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On Lyapunov-Type Inequalities for Nonlinear Hamiltonian-Type Problems

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Abstract. In this paper, we investigate new Lyapunov-type inequalities for Hamiltonian-type problem

$$\begin{cases} w' = \eta(x) w + \dfrac{|y|^{p-2} y}{\left(1+|y|^p\right)^{\frac{p-1}{p}}} \\ y' = -\sigma(x) \dfrac{|w|^{\beta-2} w}{\left(1+|w|^\beta\right)^{\frac{\beta-1}{\beta}}} - \eta(x) y \end{cases}$$

involving the p-prescribed curvature operator

$$\Phi_p(v) = \frac{|v|^{p-2} v}{\left(1+|v|^p\right)^{\frac{p-1}{p}}}, p > 1,$$

under Dirichlet boundary condition.

1. Introduction

In the last two decades, the investigation of the Emden-Fowler type differential equation

$$\left(|w'|^{q-2} w'\right)' + \sigma(x) |w|^{\beta-2} w = 0, \beta, q > 1, \tag{1}$$

has attracted considerable attention. Substituting $y = |w'|^{q-2} w'$ in Eq. (1), we get the system

$$\begin{cases} w' = |y|^{p-2} y, \\ y' = -\sigma(x) |w|^{\beta-2} w, \end{cases} \tag{2}$$

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where $\beta, p > 1$ and p is the conjugate number of q, i.e., $\frac{1}{p}+\frac{1}{q}=1$. Note that the inverse function of the p-Laplacian operator $\phi_p(v)=|v|^{p-2}v$ [1] is the q-Laplacian operator

$$\phi_q(v) := \phi_p^{-1}(v) = |v|^{q-2}v, \frac{1}{p}+\frac{1}{q}=1. \tag{3}$$

In this paper, we call

$$\Phi_p(v) = \frac{|v|^{p-2}v}{\left(1+|v|^p\right)^{\frac{p-1}{p}}}, v\in\mathbb{R}, \left|\Phi_p\right|<1, \tag{4}$$

the p-prescribed curvature operator inspired by the papers [2–5] instead of p-Laplacian operator ϕ_p. It can be easily seen that the inverse function of the p-prescribed curvature operator $\Phi_p(v)$ is the q-relativistic operator

$$\Phi_q(v) := \Phi_p^{-1}(v) = \frac{|v|^{q-2}v}{\left(1-|v|^q\right)^{\frac{q-1}{q}}}, |v|<1, \frac{1}{p}+\frac{1}{q}=1. \tag{5}$$

Then, we concern with the problem of finding new Lyapunov-type inequalities for Hamiltonian-type problem involving the p-prescribed curvature operator Φ_p under Dirichlet boundary condition. As far as we are aware, this is the first paper that uses the p-prescribed curvature operator Φ_p to study the below problem.

If we take $p=2$ in the operator Φ_p, it turns out to be the prescribed curvature operator, which attracts attention in fields of differential geometry and partial differential equation. It has many essential applications in physics, biology, and other interdisciplines [1, 6].

We know that half-linear equations are closely related to the partial differential equations with p-Laplacian operator ϕ_p [1]. To mention the importance of the p-prescribed curvature operator Φ_p in the partial differential equations, we can consider a partial differential operator of the form

$$\Delta_p u := div\left(\frac{\|\nabla u\|^{p-2}\nabla u}{\left(1+\|\nabla u\|^p\right)^{\frac{p-1}{p}}}\right), p>1, \tag{6}$$

where (for $u=u(t)=u(t_1,t_2,...,t_N)$) $\nabla u = \left(\frac{\partial}{\partial t_1},\frac{\partial}{\partial t_2},...,\frac{\partial}{\partial t_N}\right)$ is the Hamiltonian nabla operator and (for $v(t)=(v_1(t),v_2(t),...,v_N(t))$ $div\, v(t)=\sum_{i=1}^{N}\frac{\partial v(t)}{\partial t_i}$ is the usual divergence operator. If u is a radially symmetric function, i.e., $u(t)=y(x)$, $x=\|t\|=\left(\sum_{i=1}^{N}t_i^2\right)^{1/2}$, $\|\cdot\|$ being the Euclidean norm in $\mathbb{R}^N$, the (partial) differential operator Δ_p can be reduced to the ordinary differential operator

$$\Delta_p u(t) = \left(x^{N-1}\frac{\left|y'\right|^{p-2}y'}{\left(1+\left|y'\right|^p\right)^{\frac{p-1}{p}}}\right)', p>1. \tag{7}$$

