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Solvable Three-Dimensional System of Higher-Order Nonlinear Difference Equations

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Abstract. In this work, we indicate three-dimensional system of difference equations

$$x_n = ay_{n-k} + \frac{dy_{n-k}x_{n-k-l}}{\widehat{b}x_{n-k-l} + \widehat{c}z_{n-l}},\ y_n = \alpha z_{n-k} + \frac{\delta z_{n-k}y_{n-k-l}}{\widehat{\beta}y_{n-k-l} + \widehat{\gamma}x_{n-l}},\ z_n = ex_{n-k} + \frac{hx_{n-k}z_{n-k-l}}{\widehat{f}z_{n-k-l} + \widehat{g}y_{n-l}},\ n \in \mathbb{N}_0,$$

where k and l are positive integers, the parameters $a, \widehat{b}, \widehat{c}, d, \alpha, \widehat{\beta}, \widehat{\gamma}, \delta, e, \widehat{f}, \widehat{g}, h$ and the initial values x_{-j}, y_{-j}, z_{-j} $j = \overline{1, k+l}$, are non-zero real numbers, can be solved in closed form. In addition, we obtain explicit formulas for the well-defined solutions of the aforementioned system for the case $l = 1$. Also, the set of undefinable solutions of the system is found. Finally, an application about a three-dimensional system of difference equations is given.

1. Introduction and Preliminaries

First, remind that $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$, stand for natural, non-negative integer, integer, real and complex numbers, respectively. If $m, n \in \mathbb{Z}$, $m \le n$ the notation $i = \overline{m, n}$ stands for $\{i \in \mathbb{Z} : m \le i \le n\}$.

Difference equations emerge from the study of the evolution of naturally occurring events. There is no doubt that the theory of difference equations will proceed to play an important role in mathematics. Especially, the focus of interest for most authors is non-linear difference equations and their systems (see, e.g. [4, 5, 13–18, 20, 22, 30–32])
One of important non-linear solvable difference equation is the following

$$x_n = \alpha x_{n-k} + \frac{\delta x_{n-k}x_{n-(k+l)}}{\beta x_{n-(k+l)} + \gamma x_{n-l}},\ n \in \mathbb{N}_0, \tag{1}$$

where k and l are fixed natural numbers, $\alpha, \beta, \gamma, \delta \in \mathbb{R}$, and the initial values x_{-i}, $i = \overline{1, k+l}$, are real numbers. Tollu et al. solved equation (1) in closed-form in [28]. Some authors studied the case $k = 1$, $l = 1, 2, 3, 4$, in equation (1) with the special choices of $\alpha, \beta, \gamma, \delta$, in [1, 2, 6, 7, 10, 21, 23, 25]. In addition, for the case $k = 2$,

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$l = 1, 2$, with the special choices of $\alpha, \beta, \gamma, \delta$ in equation (1) are investigated in [6, 7, 11].
Recently, in [19] we showed the solvability of the following two-dimensional relative to equation (1)

$$x_n = ay_{n-k} + \frac{dy_{n-k}x_{n-(k+l)}}{bx_{n-(k+l)} + cy_{n-l}},\ y_n = \alpha x_{n-k} + \frac{\delta x_{n-k}y_{n-(k+l)}}{\beta y_{n-(k+l)} + \gamma x_{n-l}},\ n \in \mathbb{N}_0, \tag{2}$$

where k and l are positive integers, $a, b, c, d, \alpha, \beta, \gamma, \delta \in \mathbb{R}$, and the initial values $x_{-i}, y_{-i}, i = \overline{1, k+l}$, are real numbers. System (2) is a natural generalization of the systems given in [3, 8, 9, 24, 27, 29]. In these papers, authors studied the case $a = 0, \alpha = 0$ in system (2) with the special choices of $k, l, b, c, d, \beta, \gamma, \delta$. In addition, authors found solutions of these systems which are associated to Fibonacci numbers in these papers.
A few years ago, in [12] the following systems of difference equations was studied:

$$x_{n+1} = \frac{y_n x_{n-2}}{x_{n-2} \pm z_{n-1}},\ y_{n+1} = \frac{z_n y_{n-2}}{y_{n-2} \pm x_{n-1}},\ z_{n+1} = \frac{x_n z_{n-2}}{z_{n-2} \pm y_{n-1}},\ n \in \mathbb{N}_0, \tag{3}$$

where the initial values are non-zero real numbers and solved by using induction principle. But induction method didn't give much detail on how solutions were obtained. Note that, the system (3) is the extended version of three-dimensional the equations in [1, 6, 7] and systems in [3, 24, 27].
A natural question is to study both three-dimensional form of equation (1), system (2) and more general system of (3) solvable in closed form. Here we study such a system. That is, we deal with the following system of difference equations

$$x_n = ay_{n-k} + \frac{dy_{n-k}x_{n-k-l}}{\widehat{b}x_{n-k-l} + \widehat{c}z_{n-l}},\ y_n = \alpha z_{n-k} + \frac{\delta z_{n-k}y_{n-k-l}}{\widehat{\beta}y_{n-k-l} + \widehat{\gamma}x_{n-l}},\ z_n = ex_{n-k} + \frac{hx_{n-k}z_{n-k-l}}{\widehat{f}z_{n-k-l} + \widehat{g}y_{n-l}},\ n \in \mathbb{N}_0, \tag{4}$$

where k and l are positive integers, the parameters $a, \widehat{b}, \widehat{c}, d, \alpha, \widehat{\beta}, \widehat{\gamma}, \delta, e, \widehat{f}, \widehat{g}, h$ and the initial values $x_{-j}, y_{-j}, z_{-j}\ j = \overline{1, k+l}$, are non-zero real numbers.

Remark 1.1. *We may assume that* $\frac{\widehat{b}}{d} = b$, $\frac{\widehat{c}}{d} = c$, $\frac{\widehat{\beta}}{\delta} = \beta$, $\frac{\widehat{\gamma}}{\delta} = \gamma$, $\frac{\widehat{f}}{h} = f$ *and* $\frac{\widehat{g}}{h} = g$, *from system* (4) *we get*

$$x_n = ay_{n-k} + \frac{y_{n-k}x_{n-k-l}}{bx_{n-k-l} + cz_{n-l}},\ y_n = \alpha z_{n-k} + \frac{z_{n-k}y_{n-k-l}}{\beta y_{n-k-l} + \gamma x_{n-l}},\ z_n = ex_{n-k} + \frac{x_{n-k}z_{n-k-l}}{fz_{n-k-l} + gy_{n-l}},\ n \in \mathbb{N}_0. \tag{5}$$

From now on, we will consider system (5) *instead of system* (4).

In this paper, we show that system (5) is solvable in closed form. Also, we give the forbidden set of the initial values of system (5). Finally, an application that guarantees the accuracy of the results, is given.

2. Main Results

The first result is an auxiliary one which will be used for in solutions in this paper.

Lemma 2.1. *[26] Consider*

$$x_{n+1} = \frac{ax_n + b}{cx_n + d},\ n \in \mathbb{N}_0, \tag{6}$$

for $c \neq 0$, $ad \neq bc$, *where parameters* a, b, c, d *and the initial value* x_0 *are real numbers, which called Riccati difference equation. Indeed, equation* (6) *has the general solution can be written in the following form*

$$x_n = \frac{x_0 (bc - ad) s_{n-1} + (ax_0 + b) s_n}{(cx_0 - a) s_n + s_{n+1}},\quad n \in \mathbb{N}, \tag{7}$$

where $(s_n)_{n\in\mathbb{N}_0}$ *is the sequence satisfying*

$$s_{n+1} - (a + d) s_n - (bc - ad) s_{n-1} = 0,\quad n \in \mathbb{N}, \tag{8}$$

where $s_0 = 0$, $s_1 = 1$.

In other result, we show that system (5) is solvable in closed form. First, we write the system as follows:

$$\frac{x_n}{y_{n-k}} = \frac{ac\frac{z_{n-l}}{x_{n-k-l}} + ab + 1}{c\frac{z_{n-l}}{x_{n-k-l}} + b}, \ \frac{y_n}{z_{n-k}} = \frac{\alpha\gamma\frac{x_{n-l}}{y_{n-k-l}} + \alpha\beta + 1}{\gamma\frac{x_{n-l}}{y_{n-k-l}} + \beta}, \ \frac{z_n}{x_{n-k}} = \frac{eg\frac{y_{n-l}}{z_{n-k-l}} + ef + 1}{g\frac{y_{n-l}}{z_{n-k-l}} + f}, \ n \in \mathbb{N}_0.$$

Putting

$$u_n = \frac{x_n}{y_{n-k}}, \ v_n = \frac{y_n}{z_{n-k}}, \ w_n = \frac{z_n}{x_{n-k}}, n \geq -l, \tag{9}$$

in the last expressions, we get the system of equations

$$u_n = \frac{acw_{n-l} + ab + 1}{cw_{n-l} + b}, \ v_n = \frac{\alpha\gamma u_{n-l} + \alpha\beta + 1}{\gamma u_{n-l} + \beta}, \ w_n = \frac{egv_{n-l} + ef + 1}{gv_{n-l} + f}, \ n \in \mathbb{N}_0, \tag{10}$$

where the parameters $a, b, c, \alpha, \beta, \gamma, e, f, g$, in the new variables u_n, v_n and w_n. System (10) can be written as

$$u_n = \frac{(aceg + abg + g)\, v_{n-2l} + acef + abf + ac + f}{(ceg + bg)\, v_{n-2l} + cef + bf + c}, n \geq l, \tag{11}$$

$$v_n = \frac{(\alpha\gamma ac + \alpha\beta c + c)\, w_{n-2l} + \alpha\gamma ab + \alpha\beta b + \alpha\gamma + b}{(\gamma ac + \beta c)\, w_{n-2l} + \gamma ab + \beta b + \gamma}, n \geq l, \tag{12}$$

and

$$w_n = \frac{(eg\alpha\gamma + ef\gamma + \gamma)\, u_{n-2l} + eg\alpha\beta + ef\beta + eg + \beta}{(g\alpha\gamma + f\gamma)\, u_{n-2l} + g\alpha\beta + f\beta + g}, n \geq l. \tag{13}$$

Let

$$A_1 := abg\alpha\gamma + g\alpha\gamma + aceg\alpha\gamma + abf\gamma + f\gamma + acef\gamma + ac\gamma,$$

$$B_1 := abg\alpha\beta + g\alpha\beta + aceg\alpha\beta + abg + g + aceg + abf\beta + f\beta + acef\beta + ac\beta,$$
$$C_1 := bg\alpha\gamma + ceg\alpha\gamma + bf\gamma + cef\gamma + c\gamma,$$
$$D_1 := bg\alpha\beta + ceg\alpha\beta + bg + ceg + bf\beta + cef\beta + c\beta,$$
$$A_2 := \alpha\beta ceg + ceg + \alpha\gamma aceg + \alpha\beta bg + bg + \alpha\gamma abg + \alpha\gamma g,$$
$$B_2 := \alpha\beta cef + cef + \alpha\gamma acef + \alpha\beta c + c + \alpha\gamma ac + \alpha\beta bf + bf + \alpha\gamma abf + \alpha\gamma f,$$
$$C_2 := \beta ceg + \gamma aceg + \beta bg + \gamma abg + \gamma g,$$
$$D_2 := \beta cef + \gamma acef + \beta c + \gamma ac + \beta bf + \gamma abf + \gamma f,$$
$$A_3 := ef\gamma ac + \gamma ac + eg\alpha\gamma ac + ef\beta c + \beta c + eg\alpha\beta c + egc,$$
$$B_3 := ef\gamma ab + \gamma ab + eg\alpha\gamma ab + ef\gamma + \gamma + eg\alpha\gamma + ef\beta b + \beta b + eg\alpha\beta b + egb,$$
$$C_3 := f\gamma ac + g\alpha\gamma ac + f\beta c + g\alpha\beta c + gc,$$
$$D_3 := f\gamma ab + g\alpha\gamma ab + f\gamma + g\alpha\gamma + f\beta b + g\alpha\beta b + gb.$$

