otimes}\mathcal{A} \to \mathcal{A}$ is completely contractive where $\widehat{\otimes}$ means the operator projective tensor product. An $\mathcal{A}$-bimodule V is called an *operator $\mathcal{A}$-bimodule* if V is an operator space and the $\mathcal{A}$-bimodule operations $\mathcal{A}\widehat{\otimes}V \to V$, $a \otimes v \mapsto a \cdot v$ and $V\widehat{\otimes}\mathcal{A} \to V$, $v \otimes a \mapsto v \cdot a$ are completely bounded. For an operator $\mathcal{A}$-bimodule V with the operator dual V^*, there is a natural operator $\mathcal{A}$-bimodule structure on V^*, namely, if $\phi \in V^*$ and $a \in \mathcal{A}$, we define $a \cdot \phi$ and $\phi \cdot a$ by

$$(a \cdot \phi)(v) = \phi(va) \quad \text{and} \quad (\phi \cdot a)(v) = \phi(av), \qquad (v \in V). \tag{4}$$

See [12] for more details. We call $\mathcal{A}$ *operator amenable* if for any operator $\mathcal{A}$-bimodule V, every completely bounded derivation from $\mathcal{A}$ into the dual $\mathcal{A}$-bimodule V^* is inner.

Since every completely contractive Banach algebra is a Banach algebra, we see that if a completely contractive Banach algebra $\mathcal{A}$ is amenable as a Banach algebra, then $\mathcal{A}$ must be operator amenable. However, in general, the converse is not true [9].

In the remaining of this paper, we denote by $\mathcal{A}$ and $\mathcal{A}\widehat{\otimes}\mathcal{A}$ a completely contractive Banach algebra and the operator projective tensor product, respectively, unless specified otherwise. The following theorem gives an intrinsic characterization for operator amenability in terms of approximate diagonals.

Theorem 3.1. *([12, Proposition 2.4]) Let $\mathcal{A}$ be a completely contractive Banach algebra. The followings are equivalent:*

(i) $\mathcal{A}$ is operator amenable.

(ii) $\mathcal{A}$ has a operator virtual diagonal, that is, there exists $U \in (\mathcal{A}\widehat{\otimes}\mathcal{A})^{**}$ such that $a \cdot U = U \cdot a$ and $m^{**}(U)a = a$ for every $a \in \mathcal{A}$.

(iii) $\mathcal{A}$ has a bounded approximate diagonal, that is, there is a net $\{x_\alpha\}_{\alpha\in I}$ of bounded elements in $\mathcal{A}\widehat{\otimes}\mathcal{A}$ such that for every $a \in \mathcal{A}$

(a) $\|a \cdot x_\alpha - x_\alpha \cdot a\| \to 0$,

(b) $\|m(x_\alpha) \cdot a - a\| \to 0$.

On $\mathcal{A}\widehat{\otimes}\mathcal{A}$, the *flip map* is defined by $(a \otimes b)^\bullet = b \otimes a$. An element $x \in \mathcal{A}\widehat{\otimes}\mathcal{A}$ is called *symmetric* if $x^\bullet = x$. Now we can define a weaker notion of symmetric amenability and operator amenability in the category of completely contractive Banach algebras.

Definition 3.2. *A completely contractive Banach algebra $\mathcal{A}$ is symmetrically operator amenable if it has a bounded approximate diagonal consisting of symmetric elements in $\mathcal{A}\widehat{\otimes}\mathcal{A}$.*

We also see that if $\mathcal{A}$ is symmetrically amenable as a Banach algebra, then it is symmetrically operator amenable. If $\mathcal{A}^\circ$ is the *opposite algebra* of $\mathcal{A}$, we can rewrite (a) and (b) in the condition (iii) of Theorem 3.1 for $\mathcal{A}^\circ$ as follows; there is a bounded net $\{x_\alpha\}$ in $\mathcal{A}\widehat{\otimes}\mathcal{A}$ such that