The concern of this paper is finding new Lyapunov-type inequalities for Hamiltonian-type problem

$$\begin{cases} w' = \eta(x)w + \dfrac{\left|y\right|^{p-2}y}{\left(1+\left|y\right|^p\right)^{\frac{p-1}{p}}}, \\ y' = -\sigma(x)\dfrac{|w|^{\beta-2}w}{\left(1+|w|^\beta\right)^{\frac{\beta-1}{\beta}}} - \eta(x)y, \end{cases} \tag{8}$$

where $\beta, p > 1$ are real constants, and $\eta(x)$ and $\sigma(x)$ are real-valued continuous functions for all $x \in \mathbb{R}$ under Dirichlet boundary condition.

It is easy to see that if we take $\eta(x) \equiv 0$ in the system (8), we obtain the following equation

$$\left(\frac{|w'|^{q-2}w'}{\left(1-|w'|^q\right)^{\frac{q-1}{q}}}\right)' + \sigma(x)\frac{|w|^{\beta-2}w}{\left(1+|w|^\beta\right)^{\frac{\beta-1}{\beta}}} = 0, \; |w'| < 1, \tag{9}$$

where $\frac{1}{p} + \frac{1}{q} = 1$ from (5). Moreover, if we take the transformation

$$\begin{cases} \dfrac{w}{r(x)} = u, \\ yr(x) = v, \end{cases} \tag{10}$$

where $r'(x) = \eta(x)r(x)$, i.e. $r(x) = \exp\left(\int\limits_{x_1}^{x} \eta(z)dz\right)$ in the system (8), then we obtain the following system

$$\begin{cases} u' = \dfrac{1}{r(x)} \dfrac{\left|\frac{v}{r(x)}\right|^{p-2} \frac{v}{r(x)}}{\left(1+\left|\frac{v}{r(x)}\right|^p\right)^{\frac{p-1}{p}}}, \\ v' = -\sigma(x)r(x)\dfrac{|r(x)u|^{\beta-2} r(x)u}{\left(1+|r(x)u|^\beta\right)^{\frac{\beta-1}{\beta}}}, \end{cases} \tag{11}$$

which does not have the functions $\eta(x)$ and $-\eta(x)$. Furthermore, the system (11) with $\beta = q$ is equivalent to the nonlinear equation

$$\left(r(x)\frac{|u'|^{q-2}u'}{\left(r^{-q}(x)-|u'|^q\right)^{\frac{q-1}{q}}}\right)' + \sigma(x)r(x)\frac{|u|^{q-2}u}{\left(r^{-q}(x)+|u|^q\right)^{\frac{q-1}{q}}} = 0, \; |u'| < 1, \tag{12}$$

where $\frac{1}{p} + \frac{1}{q} = 1$ from (5). If we also let $q = 2$, $r(x) = 1$ (or $\eta(x) \equiv 0$), and $u = w$, then Eq. (12) is reduced to the nonlinear equation

$$\left(\frac{w'}{\sqrt{1-w'^2}}\right)' + \sigma(x)\frac{w}{\sqrt{1+w^2}} = 0, \; |w'| < 1. \tag{13}$$

In 1907, Lyapunov [7] obtained the following inequality

$$\frac{4}{x_2 - x_1} < \int\limits_{x_1}^{x_2} |\sigma(z)|\, dz \tag{14}$$

when second order linear differential equation

$$w'' + \sigma(x)w = 0 \tag{15}$$

has a non-trivial solution $w(x)$ on $[x_1, x_2]$ such that

$$w(x_1) = 0 = w(x_2), \tag{16}$$

where $x_1, x_2 \in \mathbb{R}$ with the condition $x_1 < x_2$.