By using the second equation of system (10) in equation (11), the third equation of system (10) in equation (12), the first equation of system (10) in equation (13), we obtain the independent equations

$$u_n = \frac{A_1 u_{n-3l} + B_1}{C_1 u_{n-3l} + D_1}, \ n \geq 2l, \tag{14}$$

$$v_n = \frac{A_2 v_{n-3l} + B_2}{C_2 v_{n-3l} + D_2},\ n \geq 2l, \tag{15}$$

and

$$w_n = \frac{A_3 w_{n-3l} + B_3}{C_3 w_{n-3l} + D_3},\ n \geq 2l. \tag{16}$$

If we apply the decomposition of indexes $n \to 3l(m+1) + i$, for $m \geq -1$ and $i = \overline{-l, 2l-1}$, to (14), (15) and (16), they become

$$u_{3l(m+1)+i} = \frac{A_1 u_{3lm+i} + B_1}{C_1 u_{3lm+i} + D_1},\ m \in \mathbb{N}_0, \tag{17}$$

$$v_{3l(m+1)+i} = \frac{A_2 v_{3lm+i} + B_2}{C_2 v_{3lm+i} + D_2},\ m \in \mathbb{N}_0, \tag{18}$$

and

$$w_{3l(m+1)+i} = \frac{A_3 w_{3lm+i} + B_3}{C_3 w_{3lm+i} + D_3},\ m \in \mathbb{N}_0, \tag{19}$$

for $i = \overline{-l, 2l-1}$. Let $u_{m+1}^{(i)} = u_{3l(m+1)+i}$, $v_{m+1}^{(i)} = v_{3l(m+1)+i}$, $w_{m+1}^{(i)} = w_{3l(m+1)+i}$, for some $m \geq -1$ and $i = \overline{-l, 2l-1}$. Then equations in (17)-(19) can be written as the following

$$u_{m+1}^{(i)} = \frac{A_1 u_m^{(i)} + B_1}{C_1 u_m^{(i)} + D_1},\ m \in \mathbb{N}_0, \tag{20}$$

$$v_{m+1}^{(i)} = \frac{A_2 v_m^{(i)} + B_2}{C_2 v_m^{(i)} + D_2},\ m \in \mathbb{N}_0, \tag{21}$$

$$w_{m+1}^{(i)} = \frac{A_3 w_m^{(i)} + B_3}{C_3 w_m^{(i)} + D_3},\ m \in \mathbb{N}_0, \tag{22}$$

for $i = \overline{-l, 2l-1}$, which are essentially in the form of Riccati difference equations. From equation (7), the general solutions of (20)-(22) follow straightforwardly as

$$u_m^{(i)} = \frac{(B_1 C_1 - A_1 D_1)\, u_0^{(i)} s_{m-1} + \left(A_1 u_0^{(i)} + B_1\right) s_m}{\left(C_1 u_0^{(i)} - A_1\right) s_m + s_{m+1}},\ m \in \mathbb{N}_0, \tag{23}$$

$$v_m^{(i)} = \frac{(B_2 C_2 - A_2 D_2)\, v_0^{(i)} s_{m-1} + \left(A_2 v_0^{(i)} + B_2\right) s_m}{\left(C_2 v_0^{(i)} - A_2\right) s_m + s_{m+1}},\ m \in \mathbb{N}_0, \tag{24}$$

and

$$w_m^{(i)} = \frac{(B_3 C_3 - A_3 D_3)\, w_0^{(i)} s_{m-1} + \left(A_3 w_0^{(i)} + B_3\right) s_m}{\left(C_3 w_0^{(i)} - A_3\right) s_m + s_{m+1}},\ m \in \mathbb{N}_0, \tag{25}$$

for $i = \overline{-l, 2l-1}$, sequence of $(s_m)_{m \in \mathbb{N}_0}$ is satisfying

$$s_{m+1} - A s_m - B s_{m-1} = 0,\ m \in \mathbb{N}, \tag{26}$$

difference equation where $s_0 = 0$, $s_1 = 1$, $A = bf\beta + cef\beta + c\beta + bg\alpha\beta + ceg\alpha\beta + abf\gamma + f\gamma + acef\gamma + ac\gamma + abg\alpha\gamma + g\alpha\gamma + aceg\alpha\gamma + bg + ceg$, $B = c\gamma g$.
From (23)-(25), we get

$$u_{3lm+i} = \frac{(B_1C_1 - A_1D_1)\, u_i s_{m-1} + (A_1u_i + B_1)\, s_m}{(C_1u_i - A_1)\, s_m + s_{m+1}},\ m \in \mathbb{N}_0, \tag{27}$$

$$v_{3lm+i} = \frac{(B_2C_2 - A_2D_2)\, v_i s_{m-1} + (A_2v_i + B_2)\, s_m}{(C_2v_i - A_2)\, s_m + s_{m+1}},\ m \in \mathbb{N}_0, \tag{28}$$

and

$$w_{3lm+i} = \frac{(B_3C_3 - A_3D_3)\, w_i s_{m-1} + (A_3w_i + B_3)\, s_m}{(C_3w_i - A_3)\, s_m + s_{m+1}},\ m \in \mathbb{N}_0, \tag{29}$$

for $i = \overline{-l, 2l-1}$. Using equalities in (9), from (27)-(29) we get

$$u_{3lm+i} = \frac{c\gamma g x_i s_{m-1} + (A_1x_i + B_1y_{i-k})\, s_m}{(C_1x_i - A_1y_{i-k})\, s_m + y_{i-k}s_{m+1}},\ m \in \mathbb{N}_0, \tag{30}$$

$$v_{3lm+i} = \frac{c\gamma g y_i s_{m-1} + (A_2y_i + B_2z_{i-k})\, s_m}{(C_2y_i - A_2z_{i-k})\, s_m + z_{i-k}s_{m+1}},\ m \in \mathbb{N}_0, \tag{31}$$

and

$$w_{3lm+i} = \frac{c\gamma g z_i s_{m-1} + (A_3z_i + B_3x_{i-k})\, s_m}{(C_3z_i - A_3x_{i-k})\, s_m + x_{i-k}s_{m+1}},\ m \in \mathbb{N}_0, \tag{32}$$

for $i = \overline{-l, 2l-1}$. From (9) we have

$$\begin{aligned} x_n &= u_ny_{n-k} = u_nv_{n-k}z_{n-2k} = u_nv_{n-k}w_{n-2k}x_{n-3k}, \\ y_n &= v_nz_{n-k} = v_nw_{n-k}x_{n-2k} = v_nw_{n-k}u_{n-2k}y_{n-3k},\ n \geq 2k-l, \\ z_n &= w_nx_{n-k} = w_nu_{n-k}y_{n-2k} = w_nu_{n-k}v_{n-2k}z_{n-3k}, \end{aligned} \tag{33}$$

$$x_{3km+j_1} = u_{3km+j_1}v_{3km+j_1-k}w_{3km+j_1-2k}x_{3k(m-1)+j_1},\ m \in \mathbb{N}_0, \tag{34}$$

$$y_{3km+j_1} = v_{3km+j_1}w_{3km+j_1-k}u_{3km+j_1-2k}y_{3k(m-1)+j_1},\ m \in \mathbb{N}_0, \tag{35}$$

$$z_{3km+j_1} = w_{3km+j_1}u_{3km+j_1-k}v_{3km+j_1-2k}z_{3k(m-1)+j_1},\ m \in \mathbb{N}_0, \tag{36}$$

for $j_1 = \overline{2k-l, 5k-l-1}$, from which it follows that

$$x_{3km+j_1} = x_{j_1-3k}\prod_{j=0}^{m} u_{3kj+j_1}v_{(3j-1)k+j_1}w_{(3j-2)k+j_1}, \tag{37}$$

$$y_{3km+j_1} = y_{j_1-3k}\prod_{j=0}^{m} v_{3kj+j_1}w_{(3j-1)k+j_1}u_{(3j-2)k+j_1}, \tag{38}$$

$$z_{3km+j_1} = z_{j_1-3k}\prod_{j=0}^{m} w_{3kj+j_1}u_{(3j-1)k+j_1}v_{(3j-2)k+j_1}, \tag{39}$$

for $m \in \mathbb{N}_0$ and $j_1 = \overline{2k-l, 5k-l-1}$. By the help of the well-known quotient remainder theorem, there exists $l \in \mathbb{N}$ and $j \in \mathbb{N}_0$ such that $n = 3lj + j_2$ and $j_2 \in \{0, 1, \ldots, 3l-1\}$. From this and equations in (37)-(39), we can write

$$x_{3klm+3lj+j_2} = x_{3lj+j_2-3kl} \prod_{p=0}^{m} \prod_{n=1}^{l} u_{3klp+3lj+j_2-(3n-3)k} v_{3klp+3lj+j_2-(3n-2)k} w_{3klp+3lj+j_2-(3n-1)k}, \tag{40}$$

$$y_{3klm+3lj+j_2} = y_{3lj+j_2-3kl} \prod_{p=0}^{m} \prod_{n=1}^{l} v_{3klp+3lj+j_2-(3n-3)k} w_{3klp+3lj+j_2-(3n-2)k} u_{3klp+3lj+j_2-(3n-1)k}, \tag{41}$$

$$z_{3klm+3lj+j_2} = z_{3lj+j_2-3kl} \prod_{p=0}^{m} \prod_{n=1}^{l} w_{3klp+3lj+j_2-(3n-3)k} u_{3klp+3lj+j_2-(3n-2)k} v_{3klp+3lj+j_2-(3n-1)k}, \tag{42}$$

where $m \in \mathbb{N}_0$ and $3lj + j_2 \in \{3kl-k-l, 3kl-k-l+1, \ldots, 6kl-k-l-1\}$.
Let

$$K_1 := -\frac{\beta f + g\alpha\beta + g}{\gamma f + g\alpha\gamma},$$

$$L_1 := -\frac{\beta f b + \beta c e f + \beta c + \alpha\beta g b + \alpha\beta c e g + g b + c e g}{\gamma f b + \gamma c e f + \gamma c + \alpha\gamma g b + \alpha\gamma c e g},$$

$$K_2 := -\frac{fb + cef + c}{gb + ceg},$$

$$L_2 := -\frac{fb\beta + fab\gamma + f\gamma + efc\beta + ef\gamma ac + c\beta + \gamma ac}{gb\beta + gab\gamma + g\gamma + c\beta eg + \gamma aceg},$$

$$K_3 := -\frac{b\beta + ab\gamma + \gamma}{c\beta + \gamma ac},$$

$$L_3 := -\frac{b\beta f + bg\alpha\beta + bg + ab\gamma f + abg\alpha\gamma + \gamma f + g\alpha\gamma}{c\beta f + cg\alpha\beta + cg + ac\gamma f + acg\alpha\gamma}.$$

By the following theorem, we characterize the forbidden set of the initial values for system (5).