(a′) $\|a \bullet x_\alpha - x_\alpha \bullet a\| \to 0$,

(b′) $\|a \cdot m^\bullet(x_\alpha) - a\| \to 0$

where $a \bullet (b \otimes c) = b \otimes ac$, $(b \otimes c) \bullet a = ba \otimes c$ and $m^\bullet(b \otimes c) = cb$. Such a net $\{x_\alpha\}$ is called a bounded approximate diagonal for $\mathcal{A}^\circ$. Then $\{x_\alpha\}$ is a bounded approximate diagonal for $\mathcal{A}$ if and only if $\{x_\alpha^\bullet\}$ is a bounded approximate diagonal for $\mathcal{A}^\circ$. Indeed, we have that

$$\|a \bullet x_\alpha^\bullet - x_\alpha^\bullet \bullet a\| = \|a \cdot x_\alpha - x_\alpha \cdot a\|, \quad \|a \bullet m^\bullet(x_\alpha^\bullet) - a\| = \|m(x_\alpha) \cdot a - a\|$$

since $a \bullet x^\bullet = (a \cdot x)^\bullet$ and $x^\bullet \bullet a = (x \cdot a)^\bullet$ for any $a \in \mathcal{A}$ and $x \in \mathcal{A}\widehat{\otimes}\mathcal{A}$. Hence the operator amenability of $\mathcal{A}$ is equivalent to that of $\mathcal{A}^\circ$.

Example 3.3. *Let $M_n(\mathbb{C})$ be the (completely contractive) Banach algebra of all $n \times n$ matrices over $\mathbb{C}$ and $\mathcal{U} = \{e_{ij} : i, j = 1, \dots, n\}$ be the canonical matrix units of $M_n(\mathbb{C})$. Let $x_\alpha := \frac{1}{n}\sum_{i,j=1}^{n} e_{ij} \otimes e_{ji}$ for any α. For any $e_{kl} \in \mathcal{U}$, we have that*

$$e_{kl} \cdot x_\alpha = \frac{1}{n}\sum_{i,j=1}^{n} e_{kl}e_{ij} \otimes e_{ji} = \frac{1}{n}\sum_{j=1}^{n} e_{kj} \otimes e_{jl} = \frac{1}{n}\sum_{i,j=1}^{n} e_{ij} \otimes e_{ji}e_{kl} = x_\alpha \cdot e_{kl}$$

and that $m(x_\alpha) = I_n$ is the identity matrix in $M_n(\mathbb{C})$, so that $m(x_\alpha) \cdot a = a$ for all $a \in M_n(\mathbb{C})$. Since $x_\alpha^\bullet = x_\alpha$ for any α, $M_n(\mathbb{C})$ is symmetrically operator amenable. □

Proposition 3.4. *A necessary and sufficient condition for $\mathcal{A}$ to be symmetrically operator amenable is that there exists a bounded net $\{x_\alpha\}$ in $\mathcal{A}\widehat{\otimes}\mathcal{A}$ which satisfies the above properties* (a), (b), (a′) *and* (b′).

Proof. If $\mathcal{A}$ is symmetrically operator amenable, then there exists a net $\{x_\alpha\}$ such that $x_\alpha = x_\alpha^\bullet$ and $\{x_\alpha\}$ satisfies properties (a) and (b) in Theorem 3.1. Hence we have that

$$\begin{aligned} \|a \bullet x_\alpha - x_\alpha \bullet a\| &= \|(a \cdot x_\alpha - x_\alpha \cdot a)^\bullet\| \to 0 \\ \|a \cdot m^\bullet(x_\alpha) - a\| &= \|a \cdot m(x_\alpha) - a\| \to 0. \end{aligned}$$

Conversely, suppose that $\{x_\alpha\}$ satisfies properties (a), (b), (a′) and (b′). Then we easily see that $\{x_\alpha^\bullet\}$ also satisfies (a), (b), (a′) and (b′) since we have the equations

$$\begin{aligned} \|a \cdot x_\alpha^\bullet - x_\alpha^\bullet \cdot a\| &= \|(a \bullet x_\alpha - x_\alpha \bullet a)^\bullet\|, \quad \|m(x_\alpha^\bullet) \cdot a - a\| = \|a \bullet m^\bullet(x_\alpha) - a\|, \\ \|a \bullet x_\alpha^\bullet - x_\alpha^\bullet \bullet a\| &= \|(a \cdot x_\alpha - x_\alpha \cdot a)^\bullet\|, \quad \|a \cdot m^\bullet(x_\alpha^\bullet) - a\| = \|a \cdot m(x_\alpha) - a\|. \end{aligned}$$