More recently, Yang et al. [8] have shown that if we assume that

$$\sigma \in \mathcal{A} = \left\{ \sigma \in L^1_{loc}\left((x_1, x_2), [0, \infty)\right) : \int_{x_1}^{x_2} (z - x_1)(x_2 - z)\, \sigma(z)\, dz < \infty \right\} \tag{17}$$

and one-dimensional Minkowski-curvature problem

$$\left(\frac{w'}{\sqrt{1 - w'^2}} \right)' + \sigma(x) w = 0, \; |w'| < 1, \tag{18}$$

under Dirichlet boundary condition (16) has a positive solution, then the Lyapunov-type inequality

$$x_2 - x_1 < \int_{x_1}^{x_2} (z - x_1)(x_2 - z)\, \sigma(z)\, dz \tag{19}$$

holds.

In this paper, on Lyapunov-type inequalities, we derive new results for Hamiltonian-type problem (8) under Dirichlet boundary condition (16). Our motivation for this paper comes from the recent papers of Yang et al. [8] and Tiryaki et al. [9]. For recent works on Lyapunov-type inequalities for the problems under various types of boundary conditions, the reader is referred to [10–16].

2. Main Results

By using a similar technique to that of Tiryaki et al. [9], we obtain the following result for the problem (8) with $\beta = q$ under Dirichlet boundary condition (16). Beneficial to brevity, all over the paper, we denote $\sigma^+(x) = \max\{0, \sigma(x)\}$ is the non-negative part of $\sigma(x)$.

Theorem 2.1. *If the problem (8) with $\beta = q$ and (16) has a non-trivial solution $(w(x), y(x))$, then the following inequality*

$$1 \le \int_{x_1}^{x_2} \frac{\sigma^+(z)}{\left(\int_{x_1}^{z} e^{-p \int_z^v \eta(\theta) d\theta} dv \right)^{1-q} + \left(\int_{z}^{x_2} e^{-p \int_z^v \eta(\theta) d\theta} dv \right)^{1-q}} dz \tag{20}$$

holds.

Proof. Let $w(x_1) = 0 = w(x_2)$ where $x_1, x_2 \in \mathbb{R}$ with $x_1 < x_2$ and w is not identically zero on $[x_1, x_2]$. If we multiply the equations of the system (8) by $y(x)$ and $w(x)$, and then add the results, we get

$$(w(x)y(x))' = \frac{|y(x)|^p}{\left(1 + |y(x)|^p\right)^{\frac{p-1}{p}}} - \frac{|w(x)|^\beta}{\left(1 + |w(x)|^\beta\right)^{\frac{\beta-1}{\beta}}} \sigma(x). \tag{21}$$

Integrating (21) from x_1 to x_2 and using Dirichlet boundary condition (16), we have $w(x_1) = 0 = w(x_2)$ implies

$$\int_{x_1}^{x_2} \frac{|y(z)|^p}{\left(1 + |y(z)|^p\right)^{\frac{p-1}{p}}} dz = \int_{x_1}^{x_2} \frac{|w(z)|^\beta}{\left(1 + |w(z)|^\beta\right)^{\frac{\beta-1}{\beta}}} \sigma(z) dz \le \int_{x_1}^{x_2} |w(z)|^\beta \sigma^+(z) dz. \tag{22}$$