Theorem 2.2. *The forbidden set of the initial values for system (5) is combination of two sets*

$$\left\{ \overrightarrow{X} : x_{-j} = 0 \text{ or } y_{-j} = 0, \text{ or } y_{-j} = 0, \ j = \overline{1,k} \right\}$$

and

$$\begin{aligned} \bigcup_{m\in\mathbb{N}_0} \bigcup_{s=0}^{l-1} \Big\{ \overrightarrow{X} : \frac{x_{s-l}}{y_{s-k-l}} &= \left(\widetilde{f} \circ \widetilde{h} \circ \widetilde{g}\right)^{-m} \left(-\frac{\beta}{\gamma}\right) \text{ or } \left(\widetilde{f} \circ \widetilde{h} \circ \widetilde{g}\right)^{-m} (K_1) \text{ or } \left(\widetilde{f} \circ \widetilde{h} \circ \widetilde{g}\right)^{-m} (L_1) \text{ or} \\ \frac{y_{s-l}}{z_{s-k-l}} &= \left(\widetilde{g} \circ \widetilde{f} \circ \widetilde{h}\right)^{-m} \left(-\frac{f}{g}\right) \text{ or } \left(\widetilde{g} \circ \widetilde{f} \circ \widetilde{h}\right)^{-m} (K_2) \text{ or } \left(\widetilde{g} \circ \widetilde{f} \circ \widetilde{h}\right)^{-m} (L_2) \text{ or} \\ \frac{z_{s-l}}{x_{s-k-l}} &= \left(\widetilde{h} \circ \widetilde{g} \circ \widetilde{f}\right)^{-m} \left(-\frac{b}{c}\right) \text{ or } \left(\widetilde{h} \circ \widetilde{g} \circ \widetilde{f}\right)^{-m} (K_3) \text{ or } \left(\widetilde{h} \circ \widetilde{g} \circ \widetilde{f}\right)^{-m} (L_3) \Big\} \end{aligned} \tag{43}$$

where $\overrightarrow{X} = (x_{-k-l}, x_{-k-l+1}, \cdots, x_{-1}, y_{-k-l}, y_{-k-l+1}, \cdots, y_{-1}, z_{-k-l}, z_{-k-l+1}, \cdots, z_{-1})$.

Proof. Let $(x_n, y_n, z_n)_{n\geq -k-l}$ be a solution of system (5). Assume that $x_{-j} = 0$ or $y_{-j} = 0$ or $z_{-j} = 0$ for some $j = \overline{1,k}$. For example, if $x_{-k} = 0$, then $z_0 = 0$, and so x_l can not be calculated. For the dual of this case, the result is same, too. That is, if $y_{-k} = 0$ $(z_{-k} = 0)$, then $x_0 = 0$ $(y_0 = 0)$, and so y_l (z_l) can not be calculated. For the other initial values, the case is not same. Because, if $x_{-j} = 0$, $y_{-j} = 0$, $z_{-j} = 0$ for some $j = \overline{k+1, k+l}$, then $x_n \neq 0$, $y_n \neq 0$, $z_n \neq 0$ for $n \geq 0$. So, we incorporate the set

$$\left\{\overrightarrow{X} : x_{-j} = 0 \text{ or } y_{-j} = 0 \text{ or } z_{-j} = 0,\ j = \overline{1,k}\right\}$$

in the forbidden set. Now, we suppose that $x_n \neq 0$, $y_n \neq 0$ and $z_n \neq 0$. The solution $(x_n, y_n, z_n)_{n\geq -k-l}$ of system (5) is not defined if and only if $bx_{n-k-l} + cz_{n-l} = 0$, $\beta y_{n-k-l} + \gamma x_{n-l} = 0$ and $fz_{n-k-l} + gy_{n-l} = 0$ which correspond to the statements $\frac{z_{n-l}}{x_{n-k-l}} = -\frac{b}{c}$, $\frac{x_{n-l}}{y_{n-k-l}} = -\frac{\beta}{\gamma}$ and $\frac{y_{n-l}}{z_{n-k-l}} = -\frac{f}{g}$ for $n \geq 0$, respectively. Therefore, by taking into account (9), we have

$$w_{n-l} = -\frac{b}{c} \text{ and } u_{n-l} = -\frac{\beta}{\gamma} \text{ and } v_{n-l} = -\frac{f}{g} \tag{44}$$

for $n \in \mathbb{N}_0$. Now, we again consider system (10) and the functions

$$\widetilde{f}(t) = \frac{act + ab + 1}{ct + b},\ \widetilde{g}(t) = \frac{\alpha\gamma t + \alpha\beta + 1}{\gamma t + \beta},\ \widetilde{h}(t) = \frac{egt + ef + 1}{gt + f}$$

which correspond to the equations of (10). From which it follows that

$$w_{3lm+i} = \left(\widetilde{h} \circ \widetilde{g} \circ \widetilde{f}\right)^m (w_i), \tag{45}$$

$$w_{3lm+l+i} = \left(\left(\widetilde{h} \circ \widetilde{g} \circ \widetilde{f}\right)^m \circ \widetilde{h}\right)(v_i), \tag{46}$$

$$w_{3lm+2l+i} = \left(\left(\widetilde{h} \circ \widetilde{g} \circ \widetilde{f}\right)^m \circ \widetilde{h} \circ \widetilde{g}\right)(u_i), \tag{47}$$

$$v_{3lm+i} = \left(\widetilde{g} \circ \widetilde{f} \circ \widetilde{h}\right)^m (v_i), \tag{48}$$

$$v_{3lm+l+i} = \left(\left(\widetilde{g} \circ \widetilde{f} \circ \widetilde{h}\right)^m \circ \widetilde{g}\right)(u_i), \tag{49}$$

$$v_{3lm+2l+i} = \left(\left(\widetilde{g} \circ \widetilde{f} \circ \widetilde{h}\right)^m \circ \widetilde{g} \circ \widetilde{f}\right)(w_i), \tag{50}$$

$$u_{3lm+i} = \left(\widetilde{f} \circ \widetilde{h} \circ \widetilde{g}\right)^m (u_i), \tag{51}$$

$$u_{3lm+l+i} = \left(\left(\widetilde{f} \circ \widetilde{h} \circ \widetilde{g}\right)^m \circ \widetilde{f}\right)(w_i), \tag{52}$$

$$u_{3lm+2l+i} = \left(\left(\widetilde{f} \circ \widetilde{h} \circ \widetilde{g}\right)^m \circ \widetilde{f} \circ \widetilde{h}\right)(v_i), \tag{53}$$

for $m \in \mathbb{N}$, $i = \overline{-l, -1}$ and $l \in \mathbb{N}$. By using (44) and the implicit forms (45)-(53), we have

$$w_i = \left(\widetilde{h} \circ \widetilde{g} \circ \widetilde{f}\right)^{-m} \left(-\frac{b}{c}\right), \tag{54}$$

$$v_i = \left(\widetilde{g} \circ \widetilde{f} \circ \widetilde{h}\right)^{-m} \left(\widetilde{h}^{-1}\left(-\frac{b}{c}\right)\right) = \left(\widetilde{g} \circ \widetilde{f} \circ \widetilde{h}\right)^{-m} \left(-\frac{fb + cef + c}{gb + ceg}\right), \tag{55}$$

$$u_i = \left(\left(\widetilde{f}\circ\widetilde{h}\circ\widetilde{g}\right)^{-m}\circ\widetilde{g}^{-1}\circ\widetilde{h}^{-1}\right)\left(-\frac{b}{c}\right)$$
$$= \left(\widetilde{f}\circ\widetilde{h}\circ\widetilde{g}\right)^{-m}\left(-\frac{\beta fb+\beta cef+\beta c+\alpha\beta gb+\alpha\beta ceg+gb+ceg}{\gamma fb+\gamma cef+\gamma c+\alpha\gamma gb+\alpha\gamma ceg}\right), \tag{56}$$

$$v_i = \left(\widetilde{g}\circ\widetilde{f}\circ\widetilde{h}\right)^{-m}\left(-\frac{f}{g}\right), \tag{57}$$

$$u_i = \left(\widetilde{f}\circ\widetilde{h}\circ\widetilde{g}\right)^{-m}\left(\widetilde{g}^{-1}\left(-\frac{f}{g}\right)\right) = \left(\widetilde{f}\circ\widetilde{h}\circ\widetilde{g}\right)^{-m}\left(-\frac{\beta f+g\alpha\beta+g}{\gamma f+g\alpha\gamma}\right), \tag{58}$$

$$w_i = \left(\left(\widetilde{h}\circ\widetilde{g}\circ\widetilde{f}\right)^{-m}\circ\widetilde{f}^{-1}\circ\widetilde{g}^{-1}\right)\left(-\frac{f}{g}\right)$$
$$= \left(\widetilde{h}\circ\widetilde{g}\circ\widetilde{f}\right)^{-m}\left(-\frac{b\beta f+bg\alpha\beta+bg+ab\gamma f+abg\alpha\gamma+\gamma f+g\alpha\gamma}{c\beta f+cg\alpha\beta+cg+ac\gamma f+acg\alpha\gamma}\right), \tag{59}$$

$$u_i = \left(\widetilde{f}\circ\widetilde{h}\circ\widetilde{g}\right)^{-m}\left(-\frac{\beta}{\gamma}\right), \tag{60}$$

$$w_i = \left(\widetilde{h}\circ\widetilde{g}\circ\widetilde{f}\right)^{-m}\left(\widetilde{f}^{-1}\left(-\frac{\beta}{\gamma}\right)\right) = \left(\widetilde{h}\circ\widetilde{g}\circ\widetilde{f}\right)^{-m}\left(-\frac{b\beta+ab\gamma+\gamma}{c\beta+\gamma ac}\right), \tag{61}$$

$$v_i = \left(\left(\widetilde{g}\circ\widetilde{f}\circ\widetilde{h}\right)^{-m}\circ\widetilde{h}^{-1}\circ\widetilde{f}^{-1}\right)\left(-\frac{\beta}{\gamma}\right)$$
$$= \left(\widetilde{g}\circ\widetilde{f}\circ\widetilde{h}\right)^{-m}\left(-\frac{fb\beta+fab\gamma+f\gamma+efc\beta+ef\gamma ac+c\beta+\gamma ac}{gb\beta+gab\gamma+g\gamma+c\beta eg+\gamma aceg}\right), \tag{62}$$

for $m\in\mathbb{N}$, $i=\overline{-l,-1}$ and $l\in\mathbb{N}$, where

$$\widetilde{h}^{-1}(t)=\frac{-ft+ef+1}{gt-eg},\ \widetilde{g}^{-1}(t)=\frac{-\beta t+\alpha\beta+1}{\gamma t-\alpha\gamma},\ \widetilde{f}^{-1}(t)=\frac{-bt+ab+1}{ct-ac},$$

respectively. This means that if one of the conditions in (54)-(62) holds, then $3m-th$ iteration or $(3m+1)-th$ or $(3m+2)-th$ iteration in (10) can not be calculated. Consequently, desired result follows from (9). Also, note that system associated with the functions $\widetilde{f}^{-1}$, $\widetilde{g}^{-1}$ and $\widetilde{h}^{-1}$ is

$$p_n=\frac{-f\widehat{p}_{n-l}+ef+1}{g\widehat{p}_{n-l}-eg},\ \widehat{p}_n=\frac{-\beta\widetilde{p}_{n-l}+\alpha\beta+1}{\gamma\widetilde{p}_{n-l}-\alpha\gamma},\ \widetilde{p}_n=\frac{-bp_n+ab+1}{cp_n-ac},\ n\in\mathbb{N}_0,$$

and is solvable. That is, the right hand sides of the equalities in (54)-(62) can be obtained in the closed form. □