Considering the net $\{\frac{1}{2}(x_\alpha + x_\alpha^\bullet)\}$ of symmetric elements, the net satisfies (a) and (b) in Theorem 3.1, which becomes a bounded approximate identity. □

Corollary 3.5. *If $\mathcal{A}$ is commutative and operator amenable, then it is symmetrically operator amenable.*

We note that two Arens products on the second dual $\mathcal{A}^{**}$ coincide and the resulting algebra is completely isometric to a σ-weakly closed operator algebra. Thus, the second dual $\mathcal{A}^{**}$ is again a completely contractive Banach algebra. A *symmetric operator virtual diagonal* for $\mathcal{A}$ is an element U in $(\mathcal{A}\widehat{\otimes}_{\mathrm{sym}}\mathcal{A})^{**}$ such that $a \cdot U = U \cdot a$ and $m^{**}(U)a = a$ for all $a \in \mathcal{A}$ where $\mathcal{A}\widehat{\otimes}_{\mathrm{sym}}\mathcal{A}$ is the space of all symmetric elements in $\mathcal{A}\widehat{\otimes}\mathcal{A}$. Since a symmetric operator virtual diagonal has the same relation as the equation (2), we also have the same result as Proposition 2.2.

Proposition 3.6. *A necessary and sufficient condition for $\mathcal{A}$ to be symmetrically operator amenable is that it has a symmetric operator virtual diagonal in $(\mathcal{A}\widehat{\otimes}_{sym}\mathcal{A})^{**}$.*

Proof. The proof is the same as that of Proposition 2.2, so that we omit it. □

Proposition 3.7. *Suppose that $\mathcal{A}\widehat{\otimes}\mathcal{A}$ has a bounded net $\{x_\alpha\}$ which satisfies properties* (a) *and* (b′). *If z is a non-zero element in the center of $\mathcal{A}$, then there exists a functional $f \in \mathcal{A}^*$ such that $f(ab) = f(ba)$ and $f(z) = 1$.*

Proof. The proof is similar to that of [10, Proposition 2.4]. □

Remark 3.8. *If a unital algebra $\mathcal{A}$ satisfies the hypotheses in Proposition 3.7, then there is $f \in \mathcal{A}^*$ with $f(ab) = f(ba)$ and $f(1) = 1$, which implies that Cuntz algebra $\mathcal{O}_n$ $(n \geq 2)$ is not symmetrically operator amenable since $\mathcal{O}_n$ has no normalized trace. But it is well-known that $\mathcal{O}_n$ is amenable.*

Corollary 3.9. *(cf. [10, Proposition 2.6]) If there is a bounded net $\{x_\alpha\}$ in $\mathcal{A}\widehat{\otimes}\mathcal{A}$ which satisfies properties* (a), (b) *in Theorem 3.1 and* (b′), *then $\mathcal{A}$ is symmetrically operator amenable.*

Most, but not all, completely contractive Banach algebras which are operator amenable are symmetrically operator amenable. In the following theorem, we see that symmetric operator amenability has hereditary properties similar to operator amenability [12]. The proof is similar to that of [10, Theorem 3.1], but we omit the proof.

Theorem 3.10. *Let J be a closed two sided ideal in $\mathcal{A}$.*

(i) If $\mathcal{A}/J$ and J are symmetrically operator amenable, then $\mathcal{A}$ is symmetrically operator amenable.

(ii) If $\mathcal{A}$ is symmetrically operator amenable, then $\mathcal{A}/J$ is symmetrically operator amenable.

(iii) If $\mathcal{A}$ is symmetrically operator amenable and J has a bounded approximate identity, then J is symmetrically operator amenable.

Proposition 3.11. *Let $\mathcal{B}$ be completely contractive Banach algebras.*

(i) If $\mathcal{A}$ is symmetrically operator amenable and if $\phi : \mathcal{A} \to \mathcal{B}$ is a continuous homomorphism with dense range, then $\mathcal{B}$ is also symmetrically operator amenable.

(ii) If $\mathcal{B}$ contains an element b with $b \notin \overline{\{bb' - b'b : b' \in \mathcal{B}\}}$ and if $\mathcal{A}\widehat{\otimes}\mathcal{B}$ is operator amenable, then $\mathcal{A}$ is operator amenable.