Moreover, from the first equation of the system (8),

$$\left(w(x)e^{-\int_{x_1}^{x}\eta(z)dz}\right)' = e^{-\int_{x_1}^{x}\eta(z)dz}\frac{\left|y(x)\right|^{p-2}y(x)}{\left(1+\left|y(x)\right|^{p}\right)^{\frac{p-1}{p}}} \tag{23}$$

and by integrating (23) from x_1 to x and taking the absolute value, we have

$$|w(x)| \leq \int_{x_1}^{x} e^{-\int_{x}^{z}\eta(\theta)d\theta}\frac{\left|y(z)\right|^{p-1}}{\left(1+\left|y(z)\right|^{p}\right)^{\frac{p-1}{p}}}dz \tag{24}$$

for all $x \in \mathbb{R}$. If we use Hölder's inequality on the integral of the right-hand side of the inequality (24) with indices p and q, then we obtain

$$|w(x)| \leq \left(\int_{x_1}^{x} e^{-p\int_{x}^{z}\eta(\theta)d\theta}dz\right)^{1/p}\left(\int_{x_1}^{x}\frac{\left|y(z)\right|^{(p-1)q}}{\left(1+\left|y(z)\right|^{p}\right)^{\frac{p-1}{p}q}}dz\right)^{1/q} \tag{25}$$

and

$$|w(x)|^{q}\left(\int_{x_1}^{x} e^{-p\int_{x}^{z}\eta(\theta)d\theta}dz\right)^{1-q} \leq \int_{x_1}^{x}\frac{\left|y(z)\right|^{p}}{1+\left|y(z)\right|^{p}}dz, \tag{26}$$

where $\frac{1}{p}+\frac{1}{q}=1$. Similarly, from the first equation of the system (8), we get

$$|w(x)|^{q}\left(\int_{x}^{x_2} e^{-p\int_{x}^{z}\eta(\theta)d\theta}dz\right)^{1-q} \leq \int_{x}^{x_2}\frac{\left|y(z)\right|^{p}}{1+\left|y(z)\right|^{p}}dz, \tag{27}$$

where $\frac{1}{p}+\frac{1}{q}=1$. Adding (26) and (27), from

$$\left(1+\left|y(x)\right|^{p}\right)^{\frac{p-1}{p}} \leq 1+\left|y(x)\right|^{p} \text{ for all } y(x) \in \mathbb{R}, \tag{28}$$

we have

$$|w(x)|^{q} \leq \frac{\int_{x_1}^{x_2}\frac{\left|y(z)\right|^{p}}{\left(1+\left|y(z)\right|^{p}\right)^{\frac{p-1}{p}}}dz}{\left(\int_{x_1}^{x} e^{-p\int_{x}^{z}\eta(\theta)d\theta}dz\right)^{1-q}+\left(\int_{x}^{x_2} e^{-p\int_{x}^{z}\eta(\theta)d\theta}dz\right)^{1-q}} \tag{29}$$

for all $x \in \mathbb{R}$. Thus, substituting (22) with $\beta = q$ in (29), we get

$$|w(x)|^{q} \leq \frac{\int_{x_1}^{x_2}|w(z)|^{q}\sigma^{+}(z)dz}{\left(\int_{x_1}^{x} e^{-p\int_{x}^{z}\eta(\theta)d\theta}dz\right)^{1-q}+\left(\int_{x}^{x_2} e^{-p\int_{x}^{z}\eta(\theta)d\theta}dz\right)^{1-q}} \tag{30}$$

for all $x \in \mathbb{R}$. If we multiply both sides of the inequality (30) by $\sigma^{+}(x)$, and then integrate from x_1 to x_2, we get the inequality (20). Thus, the proof is completed. □

At present, another main result is given.

Theorem 2.2. *If the problem (8) with $\beta = q$ and (16) has a non-trivial solution $(w(x), y(x))$, then the following inequality*

$$1 \leq \int_{x_1}^{x_2} 2^{q-2} \left(\frac{1}{\int_{x_1}^{z} e^{-p\int_z^v \eta(\theta)d\theta} dv} + \frac{1}{\int_{z}^{x_2} e^{-p\int_z^v \eta(\theta)d\theta} dv} \right)^{1-q} \sigma^+(z)\, dz \tag{31}$$

holds.