3. A Study of Case $l = 1$

In this case, for $n \in \mathbb{N}_0$, system (5) is

$$x_n = ay_{n-k} + \frac{y_{n-k}x_{n-k-1}}{bx_{n-k-1} + cz_{n-1}},\ y_n = \alpha z_{n-k} + \frac{z_{n-k}y_{n-k-1}}{\beta y_{n-k-1} + \gamma x_{n-1}},\ z_n = ex_{n-k} + \frac{x_{n-k}z_{n-k-1}}{fz_{n-k-1} + gy_{n-1}}, \tag{63}$$

and the solution of system (63) can be written from equations in (40)-(42) (equations in (40)-(42); $k = 3t + r$, where $r \in \{0, 1, 2\}$), for $m \in \mathbb{N}_0$, $i \in \{-1, 0, 1\}$, as follows:

$$x_{3(3t)m+3j+i} = x_{3j+i-3(3t)} \prod_{p=0}^{m} u_{3(3tp+j)+i}v_{3(3tp+j-t)+i}w_{3(3tp+j-2t)+i}, \tag{64}$$

where $t \in \mathbb{N}$, $j = \overline{2t, 5t-1}$;

$$x_{3(3t+1)m+3j+i+2} = x_{3j+i+2-3(3t+1)} \prod_{p=0}^{m} u_{3((3t+1)p+j)+i+2}v_{3((3t+1)p+j-t)+i+1}w_{3((3t+1)p+j-2t)+i}, \tag{65}$$

where $t \in \mathbb{N}_0$, $j = \overline{2t, 5t}$;

$$x_{3(3t+2)m+3j+i+1} = x_{3j+i+1-3(3t+2)} \prod_{p=0}^{m} u_{3((3t+2)p+j)+i+1}v_{3((3t+2)p+j-t)+i-1}w_{3((3t+2)p+j-2t-1)+i}, \tag{66}$$

where $t \in \mathbb{N}_0$, $j = \overline{2t+1, 5t+2}$;

$$y_{3(3t)m+3j+i} = y_{3j+i-3(3t)} \prod_{p=0}^{m} v_{3(3tp+j)+i}w_{3(3tp+j-t)+i}u_{3(3tp+j-2t)+i}, \tag{67}$$

where $t \in \mathbb{N}$, $j = \overline{2t, 5t-1}$;

$$y_{3(3t+1)m+3j+i+2} = y_{3j+i+2-3(3t+1)} \prod_{p=0}^{m} v_{3((3t+1)p+j)+i+2}w_{3((3t+1)p+j-t)+i+1}u_{3((3t+1)p+j-2t)+i}, \tag{68}$$

where $t \in \mathbb{N}_0$, $j = \overline{2t, 5t}$;

$$y_{3(3t+2)m+3j+i+1} = y_{3j+i+1-3(3t+2)} \prod_{p=0}^{m} v_{3((3t+2)p+j)+i+1}w_{3((3t+2)p+j-t)+i-1}u_{3((3t+2)p+j-2t-1)+i}, \tag{69}$$

where $t \in \mathbb{N}_0$, $j = \overline{2t+1, 5t+2}$;

$$z_{3(3t)m+3j+i} = z_{3j+i-3(3t)} \prod_{p=0}^{m} w_{3(3tp+j)+i}u_{3(3tp+j-t)+i}v_{3(3tp+j-2t)+i}, \tag{70}$$

where $t \in \mathbb{N}$, $j = \overline{2t, 5t-1}$;

$$z_{3(3t+1)m+3j+i+2} = z_{3j+i+2-3(3t+1)} \prod_{p=0}^{m} w_{3\left((3t+1)p+j\right)+i+2} u_{3\left((3t+1)p+j-t\right)+i+1} v_{3\left((3t+1)p+j-2t\right)+i}, \tag{71}$$

where $t \in \mathbb{N}_0$, $j = \overline{2t, 5t}$;

$$z_{3(3t+2)m+3j+i+1} = z_{3j+i+1-3(3t+2)} \prod_{p=0}^{m} w_{3\left((3t+2)p+j\right)+i+1} u_{3\left((3t+2)p+j-t\right)+i-1} v_{3\left((3t+2)p+j-2t-1\right)+i}, \tag{72}$$

where $t \in \mathbb{N}_0$, $j = \overline{2t+1, 5t+2}$.
For $l = 1$, using (30)-(32) in (64)-(72), for $m \in \mathbb{N}_0$, $i \in \{-1, 0, 1\}$, we get

$$\begin{aligned} x_{3(3t)m+3j+i} &= x_{3j-3(3t)+i} \prod_{p=0}^{m} u_{3(3tp+j)+i} v_{3(3tp+j-t)+i} w_{3(3tp+j-2t)+i} \\ &= x_{3j-3(3t)+i} \prod_{p=0}^{m} \frac{c\gamma g x_i s_{3tp+j-1} + (A_1 x_i + B_1 y_{i-3t}) s_{3tp+j}}{(C_1 x_i - A_1 y_{i-3t}) s_{3tp+j} + y_{i-3t} s_{3tp+j+1}} \\ &\times \frac{c\gamma g y_i s_{3tp+j-t-1} + (A_2 y_i + B_2 z_{i-3t}) s_{3tp+j-t}}{(C_2 y_i - A_2 z_{i-3t}) s_{3tp+j-t} + z_{i-3t} s_{3tp+j-t+1}} \\ &\times \frac{c\gamma g z_i s_{3tp+j-2t-1} + (A_3 z_i + B_3 x_{i-3t}) s_{3tp+j-2t}}{(C_3 z_i - A_3 x_{i-3t}) s_{3tp+j-2t} + x_{i-3t} s_{3tp+j-2t+1}}, \end{aligned} \tag{73}$$

where $t \in \mathbb{N}$, $j = \overline{2t, 5t-1}$;

$$\begin{aligned} x_{3(3t+1)m+3j+1} &= x_{3j-3(3t+1)+1} \prod_{p=0}^{m} u_{3((3t+1)p+j)+1} v_{3((3t+1)p+j-t)} w_{3((3t+1)p+j-2t)-1} \\ &= x_{3j-3(3t+1)+1} \prod_{p=0}^{m} \frac{c\gamma g x_1 s_{(3t+1)p+j-1} + (A_1 x_1 + B_1 y_{1-(3t+1)}) s_{(3t+1)p+j}}{(C_1 x_1 - A_1 y_{1-(3t+1)}) s_{(3t+1)p+j} + y_{1-(3t+1)} s_{(3t+1)p+j+1}} \\ &\times \frac{c\gamma g y_0 s_{(3t+1)p+j-t-1} + (A_2 y_0 + B_2 z_{-(3t+1)}) s_{(3t+1)p+j-t}}{(C_2 y_0 - A_2 z_{-(3t+1)}) s_{(3t+1)p+j-t} + z_{-(3t+1)} s_{(3t+1)p+j-t+1}} \\ &\times \frac{c\gamma g z_{-1} s_{(3t+1)p+j-2t-1} + (A_3 z_{-1} + B_3 x_{-1-(3t+1)}) s_{(3t+1)p+j-2t}}{(C_3 z_{-1} - A_3 x_{-1-(3t+1)}) s_{(3t+1)p+j-2t} + x_{-1-(3t+1)} s_{(3t+1)p+j-2t+1}}, \end{aligned} \tag{74}$$

$$\begin{aligned} x_{3(3t+1)m+3j+2} &= x_{3j-3(3t+1)+2} \prod_{p=0}^{m} u_{3((3t+1)p+j+1)-1} v_{3((3t+1)p+j-t)+1} w_{3((3t+1)p+j-2t)} \\ &= x_{3j-3(3t+1)+2} \prod_{p=0}^{m} \frac{c\gamma g x_{-1} s_{(3t+1)p+j} + (A_1 x_{-1} + B_1 y_{-1-(3t+1)}) s_{(3t+1)p+j+1}}{(C_1 x_{-1} - A_1 y_{-1-(3t+1)}) s_{(3t+1)p+j+1} + y_{-1-(3t+1)} s_{(3t+1)p+j+2}} \\ &\times \frac{c\gamma g y_1 s_{(3t+1)p+j-t-1} + (A_2 y_1 + B_2 z_{1-(3t+1)}) s_{(3t+1)p+j-t}}{(C_2 y_1 - A_2 z_{1-(3t+1)}) s_{(3t+1)p+j-t} + z_{1-(3t+1)} s_{(3t+1)p+j-t+1}} \\ &\times \frac{c\gamma g z_0 s_{(3t+1)p+j-2t-1} + (A_3 z_0 + B_3 x_{-(3t+1)}) s_{(3t+1)p+j-2t}}{(C_3 z_0 - A_3 x_{-(3t+1)}) s_{(3t+1)p+j-2t} + x_{-(3t+1)} s_{(3t+1)p+j-2t+1}}, \end{aligned} \tag{75}$$

$$\begin{aligned}x_{3(3t+1)m+3j+3} &= x_{3j-3(3t+1)+3}\prod_{p=0}^{m} u_{3((3t+1)p+j+1)}v_{3((3t+1)p+j-t+1)-1}w_{3((3t+1)p+j-2t)+1}\\ &= x_{3j-3(3t+1)+3}\prod_{p=0}^{m}\frac{c\gamma g x_0 s_{(3t+1)p+j}+(A_1x_0+B_1y_{-(3t+1)})s_{(3t+1)p+j+1}}{(C_1x_0-A_1y_{-(3t+1)})s_{(3t+1)p+j+1}+y_{-(3t+1)}s_{(3t+1)p+j+2}}\\ &\times \frac{c\gamma g y_{-1}s_{(3t+1)p+j-t}+(A_2y_{-1}+B_2z_{-1-(3t+1)})s_{(3t+1)p+j-t+1}}{(C_2y_{-1}-A_2z_{-1-(3t+1)})s_{(3t+1)p+j-t+1}+z_{-1-(3t+1)}s_{(3t+1)p+j-t+2}}\\ &\times \frac{c\gamma g z_1 s_{(3t+1)p+j-2t-1}+(A_3z_1+B_3x_{1-(3t+1)})s_{(3t+1)p+j-2t}}{(C_3z_1-A_3x_{1-(3t+1)})s_{(3t+1)p+j-2t}+x_{1-(3t+1)}s_{(3t+1)p+j-2t+1}},\end{aligned} \tag{76}$$

where $t\in\mathbb{N}_0$, $j=\overline{2t,5t}$;

$$\begin{aligned}x_{3(3t+2)m+3j} &= x_{3j-3(3t+2)}\prod_{p=0}^{m} u_{3((3t+2)p+j)}v_{3((3t+2)p+j-t-1)+1}w_{3((3t+2)p+j-2t-1)-1}\\ &= x_{3j-3(3t+2)}\prod_{p=0}^{m}\frac{c\gamma g x_0 s_{(3t+2)p+j-1}+(A_1x_0+B_1y_{-(3t+2)})s_{(3t+2)p+j}}{(C_1x_0-A_1y_{-(3t+2)})s_{(3t+2)p+j}+y_{-(3t+2)}s_{(3t+2)p+j+1}}\\ &\times \frac{c\gamma g y_{1}s_{(3t+2)p+j-t-2}+(A_2y_{1}+B_2z_{1-(3t+2)})s_{(3t+2)p+j-t-1}}{(C_2y_{1}-A_2z_{1-(3t+2)})s_{(3t+2)p+j-t-1}+z_{1-(3t+2)}s_{(3t+2)p+j-t}}\\ &\times \frac{c\gamma g z_{-1} s_{(3t+2)p+j-2t-2}+(A_3z_{-1}+B_3x_{-1-(3t+2)})s_{(3t+2)p+j-2t-1}}{(C_3z_{-1}-A_3x_{-1-(3t+2)})s_{(3t+2)p+j-2t-1}+x_{-1-(3t+2)}s_{(3t+2)p+j-2t}},\end{aligned} \tag{77}$$

$$\begin{aligned}x_{3(3t+2)m+3j+1} &= x_{3j-3(3t+2)+1}\prod_{p=0}^{m} u_{3((3t+2)p+j)+1}v_{3((3t+2)p+j-t)-1}w_{3((3t+2)p+j-2t-1)}\\ &= x_{3j-3(3t+2)+1}\prod_{p=0}^{m}\frac{c\gamma g x_1 s_{(3t+2)p+j-1}+(A_1x_1+B_1y_{1-(3t+2)})s_{(3t+2)p+j}}{(C_1x_1-A_1y_{1-(3t+2)})s_{(3t+2)p+j}+y_{1-(3t+2)}s_{(3t+2)p+j+1}}\\ &\times \frac{c\gamma g y_{-1}s_{(3t+2)p+j-t-1}+(A_2y_{-1}+B_2z_{-1-(3t+2)})s_{(3t+2)p+j-t}}{(C_2y_{-1}-A_2z_{-1-(3t+2)})s_{(3t+2)p+j-t}+z_{-1-(3t+2)}s_{(3t+2)p+j-t+1}}\\ &\times \frac{c\gamma g z_{0} s_{(3t+2)p+j-2t-2}+(A_3z_{0}+B_3x_{-(3t+2)})s_{(3t+2)p+j-2t-1}}{(C_3z_{0}-A_3x_{-(3t+2)})s_{(3t+2)p+j-2t-1}+x_{-(3t+2)}s_{(3t+2)p+j-2t}},\end{aligned} \tag{78}$$