(iii) If $\mathcal{A}\widehat{\otimes}\mathcal{B}$ is symmetrically operator amenable and if there are $b_0 \in \mathcal{Z}(\mathcal{B})$ and $f \in \mathcal{B}^*$ such that $f(b_0) = 1$ and $f(bb') = f(b'b)$ for all $b, b' \in \mathcal{B}$, then $\mathcal{A}$ is symmetrically operator amenable.

As a corollary, we see that if $\mathcal{A}$ and $\mathcal{B}$ are symmetrically operator amenable, then $\mathcal{A}\widehat{\otimes}\mathcal{B}$ is also symmetrically operator amenable. Indeed, let $\{x_\lambda\}_{\lambda\in\Lambda}$ and $\{x_\mu\}_{\mu\in\Upsilon}$ be symmetrically bounded approximate diagonals for $\mathcal{A}$ and $\mathcal{B}$, respectively. Then the family $\{(I_{\mathcal{A}} \otimes \sigma \otimes I_{\mathcal{B}})(x_\lambda \otimes x_\mu)\}$ is a symmetrically bounded approximate diagonal for $\mathcal{A}\widehat{\otimes}\mathcal{B}$ where σ is a flip map from $\mathcal{A}\widehat{\otimes}\mathcal{B} \to \mathcal{B}\widehat{\otimes}\mathcal{A}$.

4. Fourier Algebras and Symmetric Operator Diagonals

Let G be a locally compact group with a left Haar measure μ and let Δ be the modular function of G into $\mathbb{R}^+$. For any $s, t \in G$, we have

$$d\mu(st) = d\mu(t), \;\; d\mu(ts) = \Delta(s)d\mu(t) \;\text{ and }\; d\mu(t^{-1}) = \Delta(t)^{-1}d\mu(t).$$

Let $L : G \to B(L^2(G))$ be a left regular representation defined by $[L(s)\xi](t) = \xi(st)$ for any $\xi \in L^2(G)$. The *group von Neumann algebra* $L(G)$ is the weak operator closure of the set $\{L(s) : s \in G\}$ in $B(L^2(G))$. Then $L(G)$ becomes a Hopf von Neumann algebra with the coassociative comultiplication $\Gamma : L(G) \to L(G)\bar{\otimes}L(G)$ given by

$$\Gamma(L(s)) = L(s) \otimes L(s) = W^*(L(s) \otimes 1)W$$

where $W\psi(s,t) = \psi(s, st)$ is a distinguished unitary operator on $L^2(G \times G)$. Then we have that $(\Gamma \otimes \mathrm{id})\Gamma = (\mathrm{id} \otimes \Gamma)\Gamma$.

The Fourier algebra $A(G)$ consists of all coefficient functions of the left regular representation L of G, that is, $A(G) = \left\{\omega(\cdot) = \langle L(\cdot)\xi, \eta\rangle : \xi, \eta \in L^2(G)\right\}$. The norm of $\omega \in A(G)$ is defined by $\|\omega\|_{A(G)} = \inf\{\|\xi\|\|\eta\|\}$ where the infimum is taken over all possible representations $\omega(\cdot) = \langle L(\cdot)\xi, \eta\rangle$. It was proved by Eymard [4] that $A(G)$, up to isomorphism, is the predual of the group von Neumann algebra $L(G)$, so that $A(G) = L(G)_*$ has a natural operator matrix norm and $A(G)\widehat{\otimes}A(G) \simeq A(G \times G) = L(G \times G)_*$. The multiplication $m = \Gamma_* : A(G)\widehat{\otimes}A(G) \to A(G)$ is completely contractive, so that $A(G)$ is a completely contractive Banach algebra which is commutative.

Theorem 4.1. *Let G be a locally compact group. Then $A(G)$ is symmetrically operator amenable if and only if G is amenable.*

Proof. If $A(G)$ is symmetrically operator amenable, then $A(G)$ is also operator amenable, so that by Ruan's result [12] G is amenable. Conversely, suppose that G is amenable. Then $A(G)$ is operator amenable by Ruan's result [12]. It follows from Corollary 3.5 that $A(G)$ is symmetrically operator amenable. □

An element $u \in \mathcal{A} \otimes \mathcal{A}$ is a *symmetric operator diagonal* for $\mathcal{A}$ if $a \bullet u = u \bullet a$ and $am^{\bullet}(u) = a$ for all $a \in \mathcal{A}$ where $\mathcal{A} \otimes \mathcal{A}$ is the algebraic tensor product.