Proof. Let $w(x_1) = 0 = w(x_2)$ where $x_1, x_2 \in \mathbb{R}$ with $x_1 < x_2$ and w is not identically zero on $[x_1, x_2]$. If we proceeded as in the proof of the Theorem 2.1, we obtain the inequalities (26) and (27). Then we have the inequalities

$$|w(x)|^q \leq \left(\int_{x_1}^{x} e^{-p\int_x^z \eta(\theta)d\theta} dz \right)^{q-1} \int_{x_1}^{x} \frac{|y(z)|^p}{1 + |y(z)|^p} dz \tag{32}$$

and

$$|w(x)|^q \leq \left(\int_{x}^{x_2} e^{-p\int_x^z \eta(\theta)d\theta} dz \right)^{q-1} \int_{x}^{x_2} \frac{|y(z)|^p}{1 + |y(z)|^p} dz, \tag{33}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Since $\int_{x_1}^{x} e^{-p\int_x^z \eta(\theta)d\theta} dz$ is non-decreasing and $\int_{x}^{x_2} e^{-p\int_x^z \eta(\theta)d\theta} dz$ is non-increasing for $x \in (x_1, x_2)$, there exists at least one $x_* \in (x_1, x_2)$ such that

$$\int_{x_1}^{x_*} e^{-p\int_{x_*}^z \eta(\theta)d\theta} dz = \int_{x_*}^{x_2} e^{-p\int_{x_*}^z \eta(\theta)d\theta} dz > 0. \tag{34}$$

Thus,

$$\int_{x_1}^{x} e^{-p\int_x^z \eta(\theta)d\theta} dz \leq \int_{x}^{x_2} e^{-p\int_x^z \eta(\theta)d\theta} dz \text{ for } x \in [x_1, x_*] \tag{35}$$

and

$$\int_{x}^{x_2} e^{-p\int_x^z \eta(\theta)d\theta} dz \leq \int_{x_1}^{x} e^{-p\int_x^z \eta(\theta)d\theta} dz \text{ for } x \in [x_*, x_2] \tag{36}$$

hold. For $x \in [x_1, x_*]$ and $x \in [x_*, x_2]$, by using

$$\int_{x_1}^{x} e^{-p\int_x^z \eta(\theta)d\theta} dz \leq \frac{2\int_{x_1}^{x} e^{-p\int_x^z \eta(\theta)d\theta} dz \int_{x}^{x_2} e^{-p\int_x^z \eta(\theta)d\theta} dz}{\int_{x_1}^{x} e^{-p\int_x^z \eta(\theta)d\theta} dz + \int_{x}^{x_2} e^{-p\int_x^z \eta(\theta)d\theta} dz} \tag{37}$$

and

$$\int_{x}^{x_2} e^{-p\int_{x}^{z}\eta(\theta)d\theta}dz \le \frac{2\int_{x_1}^{x} e^{-p\int_{x}^{z}\eta(\theta)d\theta}dz \int_{x}^{x_2} e^{-p\int_{x}^{z}\eta(\theta)d\theta}dz}{\int_{x_1}^{x_2} e^{-p\int_{x}^{z}\eta(\theta)d\theta}dz} \tag{38}$$

in the inequalities (32) and (33), respectively, we have the following inequalities

$$|w(x)|^q \le \left(\frac{2\int_{x_1}^{x} e^{-p\int_{x}^{z}\eta(\theta)d\theta}dz \int_{x}^{x_2} e^{-p\int_{x}^{z}\eta(\theta)d\theta}dz}{\int_{x_1}^{x_2} e^{-p\int_{x}^{z}\eta(\theta)d\theta}dz} \right)^{q-1} \int_{x_1}^{x_*} \frac{\left|y(z)\right|^p}{1+\left|y(z)\right|^p}dz \tag{39}$$

and

$$|w(x)|^q \le \left(\frac{2\int_{x_1}^{x} e^{-p\int_{x}^{z}\eta(\theta)d\theta}dz \int_{x}^{x_2} e^{-p\int_{x}^{z}\eta(\theta)d\theta}dz}{\int_{x_1}^{x_2} e^{-p\int_{x}^{z}\eta(\theta)d\theta}dz} \right)^{q-1} \int_{x_*}^{x_2} \frac{\left|y(z)\right|^p}{1+\left|y(z)\right|^p}dz. \tag{40}$$