$$\begin{aligned}x_{3(3t+2)m+3j+2} &= x_{3j-3(3t+2)+2}\prod_{p=0}^{m} u_{3((3t+2)p+j+1)-1}v_{3((3t+2)p+j-t)}w_{3((3t+2)p+j-2t-1)+1}\\ &= x_{3j-3(3t+2)+2}\prod_{p=0}^{m}\frac{c\gamma g x_{-1} s_{(3t+2)p+j}+(A_1x_{-1}+B_1y_{-1-(3t+2)})s_{(3t+2)p+j+1}}{(C_1x_{-1}-A_1y_{-1-(3t+2)})s_{(3t+2)p+j+1}+y_{-1-(3t+2)}s_{(3t+2)p+j+2}}\\ &\times \frac{c\gamma g y_{0}s_{(3t+2)p+j-t-1}+(A_2y_{0}+B_2z_{-(3t+2)})s_{(3t+2)p+j-t}}{(C_2y_{0}-A_2z_{-(3t+2)})s_{(3t+2)p+j-t}+z_{-(3t+2)}s_{(3t+2)p+j-t+1}}\\ &\times \frac{c\gamma g z_{1} s_{(3t+2)p+j-2t-2}+(A_3z_{1}+B_3x_{1-(3t+2)})s_{(3t+2)p+j-2t-1}}{(C_3z_{1}-A_3x_{1-(3t+2)})s_{(3t+2)p+j-2t-1}+x_{1-(3t+2)}s_{(3t+2)p+j-2t}},\end{aligned} \tag{79}$$

where $t \in \mathbb{N}_0$, $j = \overline{2t+1, 5t+2}$;

$$\begin{aligned} y_{3(3t)m+3j+i} &= y_{3j-3(3t)+i} \prod_{p=0}^{m} v_{3(3tp+j)+i} w_{3(3tp+j-t)+i} u_{3(3tp+j-2t)+i} \\ &= y_{3j-3(3t)+i} \prod_{p=0}^{m} \frac{c\gamma g y_i s_{3tp+j-1} + (A_2 y_i + B_2 z_{i-3t}) s_{3tp+j}}{(C_2 y_i - A_2 z_{i-3t}) s_{3tp+j} + z_{i-3t} s_{3tp+j+1}} \\ &\times \frac{c\gamma g z_i s_{3tp+j-t-1} + (A_3 z_i + B_3 x_{i-3t}) s_{3tp+j-t}}{(C_3 z_i - A_3 x_{i-3t}) s_{3tp+j-t} + x_{i-3t} s_{3tp+j-t+1}} \\ &\times \frac{c\gamma g x_i s_{3tp+j-2t-1} + (A_1 x_i + B_1 y_{i-3t}) s_{3tp+j-2t}}{(C_1 x_i - A_1 y_{i-3t}) s_{3tp+j-2t} + y_{i-3t} s_{3tp+j-2t+1}}, \end{aligned} \tag{80}$$

where $t \in \mathbb{N}$, $j = \overline{2t, 5t-1}$;

$$\begin{aligned} y_{3(3t+1)m+3j+1} &= y_{3j-3(3t+1)+1} \prod_{p=0}^{m} v_{3((3t+1)p+j)+1} w_{3((3t+1)p+j-t)} u_{3((3t+1)p+j-2t)-1} \\ &= y_{3j-3(3t+1)+1} \prod_{p=0}^{m} \frac{c\gamma g y_1 s_{(3t+1)p+j-1} + (A_2 y_1 + B_2 z_{1-(3t+1)}) s_{(3t+1)p+j}}{(C_2 y_1 - A_2 z_{1-(3t+1)}) s_{(3t+1)p+j} + z_{1-(3t+1)} s_{(3t+1)p+j+1}} \\ &\times \frac{c\gamma g z_0 s_{(3t+1)p+j-t-1} + (A_3 z_0 + B_3 x_{-(3t+1)}) s_{(3t+1)p+j-t}}{(C_3 z_0 - A_3 x_{-(3t+1)}) s_{(3t+1)p+j-t} + x_{-(3t+1)} s_{(3t+1)p+j-t+1}} \\ &\times \frac{c\gamma g x_{-1} s_{(3t+1)p+j-2t-1} + (A_1 x_{-1} + B_1 y_{-1-(3t+1)}) s_{(3t+1)p+j-2t}}{(C_1 x_{-1} - A_1 y_{-1-(3t+1)}) s_{(3t+1)p+j-2t} + y_{-1-(3t+1)} s_{(3t+1)p+j-2t+1}}, \end{aligned} \tag{81}$$

$$\begin{aligned} y_{3(3t+1)m+3j+2} &= y_{3j-3(3t+1)+2} \prod_{p=0}^{m} v_{3((3t+1)p+j+1)-1} w_{3((3t+1)p+j-t)+1} u_{3((3t+1)p+j-2t)} \\ &= y_{3j-3(3t+1)+2} \prod_{p=0}^{m} \frac{c\gamma g y_{-1} s_{(3t+1)p+j} + (A_2 y_{-1} + B_2 z_{-1-(3t+1)}) s_{(3t+1)p+j+1}}{(C_2 y_{-1} - A_2 z_{-1-(3t+1)}) s_{(3t+1)p+j+1} + z_{-1-(3t+1)} s_{(3t+1)p+j+2}} \\ &\times \frac{c\gamma g z_1 s_{(3t+1)p+j-t-1} + (A_3 z_1 + B_3 x_{1-(3t+1)}) s_{(3t+1)p+j-t}}{(C_3 z_1 - A_3 x_{1-(3t+1)}) s_{(3t+1)p+j-t} + x_{1-(3t+1)} s_{(3t+1)p+j-t+1}} \\ &\times \frac{c\gamma g x_0 s_{(3t+1)p+j-2t-1} + (A_1 x_0 + B_1 y_{-(3t+1)}) s_{(3t+1)p+j-2t}}{(C_1 x_0 - A_1 y_{-(3t+1)}) s_{(3t+1)p+j-2t} + y_{-(3t+1)} s_{(3t+1)p+j-2t+1}}, \end{aligned} \tag{82}$$

$$\begin{aligned} y_{3(3t+1)m+3j+3} &= y_{3j-3(3t+1)+3} \prod_{p=0}^{m} v_{3((3t+1)p+j+1)} w_{3((3t+1)p+j-t+1)-1} u_{3((3t+1)p+j-2t)+1} \\ &= y_{3j-3(3t+1)+3} \prod_{p=0}^{m} \frac{c\gamma g y_0 s_{(3t+1)p+j} + (A_2 y_0 + B_2 z_{-(3t+1)}) s_{(3t+1)p+j+1}}{(C_2 y_0 - A_2 z_{-(3t+1)}) s_{(3t+1)p+j+1} + z_{-(3t+1)} s_{(3t+1)p+j+2}} \\ &\times \frac{c\gamma g z_{-1} s_{(3t+1)p+j-t} + (A_3 z_{-1} + B_3 x_{-1-(3t+1)}) s_{(3t+1)p+j-t+1}}{(C_3 z_{-1} - A_3 x_{-1-(3t+1)}) s_{(3t+1)p+j-t+1} + x_{-1-(3t+1)} s_{(3t+1)p+j-t+2}} \\ &\times \frac{c\gamma g x_1 s_{(3t+1)p+j-2t-1} + (A_1 x_1 + B_1 y_{1-(3t+1)}) s_{(3t+1)p+j-2t}}{(C_1 x_1 - A_1 y_{1-(3t+1)}) s_{(3t+1)p+j-2t} + y_{1-(3t+1)} s_{(3t+1)p+j-2t+1}}, \end{aligned} \tag{83}$$

where $t \in \mathbb{N}_0$, $j = \overline{2t, 5t}$;

$$\begin{aligned} y_{3(3t+2)m+3j} &= y_{3j-3(3t+2)} \prod_{p=0}^{m} v_{3((3t+2)p+j)} w_{3((3t+2)p+j-t-1)+1} u_{3((3t+2)p+j-2t-1)-1} \\ &= y_{3j-3(3t+2)} \prod_{p=0}^{m} \frac{c\gamma g y_0 s_{(3t+2)p+j-1} + (A_2 y_0 + B_2 z_{-(3t+2)}) s_{(3t+2)p+j}}{(C_2 y_0 - A_2 z_{-(3t+2)}) s_{(3t+2)p+j} + z_{-(3t+2)} s_{(3t+2)p+j+1}} \\ &\times \frac{c\gamma g z_1 s_{(3t+2)p+j-t-2} + (A_3 z_1 + B_3 x_{1-(3t+2)}) s_{(3t+2)p+j-t-1}}{(C_3 z_1 - A_3 x_{1-(3t+2)}) s_{(3t+2)p+j-t-1} + x_{1-(3t+2)} s_{(3t+2)p+j-t}} \\ &\times \frac{c\gamma g x_{-1} s_{(3t+2)p+j-2t-2} + (A_1 x_{-1} + B_1 y_{-1-(3t+2)}) s_{(3t+2)p+j-2t-1}}{(C_1 x_{-1} - A_1 y_{-1-(3t+2)}) s_{(3t+2)p+j-2t-1} + y_{-1-(3t+2)} s_{(3t+2)p+j-2t}}, \end{aligned} \tag{84}$$

$$\begin{aligned} y_{3(3t+2)m+3j+1} &= y_{3j-3(3t+2)+1} \prod_{p=0}^{m} v_{3((3t+2)p+j)+1} w_{3((3t+2)p+j-t)-1} u_{3((3t+2)p+j-2t-1)} \\ &= y_{3j-3(3t+2)+1} \prod_{p=0}^{m} \frac{c\gamma g y_1 s_{(3t+2)p+j-1} + (A_2 y_1 + B_2 z_{1-(3t+2)}) s_{(3t+2)p+j}}{(C_2 y_1 - A_2 z_{1-(3t+2)}) s_{(3t+2)p+j} + z_{1-(3t+2)} s_{(3t+2)p+j+1}} \\ &\times \frac{c\gamma g z_{-1} s_{(3t+2)p+j-t-1} + (A_3 z_{-1} + B_3 x_{-1-(3t+2)}) s_{(3t+2)p+j-t}}{(C_3 z_{-1} - A_3 x_{-1-(3t+2)}) s_{(3t+2)p+j-t} + x_{-1-(3t+2)} s_{(3t+2)p+j-t+1}} \\ &\times \frac{c\gamma g x_0 s_{(3t+2)p+j-2t-2} + (A_1 x_0 + B_1 y_{-(3t+2)}) s_{(3t+2)p+j-2t-1}}{(C_1 x_0 - A_1 y_{-(3t+2)}) s_{(3t+2)p+j-2t-1} + y_{-(3t+2)} s_{(3t+2)p+j-2t}}, \end{aligned} \tag{85}$$

$$\begin{aligned} y_{3(3t+2)m+3j+2} &= y_{3j-3(3t+2)+2} \prod_{p=0}^{m} v_{3((3t+2)p+j+1)-1} w_{3((3t+2)p+j-t)} u_{3((3t+2)p+j-2t-1)+1} \\ &= y_{3j-3(3t+2)+2} \prod_{p=0}^{m} \frac{c\gamma g y_{-1} s_{(3t+2)p+j} + (A_2 y_{-1} + B_2 z_{-1-(3t+2)}) s_{(3t+2)p+j+1}}{(C_2 y_{-1} - A_2 z_{-1-(3t+2)}) s_{(3t+2)p+j+1} + z_{-1-(3t+2)} s_{(3t+2)p+j+2}} \\ &\times \frac{c\gamma g z_0 s_{(3t+2)p+j-t-1} + (A_3 z_0 + B_3 x_{-(3t+2)}) s_{(3t+2)p+j-t}}{(C_3 z_0 - A_3 x_{-(3t+2)}) s_{(3t+2)p+j-t} + x_{-(3t+2)} s_{(3t+2)p+j-t+1}} \\ &\times \frac{c\gamma g x_1 s_{(3t+2)p+j-2t-2} + (A_1 x_1 + B_1 y_{1-(3t+2)}) s_{(3t+2)p+j-2t-1}}{(C_1 x_1 - A_1 y_{1-(3t+2)}) s_{(3t+2)p+j-2t-1} + y_{1-(3t+2)} s_{(3t+2)p+j-2t}}, \end{aligned} \tag{86}$$