Example 4.2. *Let $\mathcal{A} = M_n$ be the set of all $n \times n$ matrices over $\mathbb{C}$ and let $\{e_{ij}\}$ be the set of canonical matrix units for $\mathcal{A}$. Then the element $u = \frac{1}{n}\sum_{i,j} e_{ij} \otimes e_{ji}$ is a symmetric operator diagonal for $\mathcal{A}$. Indeed, we have that*

$$m^{\bullet}(u) = \frac{i}{n}\sum_{i,j} e_{ji}e_{ij} = \frac{i}{n}\sum_{i,j} e_{jj} = \sum_{j} e_{jj} = I.$$

For any $1 \le k, l \le n$, we also have that

$$e_{kl} \bullet u = \frac{1}{n}\sum_{i,j} e_{il} \otimes e_{ki}, \quad u \bullet e_{kl} = \frac{1}{n}\sum_{i,j} e_{il} \otimes e_{ki},$$

so that $e_{kl} \bullet u = u \bullet e_{kl}$. Hence $a \bullet u = u \bullet a$ for any $a \in \mathcal{A}$. □

If $\sum_i a_i \otimes b_i \in \mathcal{A} \otimes \mathcal{A}$ is a symmetric operator diagonal for $\mathcal{A}$, then we see that $\sum_i a_i \otimes cb_i = \sum_i a_i c \otimes b_i$ for all $c \in \mathcal{A}$. We consider a bilinear mapping $F : \mathcal{A} \otimes \mathcal{A} \to V$ satisfying

$$\sum_i F(a_i, cb_i) = \sum_i F(a_i c, b_i) \quad \text{for all } c \in \mathcal{A}$$

where V is a vector space. The following theorem is the symmetric version of [3, Theorem 16.1.3], whose proof follows from Example 4.2 and [3, Theorem 16.1.3]. However, we will give a sketch of proof for reader's convenience.

Theorem 4.3. *$\mathcal{A}$ has a symmetric operator diagonal if and only if there exist positive integers $n_1, \ldots, n_r \in \mathbb{N}$ such that $\mathcal{A} \simeq M_{n_1} \oplus \cdots \oplus M_{n_r}$.*

Proof. Suppose that $\mathcal{A} \simeq M_{n_1} \oplus \cdots \oplus M_{n_r}$. As in Example 4.2, we may choose a symmetric operator diagonal $u_k \in M_{n_k} \otimes M_{n_k}$ for each $k = 1, \ldots, r$. We put $u := \oplus_k u_k \in \oplus_k M_{n_k} \otimes M_{n_k} \subset \mathcal{A} \otimes \mathcal{A}$. Then the element u is a symmetric operator diagonal in $\mathcal{A} \otimes \mathcal{A}$.

Conversely, assume that $u = \sum_i a_i \otimes b_i$ is a symmetric operator diagonal in $\mathcal{A} \otimes \mathcal{A}$. Given a right $\mathcal{A}$-module V and its submodule W, let $P : V \to W$ be the linear map given by $P(v) = \sum_{\lambda \in \Lambda_0} c_\lambda e_\lambda$. We see that $P^2 = P$ and P maps V onto W, so that P is the linear projection of V onto W. We define a linear mapping $\tilde{P} : V \to W$ by $\tilde{P}(v) = \sum_i P(vb_i)a_i$. Then $\tilde{P}$ is a right $\mathcal{A}$-module map, so that $\ker \tilde{P}$ is a right $\mathcal{A}$-submodule. Since $\tilde{P}(w) = w$ for any $w \in W$, $\tilde{P}$ is a right $\mathcal{A}$-module projection of V onto W. Moreover, we have that $V = W + \ker \tilde{P}$ and $W \cap \ker \tilde{P} = \{0\}$. Since $\ker \tilde{P}$ is a right $\mathcal{A}$-module complement of W and V is semi-simple, by Theorem 16.1.2 in [3], we obtain that $\mathcal{A} \simeq M_{n_1} \oplus \cdots \oplus M_{n_r}$. □