Adding (39) and (40), from (28), we obtain

$$2\,|w(x)|^q \le \left(\frac{2\int_{x_1}^{x} e^{-p\int_{x}^{z}\eta(\theta)d\theta}dz \int_{x}^{x_2} e^{-p\int_{x}^{z}\eta(\theta)d\theta}dz}{\int_{x_1}^{x_2} e^{-p\int_{x}^{z}\eta(\theta)d\theta}dz} \right)^{q-1} \int_{x_1}^{x_2} \frac{\left|y(z)\right|^p}{\left(1+\left|y(z)\right|^p\right)^{\frac{p-1}{p}}}dz. \tag{41}$$

From the inequality (22) with $\beta = q$, we get

$$|w(x)|^q \le \frac{1}{2}\left(\frac{2\int_{x_1}^{x} e^{-p\int_{x}^{z}\eta(\theta)d\theta}dz \int_{x}^{x_2} e^{-p\int_{x}^{z}\eta(\theta)d\theta}dz}{\int_{x_1}^{x_2} e^{-p\int_{x}^{z}\eta(\theta)d\theta}dz} \right)^{q-1} \int_{x_1}^{x_2} |w(z)|^q \,\sigma^+(z)dz. \tag{42}$$

If we multiply both sides of the inequality (42) by $\sigma^+(x)$ and integrate from x_1 to x_2, we have the Lyapunov-type inequality (31). Thus, the proof is completed. □

As $k(x) = x^{q-1}$ is a concave function for $x > 0$ and $1 < q < 2$, Jensen's inequality $k\left(\frac{\rho_1+\rho_2}{2}\right) \ge \frac{k(\rho_1)+k(\rho_2)}{2}$ with

$$\rho_1 = \frac{1}{\int_{x_1}^{x} e^{-p\int_{x}^{z}\eta(\theta)d\theta}dz} \text{ and } \rho_2 = \frac{1}{\int_{x}^{x_2} e^{-p\int_{x}^{z}\eta(\theta)d\theta}dz} \tag{43}$$

implies

$$2^{2-q}\left(\frac{1}{\int\limits_{x_1}^{x} e^{-p\int\limits_{x}^{z}\eta(\theta)d\theta}dz}+\frac{1}{\int\limits_{x}^{x_2} e^{-p\int\limits_{x}^{z}\eta(\theta)d\theta}dz}\right)^{q-1} \geq \frac{1}{\left(\int\limits_{x_1}^{x} e^{-p\int\limits_{x}^{z}\eta(\theta)d\theta}dz\right)^{q-1}}+\frac{1}{\left(\int\limits_{x}^{x_2} e^{-p\int\limits_{x}^{z}\eta(\theta)d\theta}dz\right)^{q-1}} \tag{44}$$

for $1 < q < 2$. If $q > 2$, then $k(x) = x^{q-1}$ is a convex function for $x > 0$ and hence the inequality (44) changes to

$$2^{2-q}\left(\frac{1}{\int\limits_{x_1}^{x} e^{-p\int\limits_{x}^{z}\eta(\theta)d\theta}dz}+\frac{1}{\int\limits_{x}^{x_2} e^{-p\int\limits_{x}^{z}\eta(\theta)d\theta}dz}\right)^{q-1} \leq \frac{1}{\left(\int\limits_{x_1}^{x} e^{-p\int\limits_{x}^{z}\eta(\theta)d\theta}dz\right)^{q-1}}+\frac{1}{\left(\int\limits_{x}^{x_2} e^{-p\int\limits_{x}^{z}\eta(\theta)d\theta}dz\right)^{q-1}}. \tag{45}$$

Therefore, a remark can be given as follows.