where $t \in \mathbb{N}_0$, $j = \overline{2t+1, 5t+2}$;

$$\begin{aligned} z_{3(3t)m+3j+i} &= z_{3j-3(3t)+i} \prod_{p=0}^{m} w_{3(3tp+j)+i} u_{3(3tp+j-t)+i} v_{3(3tp+j-2t)+i} \\ &= z_{3j-3(3t)+i} \prod_{p=0}^{m} \frac{c\gamma g z_i s_{3tp+j-1} + (A_3 z_i + B_3 x_{i-3t}) s_{3tp+j}}{(C_3 z_i - A_3 x_{i-3t}) s_{3tp+j} + x_{i-3t} s_{3tp+j+1}} \\ &\times \frac{c\gamma g x_i s_{3tp+j-t-1} + (A_1 x_i + B_1 y_{i-3t}) s_{3tp+j-t}}{(C_1 x_i - A_1 y_{i-3t}) s_{3tp+j-t} + y_{i-3t} s_{3tp+j-t+1}} \\ &\times \frac{c\gamma g y_i s_{3tp+j-2t-1} + (A_2 y_i + B_2 z_{i-3t}) s_{3tp+j-2t}}{(C_2 y_i - A_2 z_{i-3t}) s_{3tp+j-2t} + z_{i-3t} s_{3tp+j-2t+1}}, \end{aligned} \tag{87}$$

where $t \in \mathbb{N}$, $j = \overline{2t, 5t-1}$;

$$\begin{aligned} z_{3(3t+1)m+3j+1} &= z_{3j-3(3t+1)+1} \prod_{p=0}^{m} w_{3((3t+1)p+j)+1} u_{3((3t+1)p+j-t)} v_{3((3t+1)p+j-2t)-1} \\ &= z_{3j-3(3t+1)+1} \prod_{p=0}^{m} \frac{c\gamma g z_1 s_{(3t+1)p+j-1} + (A_3 z_1 + B_3 x_{1-(3t+1)}) s_{(3t+1)p+j}}{(C_3 z_1 - A_3 x_{1-(3t+1)}) s_{(3t+1)p+j} + x_{1-(3t+1)} s_{(3t+1)p+j+1}} \\ &\times \frac{c\gamma g x_0 s_{(3t+1)p+j-t-1} + (A_1 x_0 + B_1 y_{-(3t+1)}) s_{(3t+1)p+j-t}}{(C_1 x_0 - A_1 y_{-(3t+1)}) s_{(3t+1)p+j-t} + y_{-(3t+1)} s_{(3t+1)p+j-t+1}} \\ &\times \frac{c\gamma g y_{-1} s_{(3t+1)p+j-2t-1} + (A_2 y_{-1} + B_2 z_{-1-(3t+1)}) s_{(3t+1)p+j-2t}}{(C_2 y_{-1} - A_2 z_{-1-(3t+1)}) s_{(3t+1)p+j-2t} + z_{-1-(3t+1)} s_{(3t+1)p+j-2t+1}}, \end{aligned} \tag{88}$$

$$\begin{aligned} z_{3(3t+1)m+3j+2} &= z_{3j-3(3t+1)+2} \prod_{p=0}^{m} w_{3((3t+1)p+j+1)-1} u_{3((3t+1)p+j-t)+1} v_{3((3t+1)p+j-2t)} \\ &= z_{3j-3(3t+1)+2} \prod_{p=0}^{m} \frac{c\gamma g z_{-1} s_{(3t+1)p+j} + (A_3 z_{-1} + B_3 x_{-1-(3t+1)}) s_{(3t+1)p+j+1}}{(C_3 z_{-1} - A_3 x_{-1-(3t+1)}) s_{(3t+1)p+j+1} + x_{-1-(3t+1)} s_{(3t+1)p+j+2}} \\ &\times \frac{c\gamma g x_1 s_{(3t+1)p+j-t-1} + (A_1 x_1 + B_1 y_{1-(3t+1)}) s_{(3t+1)p+j-t}}{(C_1 x_1 - A_1 y_{1-(3t+1)}) s_{(3t+1)p+j-t} + y_{1-(3t+1)} s_{(3t+1)p+j-t+1}} \\ &\times \frac{c\gamma g y_0 s_{(3t+1)p+j-2t-1} + (A_2 y_0 + B_2 z_{-(3t+1)}) s_{(3t+1)p+j-2t}}{(C_2 y_0 - A_2 z_{-(3t+1)}) s_{(3t+1)p+j-2t} + z_{-(3t+1)} s_{(3t+1)p+j-2t+1}}, \end{aligned} \tag{89}$$

$$\begin{aligned} z_{3(3t+1)m+3j+3} &= z_{3j-3(3t+1)+3} \prod_{p=0}^{m} w_{3((3t+1)p+j+1)} u_{3((3t+1)p+j-t+1)-1} v_{3((3t+1)p+j-2t)+1} \\ &= z_{3j-3(3t+1)+3} \prod_{p=0}^{m} \frac{c\gamma g z_0 s_{(3t+1)p+j} + (A_3 z_0 + B_3 x_{-(3t+1)}) s_{(3t+1)p+j+1}}{(C_3 z_0 - A_3 x_{-(3t+1)}) s_{(3t+1)p+j+1} + x_{-(3t+1)} s_{(3t+1)p+j+2}} \\ &\times \frac{c\gamma g x_{-1} s_{(3t+1)p+j-t} + (A_1 x_{-1} + B_1 y_{-1-(3t+1)}) s_{(3t+1)p+j-t+1}}{(C_1 x_{-1} - A_1 y_{-1-(3t+1)}) s_{(3t+1)p+j-t+1} + y_{-1-(3t+1)} s_{(3t+1)p+j-t+2}} \\ &\times \frac{c\gamma g y_1 s_{(3t+1)p+j-2t-1} + (A_2 y_1 + B_2 z_{1-(3t+1)}) s_{(3t+1)p+j-2t}}{(C_2 y_1 - A_2 z_{1-(3t+1)}) s_{(3t+1)p+j-2t} + z_{1-(3t+1)} s_{(3t+1)p+j-2t+1}}, \end{aligned} \tag{90}$$

where $t \in \mathbb{N}_0$, $j = \overline{2t, 5t}$;

$$\begin{aligned} z_{3(3t+2)m+3j} &= z_{3j-3(3t+2)} \prod_{p=0}^{m} w_{3((3t+2)p+j)} u_{3((3t+2)p+j-t-1)+1} v_{3((3t+2)p+j-2t-1)-1} \\ &= z_{3j-3(3t+2)} \prod_{p=0}^{m} \frac{c\gamma g z_0 s_{(3t+2)p+j-1} + (A_3 z_0 + B_3 x_{-(3t+2)}) s_{(3t+2)p+j}}{(C_3 z_0 - A_3 x_{-(3t+2)}) s_{(3t+2)p+j} + x_{-(3t+2)} s_{(3t+2)p+j+1}} \\ &\times \frac{c\gamma g x_1 s_{(3t+2)p+j-t-2} + (A_1 x_1 + B_1 y_{1-(3t+2)}) s_{(3t+2)p+j-t-1}}{(C_1 x_1 - A_1 y_{1-(3t+2)}) s_{(3t+2)p+j-t-1} + y_{1-(3t+2)} s_{(3t+2)p+j-t}} \\ &\times \frac{c\gamma g y_{-1} s_{(3t+2)p+j-2t-2} + (A_2 y_{-1} + B_2 z_{-1-(3t+2)}) s_{(3t+2)p+j-2t-1}}{(C_2 y_{-1} - A_2 z_{-1-(3t+2)}) s_{(3t+2)p+j-2t-1} + z_{-1-(3t+2)} s_{(3t+2)p+j-2t}}, \end{aligned} \tag{91}$$

$$\begin{aligned}
z_{3(3t+2)m+3j+1} &= z_{3j-3(3t+2)+1}\prod_{p=0}^{m} w_{3((3t+2)p+j)+1}u_{3((3t+2)p+j-t)-1}v_{3((3t+2)p+j-2t-1)} \\
&= z_{3j-3(3t+2)+1}\prod_{p=0}^{m}\frac{c\gamma g z_1 s_{(3t+2)p+j-1}+(A_3z_1+B_3x_{1-(3t+2)})s_{(3t+2)p+j}}{(C_3z_1-A_3x_{1-(3t+2)})s_{(3t+2)p+j}+x_{1-(3t+2)}s_{(3t+2)p+j+1}} \\
&\times \frac{c\gamma g x_{-1} s_{(3t+2)p+j-t-1}+(A_1x_{-1}+B_1y_{-1-(3t+2)})s_{(3t+2)p+j-t}}{(C_1x_{-1}-A_1y_{-1-(3t+2)})s_{(3t+2)p+j-t}+y_{-1-(3t+2)}s_{(3t+2)p+j-t+1}} \\
&\times \frac{c\gamma g y_0 s_{(3t+2)p+j-2t-2}+(A_2y_0+B_2z_{-(3t+2)})s_{(3t+2)p+j-2t-1}}{(C_2y_0-A_2z_{-(3t+2)})s_{(3t+2)p+j-2t-1}+z_{-(3t+2)}s_{(3t+2)p+j-2t}},
\end{aligned} \tag{92}$$

$$\begin{aligned}
z_{3(3t+2)m+3j+2} &= z_{3j-3(3t+2)+2}\prod_{p=0}^{m} w_{3((3t+2)p+j+1)-1}u_{3((3t+2)p+j-t)}v_{3((3t+2)p+j-2t-1)+1} \\
&= z_{3j-3(3t+2)+2}\prod_{p=0}^{m}\frac{c\gamma g z_{-1} s_{(3t+2)p+j}+(A_3z_{-1}+B_3x_{-1-(3t+2)})s_{(3t+2)p+j+1}}{(C_3z_{-1}-A_3x_{-1-(3t+2)})s_{(3t+2)p+j+1}+x_{-1-(3t+2)}s_{(3t+2)p+j+2}} \\
&\times \frac{c\gamma g x_0 s_{(3t+2)p+j-t-1}+(A_1x_0+B_1y_{-(3t+2)})s_{(3t+2)p+j-t}}{(C_1x_0-A_1y_{-(3t+2)})s_{(3t+2)p+j-t}+y_{-(3t+2)}s_{(3t+2)p+j-t+1}} \\
&\times \frac{c\gamma g y_1 s_{(3t+2)p+j-2t-2}+(A_2y_1+B_2z_{1-(3t+2)})s_{(3t+2)p+j-2t-1}}{(C_2y_1-A_2z_{1-(3t+2)})s_{(3t+2)p+j-2t-1}+z_{1-(3t+2)}s_{(3t+2)p+j-2t}},
\end{aligned} \tag{93}$$

where $t \in \mathbb{N}_0$, $j = \overline{2t+1, 5t+2}$.