Remark 4.4. *In [10], a Banach algebra $\mathcal{B}$ is symmetrically contractible if there exists an element v of $\mathcal{B}\widehat{\otimes}\mathcal{B}$ such that $b \bullet v = v \bullet b$ and $bm^{\bullet}(v) = b$ for all $b \in \mathcal{B}$ where $\mathcal{B}\widehat{\otimes}\mathcal{B}$ is the projective tensor product. Such an element is called a symmetric diagonal for $\mathcal{B}$, which is a weaker notion than that of Effros and Ruan's book [3]. Johnson [10] studied the problem which asks if there is an infinite dimensional symmetrically contractible Banach algebras.*

5. Jordan Derivations on Operator Algebras

A *Jordan derivation* from $\mathcal{A}$ into an operator $\mathcal{A}$-bimodule is a linear map δ with

$$\delta(a^2) = a \cdot \delta(a) + \delta(a) \cdot a \quad \text{for all } a \in \mathcal{A}. \tag{5}$$

By using the equality $ab + ba = (a+b)^2 - a^2 - b^2$, the Jordan derivation identity (5) is equivalent to $\delta(ab+ba) = \delta(a) \cdot b + a \cdot \delta(b) + \delta(b) \cdot a + b \cdot \delta(a)$. A Jordan derivation is *proper* if it is not an ordinary derivation.

Example 5.1. *Let $\mathcal{A}$ be the algebra of 2×2 upper triangular matrices and let $\mathbb{C}$ be an $\mathcal{A}$-bimodule with multiplication defined by $a \cdot x = a_{22}x$ and $x \cdot a = xa_{11}$ for $a \in \mathcal{A}$ and $x \in \mathbb{C}$. Then $\delta(a) = a_{12}$ is a proper Jordan derivation. Indeed, we have that*

$$\delta(a^2) = a \cdot \delta(a) + \delta(a) \cdot a = a_{11}a_{12} + a_{12}a_{22} \quad \text{for } a = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix}.$$

However, we have that $\delta(ab) = a_{11}b_{12} + a_{12}b_{22}$ and $\delta(a) \cdot b + a \cdot \delta(b) = a_{12}b_{11} + a_{22}b_{12}$, which implies that δ is not a derivation.

Theorem 5.2. *(cf. [10]) Let X be an operator $\mathcal{A}$-bimodule. If $\mathcal{A}$ is symmetrically operator amenable, there is no proper Jordan derivation from $\mathcal{A}$ into X which is completely bounded.*

Proof. The proof is similar to that of [10, Theorem 6.2], so that we give a sketch of proof.

Since $\mathbb{C}$ is symmetrically operator amenable, so is $\tilde{\mathcal{A}} = \mathcal{A} \oplus \mathbb{C}$. We can extend a Jordan derivation δ to $\tilde{\mathcal{A}}$ by defining $\delta(1) = 0$. Thus, it suffices to prove the theorem in the case where $\mathcal{A}$ is unital and X is a unital $\mathcal{A}$-bimodule. Replacing X by the double dual X^{**} if necessary, we can assume that X is the dual of some unital $\mathcal{A}$-bimodule Y.

Let $\{z_\lambda\}_{\lambda\in\Lambda}$ be a symmetric approximate diagonal. Suppose that $\delta : \mathcal{A} \to X$ is a completely bounded Jordan derivation. We define $x \in X$ by

$$(x, y) = \mathrm{LIM}_\lambda \Big(\sum_i a_\lambda^i \delta(b_\lambda^i), y \Big)$$

where $y \in Y$, $z_\lambda = \sum_i a_\lambda^i \otimes b_\lambda^i$ and LIM denotes a generalized limit on $\mathcal{A}$. For any $a \in \mathcal{A}$ and $y \in Y$, we have that

$$(a \cdot x, y) = \mathrm{LIM}_\lambda \Big(\sum_i a_\lambda^i \delta(b_\lambda^i a), y \Big) = \big(\delta(a) + xa + D(a), y \big)$$

where $(D(a), y) := \mathrm{LIM}_\lambda \sum_i (a_\lambda^i \delta(a) b_\lambda^i, y)$. Hence, we obtain that $D(a) = a \cdot x - x \cdot a - \delta(a)$, so that D is a Jordan derivation and $aD(b) = D(b)a$. We repeat this argument with D in the place of δ and denote the resulting element of X by x_0. Then we have $a \cdot x_0 = 2D(b) + x_0 \cdot a$, so that $\delta(a) = a(x - \frac{1}{2}x_0) - (x - \frac{1}{2}x_0)a$. This implies that δ is not a proper Jordan derivation. $\square$

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