Remark 2.1. *It is obvious that from the inequality (44) if one takes $1 < q < 2$, then the Lyapunov-type inequality (31) of Theorem 2.2 is stronger than (20) of Theorem 2.1 in the sense that (20) follows from (31), but not conversely. However, from the inequality (45), if $q > 2$, then (31) is weaker than (20).*

Because $k(x) = x^{1-q}$ is a convex function for $x > 0$ and $q > 1$, Jensen's inequality $k\left(\frac{\rho_3+\rho_4}{2}\right) \leq \frac{k(\rho_3)+k(\rho_4)}{2}$ with

$$\rho_3 = \int\limits_{x_1}^{x} e^{-p\int\limits_{x}^{z}\eta(\theta)d\theta}dz > 0 \text{ and } \rho_4 = \int\limits_{x}^{x_2} e^{-p\int\limits_{x}^{z}\eta(\theta)d\theta}dz > 0 \tag{46}$$

implies

$$\left(\int\limits_{x_1}^{x} e^{-p\int\limits_{x}^{z}\eta(\theta)d\theta}dz\right)^{1-q}+\left(\int\limits_{x}^{x_2} e^{-p\int\limits_{x}^{z}\eta(\theta)d\theta}dz\right)^{1-q} \geq 2^{q}\left(\int\limits_{x_1}^{x_2} e^{-p\int\limits_{x}^{z}\eta(\theta)d\theta}dz\right)^{1-q}. \tag{47}$$

If we also use the inequality $4\rho_3\rho_4 \leq (\rho_3 + \rho_4)^2$, where ρ_3 and ρ_4 are given in (46), in the term

$$2^{q-2}\left(\frac{1}{\int\limits_{x_1}^{x} e^{-p\int\limits_{x}^{z}\eta(\theta)d\theta}dz}+\frac{1}{\int\limits_{x}^{x_2} e^{-p\int\limits_{x}^{z}\eta(\theta)d\theta}dz}\right)^{1-q}, \tag{48}$$

then we obtain the following inequality

$$2^{q-2}\left(\frac{1}{\int\limits_{x_1}^{x} e^{-p\int\limits_{x}^{z}\eta(\theta)d\theta}dz}+\frac{1}{\int\limits_{x}^{x_2} e^{-p\int\limits_{x}^{z}\eta(\theta)d\theta}dz}\right)^{1-q} \leq 2^{-q}\left(\int\limits_{x_1}^{x_2} e^{-p\int\limits_{x}^{z}\eta(\theta)d\theta}dz\right)^{q-1}. \tag{49}$$

Thus, if we use the inequalities (47) and (49) in Theorems 2.1 and 2.2, respectively, a result is obtained as follows.

Corollary 2.1. *If the problem (8) with $\beta = q$ and (16) has a non-trivial solution $(w(x), y(x))$, then the following inequality*

$$1 \leq \int_{x_1}^{x_2} 2^{-q} \left(\int_{x_1}^{x_2} e^{-p\int_z^v \eta(\theta)d\theta} dv \right)^{q-1} \sigma^+(z)\, dz \tag{50}$$

holds.

For integrals in the inequality (50) if the Second Mean Value Theorem is used, then a result is obtained as follows.

Corollary 2.2. *If the problem (8) with $\beta = q$ and (16) has a non-trivial solution $(w(x), y(x))$, then there exists some point $\varsigma^* \in (x_1, x_2)$ such that*

$$1 \leq 2^{-q} \left(\int_{x_1}^{x_2} e^{-p\int_{\varsigma^*}^z \eta(\theta)d\theta} dz \right)^{q-1} \int_{x_1}^{x_2} \sigma^+(z)\, dz \tag{51}$$

holds.

Up to this point, we have only considered the case $\beta = q$ in the problem (8) with (16). At present, we start considering the problem (8) with (16) without any restriction on β and q. The theorems below associate with the consecutive zeros x_1 and x_2 to the maximum value C of $|w(x)|$ on (x_1, x_2). The proof of the next theorem is similar to that of Theorem 2.1.