4. An application

Now, we will give theoretical explanations for the formulas of solutions of difference equations systems given in [12] as an application of the main results in Section 2. First, we will derive the solution forms of the system (5) with $k = 1$, $l = 2$, $a = \alpha = e = 0$, $b = c = \beta = \gamma = f = g = 1$, that is, the system

$$x_n = \frac{y_{n-1}x_{n-3}}{x_{n-3}+z_{n-2}},\ y_n = \frac{z_{n-1}y_{n-3}}{y_{n-3}+x_{n-2}},\ z_n = \frac{x_{n-1}z_{n-3}}{z_{n-3}+y_{n-2}},\ n \in \mathbb{N}_0. \tag{94}$$

given in [12], through analytical approach. Also, the general solutions of the system (94) are expressed in terms of Fibonacci numbers. By using equations in (40)-(42) we have that every well-defined solution of system (94) can be written in the form

$$x_{6m+6j+j_2} = x_{6j+j_2-6}\prod_{p=0}^{m}\prod_{n=1}^{2} u_{6p+6j+j_2-(3n-3)}v_{6p+6j+j_2-(3n-2)}w_{6p+6j+j_2-(3n-1)}, \tag{95}$$

$$y_{6m+6j+j_2} = y_{6j+j_2-6}\prod_{p=0}^{m}\prod_{n=1}^{2} v_{6p+6j+j_2-(3n-3)}w_{6p+6j+j_2-(3n-2)}u_{6p+6j+j_2-(3n-1)}, \tag{96}$$

$$z_{6m+6j+j_2} = z_{6j+j_2-6}\prod_{p=0}^{m}\prod_{n=1}^{2} w_{6p+6j+j_2-(3n-3)}u_{6p+6j+j_2-(3n-2)}v_{6p+6j+j_2-(3n-1)}, \tag{97}$$

where $m \in \mathbb{N}_0$ and $6j + j_2 = \overline{3,8}$.
We get from (95)-(97), for $6j + j_2 = i + 5 = \overline{3,8}$,

$$x_{6m+i+5} = x_{i-1} \prod_{p=0}^{m} u_{6p+i+5} v_{6p+i+4} w_{6p+i+3} u_{6p+i+2} v_{6p+i+1} w_{6p+i}, \tag{98}$$

$$y_{6m+i+5} = y_{i-1} \prod_{p=0}^{m} v_{6p+i+5} w_{6p+i+4} u_{6p+i+3} v_{6p+i+2} w_{6p+i+1} u_{6p+i}, \tag{99}$$

$$z_{6m+i+5} = z_{i-1} \prod_{p=0}^{m} w_{6p+i+5} u_{6p+i+4} v_{6p+i+3} w_{6p+i+2} u_{6p+i+1} v_{6p+i}, \tag{100}$$

where $m \in \mathbb{N}_0$ and $i = \overline{-2,3}$.
By substituting the formulas in (30)-(32) into (98), we obtain

$$\begin{aligned} x_{6m+i+5} &= x_{i-1} \prod_{p=0}^{m} \frac{x_{i+5}s_{p-1} + (x_{i+5} + 2y_{i+4})s_p}{(2x_{i+5} - y_{i+4})s_p + y_{i+4}s_{p+1}} \frac{y_{i+4}s_{p-1} + (y_{i+4} + 2z_{i+3})s_p}{(2y_{i+4} - z_{i+3})s_p + z_{i+3}s_{p+1}} \frac{z_{i+3}s_{p-1} + (z_{i+3} + 2x_{i+2})s_p}{(2z_{i+3} - x_{i+2})s_p + x_{i+2}s_{p+1}} \\ &\times \frac{x_{i+2}s_{p-1} + (x_{i+2} + 2y_{i+1})s_p}{(2x_{i+2} - y_{i+1})s_p + y_{i+1}s_{p+1}} \frac{y_{i+1}s_{p-1} + (y_{i+1} + 2z_i)s_p}{(2y_{i+1} - z_i)s_p + z_i s_{p+1}} \frac{z_i s_{p-1} + (z_i + 2x_{i-1})s_p}{(2z_i - x_{i-1})s_p + x_{i-1}s_{p+1}}, \end{aligned} \tag{101}$$

for $m \in \mathbb{N}_0$, $i = \overline{-2,3}$. From (94), we have that

$$x_{i+2} = \frac{y_{i+1}x_{i-1}}{x_{i-1} + z_i}, \quad z_{i+3} = \frac{x_{i-1}z_i y_{i+1}}{(x_{i-1} + z_i)(z_i + y_{i+1})},$$

$$y_{i+4} = \frac{x_{i-1}z_i y_{i+1}}{(2x_{i-1} + z_i)(z_i + y_{i+1})}, \quad x_{i+5} = \frac{x_{i-1}z_i y_{i+1}}{(2x_{i-1} + z_i)(2z_i + y_{i+1})},$$

for $i = \overline{-2,3}$. From (101), after some calculations and by using the definition of the $(s_m)_{m\in\mathbb{N}_0}$ sequence, we get

$$x_{6m+i+5} = x_{i-1} \prod_{p=0}^{m} \frac{y_{i+1}s_{p-1} + (y_{i+1} + 2z_i)s_p}{y_{i+1}s_p + (2z_i + y_{i+1})s_{p+1}} \frac{z_i s_{p-1} + (z_i + 2x_{i-1})s_p}{z_i s_p + (2x_{i-1} + z_i)s_{p+1}}, \tag{102}$$

for $m \in \mathbb{N}_0$, $i = \overline{-2,3}$. From (26), we have

$$s_{m+1} - 4s_m - s_{m-1} = 0, \ m \in \mathbb{N}. \tag{103}$$

Employing $s_{-1} = s_1 - 4s_0 = 1$ in (102), we get

$$x_{6m+i+5} = \frac{x_{i-1}y_{i+1}z_i}{(2s_{m+1}z_i + (s_{m+1} + s_m)\, y_{i+1})(2s_{m+1}x_{i-1} + (s_{m+1} + s_m)\, z_i)}, \tag{104}$$

for $m \in \mathbb{N}_0$, $i = \overline{-2,3}$.

By substituting the formulas in (30)-(32) into (99), we obtain

$$\begin{aligned} y_{6m+i+5} &= y_{i-1} \prod_{p=0}^{m} \frac{y_{i+5}s_{p-1} + (y_{i+5} + 2z_{i+4})s_p}{(2y_{i+5} - z_{i+4})s_p + z_{i+4}s_{p+1}} \frac{z_{i+4}s_{p-1} + (z_{i+4} + 2x_{i+3})s_p}{(2z_{i+4} - x_{i+3})s_p + x_{i+3}s_{p+1}} \frac{x_{i+3}s_{p-1} + (x_{i+3} + 2y_{i+2})s_p}{(2x_{i+3} - y_{i+2})s_p + y_{i+2}s_{p+1}} \\ &\times \frac{y_{i+2}s_{p-1} + (y_{i+2} + 2z_{i+1})s_p}{(2y_{i+2} - z_{i+1})s_p + z_{i+1}s_{p+1}} \frac{z_{i+1}s_{p-1} + (z_{i+1} + 2x_i)s_p}{(2z_{i+1} - x_i)s_p + x_i s_{p+1}} \frac{x_i s_{p-1} + (x_i + 2y_{i-1})s_p}{(2x_i - y_{i-1})s_p + y_{i-1}s_{p+1}}, \end{aligned} \tag{105}$$

for $m \in \mathbb{N}_0$, $i = \overline{-2,3}$. From (94), we have that

$$y_{i+2} = \frac{z_{i+1}y_{i-1}}{y_{i-1}+x_i}, \quad x_{i+3} = \frac{y_{i-1}x_iz_{i+1}}{(y_{i-1}+x_i)(x_i+z_{i+1})},$$

$$z_{i+4} = \frac{y_{i-1}x_iz_{i+1}}{(2y_{i-1}+x_i)(x_i+z_{i+1})}, \quad y_{i+5} = \frac{y_{i-1}x_iz_{i+1}}{(2y_{i-1}+x_i)(2x_i+z_{i+1})},$$

for $i = \overline{-2,3}$. From (105), after some calculations and by using the definition of the $(s_m)_{m\in\mathbb{N}_0}$ sequence, we get

$$y_{6m+i+5} = y_{i-1}\prod_{p=0}^{m}\frac{z_{i+1}s_{p-1}+(z_{i+1}+2x_i)s_p}{z_{i+1}s_p+(2x_i+z_{i+1})s_{p+1}}\frac{x_is_{p-1}+(x_i+2y_{i-1})s_p}{x_is_p+(2y_{i-1}+x_i)s_{p+1}}, \tag{106}$$

for $m \in \mathbb{N}_0$, $i = \overline{-2,3}$. By using (103), we get

$$y_{6m+i+5} = \frac{y_{i-1}z_{i+1}x_i}{(2s_{m+1}x_i+(s_{m+1}+s_m)z_{i+1})(2s_{m+1}y_{i-1}+(s_{m+1}+s_m)x_i)}, \tag{107}$$

for $m \in \mathbb{N}_0$, $i = \overline{-2,3}$.

By substituting the formulas in (30)-(32) into (100), we obtain

$$\begin{aligned} z_{6m+i+5} &= z_{i-1}\prod_{p=0}^{m}\frac{z_{i+5}s_{p-1}+(z_{i+5}+2x_{i+4})s_p}{(2z_{i+5}-x_{i+4})s_p+x_{i+4}s_{p+1}}\frac{x_{i+4}s_{p-1}+(x_{i+4}+2y_{i+3})s_p}{(2x_{i+4}-y_{i+3})s_p+y_{i+3}s_{p+1}}\frac{y_{i+3}s_{p-1}+(y_{i+3}+2z_{i+2})s_p}{(2y_{i+3}-z_{i+2})s_p+z_{i+2}s_{p+1}} \\ &\times \frac{z_{i+2}s_{p-1}+(z_{i+2}+2x_{i+1})s_p}{(2z_{i+2}-x_{i+1})s_p+x_{i+1}s_{p+1}}\frac{x_{i+1}s_{p-1}+(x_{i+1}+2y_i)s_p}{(2x_{i+1}-y_i)s_p+y_is_{p+1}}\frac{y_is_{p-1}+(y_i+2z_{i-1})s_p}{(2y_i-z_{i-1})s_p+z_{i-1}s_{p+1}}, \end{aligned} \tag{108}$$

for $m \in \mathbb{N}_0$, $i = \overline{-2,3}$. From (94), we have that

$$z_{i+2} = \frac{x_{i+1}z_{i-1}}{z_{i-1}+y_i}, \quad y_{i+3} = \frac{z_{i-1}y_ix_{i+1}}{(z_{i-1}+y_i)(y_i+x_{i+1})},$$

$$x_{i+4} = \frac{z_{i-1}y_ix_{i+1}}{(2z_{i-1}+y_i)(y_i+x_{i+1})}, \quad z_{i+5} = \frac{z_{i-1}y_ix_{i+1}}{(2z_{i-1}+y_i)(2y_i+x_{i+1})},$$

for $i = \overline{-2,3}$. From (108), after some calculations and by using the definition of the $(s_m)_{m\in\mathbb{N}_0}$ sequence, we get

$$z_{6m+i+5} = z_{i-1}\prod_{p=0}^{m}\frac{x_{i+1}s_{p-1}+(x_{i+1}+2y_i)s_p}{x_{i+1}s_p+(2y_i+x_{i+1})s_{p+1}}\frac{y_is_{p-1}+(y_i+2z_{i-1})s_p}{y_is_p+(2z_{i-1}+y_i)s_{p+1}}, \tag{109}$$

for $m \in \mathbb{N}_0$, $i = \overline{-2,3}$. By using (103), we get

$$z_{6m+i+5} = \frac{z_{i-1}x_{i+1}y_i}{(2s_{m+1}y_i+(s_{m+1}+s_m)x_{i+1})(2s_{m+1}z_{i-1}+(s_{m+1}+s_m)y_i)}, \tag{110}$$

for $m \in \mathbb{N}_0$, $i = \overline{-2,3}$.
By using (104), (107) and (110) we have that every well-defined solutions of system (94) can be written in the form

$$x_{6m+3} = \frac{x_{-3}y_{-1}z_{-2}}{(2s_{m+1}z_{-2}+(s_{m+1}+s_m)y_{-1})(2s_{m+1}x_{-3}+(s_{m+1}+s_m)z_{-2})}, \tag{111}$$