Theorem 2.3. *If the problem (8) with (16) has a non-trivial solution $(w(x), y(x))$, then the following inequality*

$$1 \leq C^{\beta-q} \frac{\int_{x_1}^{x_2} \sigma^+(z)dz}{\left(\int_{x_1}^{\varsigma} e^{-p\int_{\varsigma}^z \eta(\theta)d\theta} dz \right)^{1-q} + \left(\int_{\varsigma}^{x_2} e^{-p\int_{\varsigma}^z \eta(\theta)d\theta} dz \right)^{1-q}} \tag{52}$$

holds, where $C = |w(\varsigma)| = \max_{x_1 \leq x \leq x_2} |w(x)|$.

Proof. Since $w(x_1) = 0 = w(x_2)$ where $x_1, x_2 \in \mathbb{R}$ with $x_1 < x_2$ and $w(x) \not\equiv 0$ on $[x_1, x_2]$, we can choose $\varsigma \in (x_1, x_2)$ such that $C = |w(\varsigma)| = \max_{x_1 \leq x \leq x_2} |w(x)| > 0$. If we proceed as in the proof of Theorem 2.1, then for $x = \varsigma$, the inequality (29) is also satisfied and so, we get

$$|w(\varsigma)|^q \left(\left(\int_{x_1}^{\varsigma} e^{-p\int_{\varsigma}^z \eta(\theta)d\theta} dz \right)^{1-q} + \left(\int_{\varsigma}^{x_2} e^{-p\int_{\varsigma}^z \eta(\theta)d\theta} dz \right)^{1-q} \right) \leq \int_{x_1}^{x_2} \frac{|y(z)|^p}{\left(1 + |y(z)|^p\right)^{\frac{p-1}{p}}} dz, \tag{53}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Therefore, if we substitute (22) into (53), we obtain the following inequality

$$|w(\varsigma)|^q \left(\left(\int_{x_1}^{\varsigma} e^{-p\int_{\varsigma}^z \eta(\theta)d\theta} dz \right)^{1-q} + \left(\int_{\varsigma}^{x_2} e^{-p\int_{\varsigma}^z \eta(\theta)d\theta} dz \right)^{1-q} \right) \leq \int_{x_1}^{x_2} |w(z)|^\beta \sigma^+(z)dz. \tag{54}$$

Thus, the proof is completed. □

From Theorems 2.2 and 2.3, we have the following result for the problem (8) with (16). Since its proof is similar to that of Theorem 2.2, it is omitted.

Theorem 2.4. *If the problem (8) with (16) has a non-trivial solution* $(w(x), y(x))$*, then the following inequality*

$$1 \leq C^{\beta-q} 2^{q-2} \left(\frac{1}{\int_{x_1}^{\varsigma} e^{-p\int_{\varsigma}^{z} \eta(\theta) d\theta} dz} + \frac{1}{\int_{\varsigma}^{x_2} e^{-p\int_{\varsigma}^{z} \eta(\theta) d\theta} dz} \right)^{1-q} \int_{x_1}^{x_2} \sigma^+(z) dz \tag{55}$$

holds, where $C = |w(\varsigma)| = \max_{x_1 \leq x \leq x_2} |w(x)|$.

As in Corollary 2.1, if we use the inequalities (47) and (49) in Theorems 2.3 and 2.4, respectively, the following result is obtained.

Corollary 2.3. *If the problem (8) with (16) has a non-trivial solution* $(w(x), y(x))$*, then the following inequality*

$$1 \leq C^{\beta-q} 2^{-q} \left(\int_{x_1}^{x_2} e^{-p\int_{\varsigma}^{z} \eta(\theta) d\theta} dz \right)^{q-1} \int_{x_1}^{x_2} \sigma^+(z) dz \tag{56}$$

holds, where $C = |w(\varsigma)| = \max_{x_1 \leq x \leq x_2} |w(x)|$.

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