$$x_{6m+4} = \frac{x_{-2}y_{-3}z_{-1}}{(2s_{m+1}x_{-2} + (3s_{m+1} + s_m)\, y_{-3})\,(2s_{m+1}x_{-2} + (s_{m+1} + s_m)\, z_{-1})}, \tag{112}$$

$$x_{6m+5} = \frac{x_{-1}y_{-2}z_{-3}}{(2s_{m+1}x_{-1} + (3s_{m+1} + s_m)\, y_{-2})\,(2s_{m+1}y_{-2} + (3s_{m+1} + s_m)\, z_{-3})}, \tag{113}$$

$$x_{6m+6} = \frac{x_{-3}y_{-1}z_{-2}}{((5s_{m+1} + s_m)\, x_{-3} + (3s_{m+1} + s_m)\, z_{-2})\,(2s_{m+1}y_{-1} + (3s_{m+1} + s_m)\, z_{-2})}, \tag{114}$$

$$x_{6m+7} = \frac{x_{-2}y_{-3}z_{-1}}{((5s_{m+1} + s_m)\, x_{-2} + (3s_{m+1} + s_m)\, z_{-1})\,((3s_{m+1} + s_m)\, x_{-2} + (5s_{m+1} + s_m)\, y_{-3})}, \tag{115}$$

$$x_{6m+8} = \frac{x_{-1}y_{-2}z_{-3}}{((8s_{m+1} + 2s_m)\, z_{-3} + (5s_{m+1} + s_m)\, y_{-2})\,((3s_{m+1} + s_m)\, x_{-1} + (5s_{m+1} + s_m)\, y_{-2})}, \tag{116}$$

$$y_{6m+3} = \frac{y_{-3}z_{-1}x_{-2}}{(2s_{m+1}x_{-2} + (s_{m+1} + s_m)\, z_{-1})\,(2s_{m+1}y_{-3} + (s_{m+1} + s_m)\, x_{-2})}, \tag{117}$$

$$y_{6m+4} = \frac{y_{-2}z_{-3}x_{-1}}{(2s_{m+1}y_{-2} + (3s_{m+1} + s_m)\, z_{-3})\,(2s_{m+1}y_{-2} + (s_{m+1} + s_m)\, x_{-1})}, \tag{118}$$

$$y_{6m+5} = \frac{y_{-1}z_{-2}x_{-3}}{(2s_{m+1}y_{-1} + (3s_{m+1} + s_m)\, z_{-2})\,(2s_{m+1}z_{-2} + (3s_{m+1} + s_m)\, x_{-3})}, \tag{119}$$

$$y_{6m+6} = \frac{y_{-3}z_{-1}x_{-2}}{((5s_{m+1} + s_m)\, y_{-3} + (3s_{m+1} + s_m)\, x_{-2})\,(2s_{m+1}z_{-1} + (3s_{m+1} + s_m)\, x_{-2})}, \tag{120}$$

$$y_{6m+7} = \frac{y_{-2}z_{-3}x_{-1}}{((5s_{m+1} + s_m)\, y_{-2} + (3s_{m+1} + s_m)\, x_{-1})\,((3s_{m+1} + s_m)\, y_{-2} + (5s_{m+1} + s_m)\, z_{-3})}, \tag{121}$$

$$y_{6m+8} = \frac{y_{-1}z_{-2}x_{-3}}{((8s_{m+1} + 2s_m)\, x_{-3} + (5s_{m+1} + s_m)\, z_{-2})\,((3s_{m+1} + s_m)\, y_{-1} + (5s_{m+1} + s_m)\, z_{-2})}, \tag{122}$$

$$z_{6m+3} = \frac{z_{-3}x_{-1}y_{-2}}{(2s_{m+1}y_{-2} + (s_{m+1} + s_m)\, x_{-1})\,(2s_{m+1}z_{-3} + (s_{m+1} + s_m)\, y_{-2})}, \tag{123}$$

$$z_{6m+4} = \frac{z_{-2}x_{-3}y_{-1}}{(2s_{m+1}z_{-2} + (3s_{m+1} + s_m)\, x_{-3})\,(2s_{m+1}z_{-2} + (s_{m+1} + s_m)\, y_{-1})}, \tag{124}$$

$$z_{6m+5} = \frac{z_{-1}x_{-2}y_{-3}}{(2s_{m+1}z_{-1} + (3s_{m+1} + s_m)\, x_{-2})\,(2s_{m+1}x_{-2} + (3s_{m+1} + s_m)\, y_{-3})}, \tag{125}$$

$$z_{6m+6} = \frac{z_{-3}x_{-1}y_{-2}}{((5s_{m+1} + s_m)\, z_{-3} + (3s_{m+1} + s_m)\, y_{-2})\,(2s_{m+1}x_{-1} + (3s_{m+1} + s_m)\, y_{-2})}, \tag{126}$$

$$z_{6m+7} = \frac{z_{-2}x_{-3}y_{-1}}{((5s_{m+1} + s_m)\, z_{-2} + (3s_{m+1} + s_m)\, y_{-1})\,((3s_{m+1} + s_m)\, z_{-2} + (5s_{m+1} + s_m)\, x_{-3})}, \tag{127}$$

and

$$z_{6m+8} = \frac{z_{-1}x_{-2}y_{-3}}{\left((8s_{m+1}+2s_m)\,y_{-3}+(5s_{m+1}+s_m)\,x_{-2}\right)\left((3s_{m+1}+s_m)\,z_{-1}+(5s_{m+1}+s_m)\,x_{-2}\right)}, \tag{128}$$

for $m \in \mathbb{N}_0$, where sequence of $(s_m)_{m\in\mathbb{N}_0}$ is satisfying in (103) difference equation with the initial conditions $s_0 = 0$ and $s_1 = 1$.
Binet Formula for (103)

$$s_m = \frac{\left(2+\sqrt{5}\right)^m - \left(2-\sqrt{5}\right)^m}{\left(2+\sqrt{5}\right)-\left(2-\sqrt{5}\right)}, \quad m \in \mathbb{N}_0. \tag{129}$$

Note that

$$\left(\frac{1\pm\sqrt{5}}{2}\right)^3 = 2 \pm \sqrt{5}.$$

Using this in (129) we obtain

$$s_m = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{3m} - \left(\frac{1-\sqrt{5}}{2}\right)^{3m}}{\left(\frac{1+\sqrt{5}}{2}\right)^{3} - \left(\frac{1-\sqrt{5}}{2}\right)^{3}} = \frac{f_{3m}}{2}, \quad m \in \mathbb{N}_0. \tag{130}$$

Using (130) into (111)-(128), we get

$$x_{6m+3} = \frac{x_{-3}y_{-1}z_{-2}}{\left(f_{3m+3}z_{-2}+f_{3m+2}y_{-1}\right)\left(f_{3m+3}x_{-3}+f_{3m+2}z_{-2}\right)}, \tag{131}$$

$$x_{6m+4} = \frac{x_{-2}y_{-3}z_{-1}}{\left(f_{3m+3}x_{-2}+f_{3m+4}y_{-3}\right)\left(f_{3m+3}x_{-2}+f_{3m+2}z_{-1}\right)}, \tag{132}$$

$$x_{6m+5} = \frac{x_{-1}y_{-2}z_{-3}}{\left(f_{3m+3}x_{-1}+f_{3m+4}y_{-2}\right)\left(f_{3m+3}y_{-2}+f_{3m+4}z_{-3}\right)}, \tag{133}$$

$$x_{6m+6} = \frac{x_{-3}y_{-1}z_{-2}}{\left(f_{3m+5}x_{-3}+f_{3m+4}z_{-2}\right)\left(f_{3m+3}y_{-1}+f_{3m+4}z_{-2}\right)}, \tag{134}$$

$$x_{6m+7} = \frac{x_{-2}y_{-3}z_{-1}}{\left(f_{3m+5}x_{-2}+f_{3m+4}z_{-1}\right)\left(f_{3m+4}x_{-2}+f_{3m+5}y_{-3}\right)}, \tag{135}$$

$$x_{6m+8} = \frac{x_{-1}y_{-2}z_{-3}}{\left(f_{3m+6}z_{-3}+f_{3m+5}y_{-2}\right)\left(f_{3m+4}x_{-1}+f_{3m+5}y_{-2}\right)}, \tag{136}$$

$$y_{6m+3} = \frac{y_{-3}z_{-1}x_{-2}}{\left(f_{3m+3}x_{-2}+f_{3m+2}z_{-1}\right)\left(f_{3m+3}y_{-3}+f_{3m+2}x_{-2}\right)}, \tag{137}$$

$$y_{6m+4} = \frac{y_{-2}z_{-3}x_{-1}}{\left(f_{3m+3}y_{-2}+f_{3m+4}z_{-3}\right)\left(f_{3m+3}y_{-2}+f_{3m+2}x_{-1}\right)}, \tag{138}$$

$$y_{6m+5} = \frac{y_{-1}z_{-2}x_{-3}}{\left(f_{3m+3}y_{-1}+f_{3m+4}z_{-2}\right)\left(f_{3m+3}z_{-2}+f_{3m+4}x_{-3}\right)}, \tag{139}$$

$$y_{6m+6} = \frac{y_{-3}z_{-1}x_{-2}}{(f_{3m+5}y_{-3} + f_{3m+4}x_{-2})(f_{3m+3}z_{-1} + f_{3m+4}x_{-2})}, \tag{140}$$

$$y_{6m+7} = \frac{y_{-2}z_{-3}x_{-1}}{(f_{3m+5}y_{-2} + f_{3m+4}x_{-1})(f_{3m+4}y_{-2} + f_{3m+5}z_{-3})}, \tag{141}$$

$$y_{6m+8} = \frac{y_{-1}z_{-2}x_{-3}}{(f_{3m+6}x_{-3} + f_{3m+5}z_{-2})(f_{3m+4}y_{-1} + f_{3m+5}z_{-2})}, \tag{142}$$

$$z_{6m+3} = \frac{z_{-3}x_{-1}y_{-2}}{(f_{3m+3}y_{-2} + f_{3m+2}x_{-1})(f_{3m+3}z_{-3} + f_{3m+2}y_{-2})}, \tag{143}$$

$$z_{6m+4} = \frac{z_{-2}x_{-3}y_{-1}}{(f_{3m+3}z_{-2} + f_{3m+4}x_{-3})(f_{3m+3}z_{-2} + f_{3m+2}y_{-1})}, \tag{144}$$

$$z_{6m+5} = \frac{z_{-1}x_{-2}y_{-3}}{(f_{3m+3}z_{-1} + f_{3m+4}x_{-2})(f_{3m+3}x_{-2} + f_{3m+4}y_{-3})}, \tag{145}$$

$$z_{6m+6} = \frac{z_{-3}x_{-1}y_{-2}}{(f_{3m+5}z_{-3} + f_{3m+4}y_{-2})(f_{3m+3}x_{-1} + f_{3m+4}y_{-2})}, \tag{146}$$

$$z_{6m+7} = \frac{z_{-2}x_{-3}y_{-1}}{(f_{3m+5}z_{-2} + f_{3m+4}y_{-1})(f_{3m+4}z_{-2} + f_{3m+5}x_{-3})}, \tag{147}$$

and

$$z_{6m+8} = \frac{z_{-1}x_{-2}y_{-3}}{(f_{3m+6}y_{-3} + f_{3m+5}x_{-2})(f_{3m+4}z_{-1} + f_{3m+5}x_{-2})}, \tag{148}$$

for $m \in \mathbb{N}_0$, where f_m is m-th Fibonacci number.
If we take $n-1$ instead of m in (131)-(148) and consider $\{f_m\}_{m=0}^{\infty} = \{0, 1, 1, 2, 3, 5, 8, 13, \dots\}$, the formulas of the solutions in (131)-(148) are the same as in Theorem 1 in [12].

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