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# On Dirac Systems with Multi-Eigenparameter-Dependent Transmission Conditions

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**Abstract.** In this work, we investigate a Dirac system which has discontinuities at finite interior points and contains eigenparameter in both boundary and transmission conditions. By defining a suitable Hilbert space $\mathfrak{H}$ associated with the problem, we generate a self-adjoint operator $\mathcal{T}$ such that the eigenvalues of the considered problem coincide with those of $\mathcal{T}$. We construct the fundamental system of solutions of the problem and get the asymptotic formulas for the fundamental solutions, eigenvalues and eigen-vector-functions. Also, we examine the asymptotic behaviour for the norm of eigenvectors corresponding to the operator $\mathcal{T}$. We construct Green's matrix, and derive the resolvent of the operator $\mathcal{T}$ in terms of Green's matrix. Finally, we estimate the norm of resolvent of the operator $\mathcal{T}$. In the special case, when our problem has no eigenparameter in both boundary and transmission conditions, the obtained results coincide with the corresponding results in Tharwat (Boundary Value Problems, DOI:10.1186/s13661-015-0515-1, 2016).

## 1. Introduction

In this section we consider some of the notations and relations that will be used in this study; then we prove some useful theorems and lemmas in the following sections. In this work we shall investigate a new class of boundary value problems which consist of the Dirac system

$$\tau(y) := P(x)y'(x) - Q(x)y(x) = \lambda y(x), \tag{1}$$

on finite number disjoint intervals $[d_0, d_1)$, $(d_{i-1}, d_i)$, $i = \overline{2, m}$ and $(d_m, d_{m+1}]$, where $a = d_0 < d_1 < d_2 < \cdots < d_{m+1} = b$, together with boundary conditions at end points $x = a, b$

$$\tau_1(y) := \mathcal{L}_\alpha(y(a)) + \lambda \mathcal{L}_{\vartheta_1}(y(a)) = 0, \tag{2}$$

$$\tau_2(y) := \mathcal{L}_\beta(y(b)) + \lambda \mathcal{L}_{\vartheta_2}(y(b)) = 0, \tag{3}$$

and the transmission conditions at interior points $d_i \in (a, b)$, $i = \overline{1, m}$,

$$\tau_{1,i}(y) := \delta_{1,i} y_1(d_i^-) - \delta'_{1,i} y_1(d_i^+) = 0, \tag{4}$$

$$\tau_{2,i}(y) := \mathcal{L}_{1,i}(y(d_i)) + \lambda \mathcal{L}_{2,i}(y(d_i)) = 0, \tag{5}$$

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where $p_i > 0$ $(i = \overline{1, m+1})$ are real numbers,

$$p(x) := \begin{cases} p_1, & x \in [a, d_1); \\ p_i, & x \in (d_{i-1}, d_i)(i = \overline{2, m}); \\ p_{m+1}, & x \in (d_m, b]; \end{cases} \tag{6}$$

$$P(x) = \begin{pmatrix} 0 & p(x) \\ -p(x) & 0 \end{pmatrix}, \quad Q(x) = \begin{pmatrix} q_1(x) & 0 \\ 0 & q_2(x) \end{pmatrix}, \quad y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}, \tag{7}$$

$$\begin{pmatrix} \mathcal{L}_\alpha(y(a)) & \mathcal{L}_{\vartheta_1}(y(a)) \\ \mathcal{L}_\beta(y(b)) & \mathcal{L}_{\vartheta_2}(y(b)) \\ \mathcal{L}_{1,i}(y(d_i)) & \mathcal{L}_{2,i}(y(d_i)) \end{pmatrix} = \begin{pmatrix} \alpha y_1(a) - \alpha' y_2(a) & \sin\vartheta_1 y_1(a) - \cos\vartheta_1 y_2(a) \\ \beta y_1(b) - \beta' y_2(b) & \sin\vartheta_2 y_1(b) - \cos\vartheta_2 y_2(b) \\ \delta_{2,i} y_2(d_i^-) - \delta'_{2,i} y_2(d_i^+) & \delta_{2,i} y_1(d_i^-) \end{pmatrix}, \quad i = \overline{1, m}, \tag{8}$$

$\lambda$ is a complex spectral parameter; $q_1(\cdot)$ and $q_2(\cdot)$ are real-valued functions which are continuous in each of the intervals $(d_{i-1}, d_i)$ $(i = \overline{1, m+1})$ and have finite limits $q_1(d_i^{\pm})$ and $q_2(d_i^{\pm})$ $(i = \overline{1, m})$ respectively; $\alpha, \alpha', \beta, \beta' \in \mathbb{R}$; $\delta_{1,i}, \delta_{2,i}, \delta'_{1,i}, \delta'_{2,i} \in \mathbb{R}$, $\delta_{1,i}, \delta_{2,i}, \delta'_{1,i}, \delta'_{2,i} \neq 0$ $(i = \overline{1, m})$; $\vartheta_1, \vartheta_2 \in [0, \pi)$ and

$$\omega := \det\begin{pmatrix} \alpha' & \alpha \\ \cos\vartheta_1 & \sin\vartheta_1 \end{pmatrix} > 0, \qquad \nu := \det\begin{pmatrix} \beta & \beta' \\ \sin\vartheta_2 & \cos\vartheta_2 \end{pmatrix} > 0. \tag{9}$$

The mathematical modeling of several practical problems in areas of mathematical physics requires solutions of boundary value problems. The boundary value problems with discontinuity at one or more interior point have become an important area of research in recent years because of the needs of modern technology, engineering and physics, see [2, 10, 14, 15, 21, 22]. Moreover, there has been a growing interest in boundary value problems with eigenparameter appears not only in the differential equation but also in the boundary conditions of the problems, see [3, 5, 7–9, 13, 16, 17, 20, 24]. The spectral analysis and some properties of Dirac systems with transmission conditions have been studied by many authors, cf. [6, 18–21]. In these direct problems there are at most two transmission points, see [6, 18, 20, 21]. In [19], Tharwat studied the Dirac system which has discontinuities at finite interior points and has no eigenparameter in both boundary and transmission conditions. Our objective is to discuss the Dirac system with multi-eigenparameter-dependent transmission conditions and contains an eigenparameter, at the same time, in all boundary conditions. The problem (1)–(5), as far as we know, does not exist. The rest of this paper is organized as follows: In the next section, we shall construct the special inner product in the Hilbert space $\mathfrak{H}$ and define a symmetric linear operator $\mathcal{T}$ in this Hilbert space $\mathfrak{H}$ such that the boundary-value-transmission problem (1)–(5) can be interpreted as the eigenvalue problem of the operator $\mathcal{T}$. Section 3 presents the construction of the fundamental system of solutions of the problem (1)–(5) and verify the simplicity of the eigenvalues of the problem (1)–(5). In Section 4, we derive the asymptotic formulas of the eigenvalues of the problem (1)–(5) and the corresponding eigen-vector-functions. In Section 5, we discuss the asymptotic behaviour fo the norm of eigenvectors corresponding to the operator $\mathcal{T}$. In the last section, we construct Green's matrix, and derive the resolvent of the operator $\mathcal{T}$ in terms of Green's matrix. Then, we estimate the norm of resolvent of the operator $\mathcal{T}$.

## 2. An Operator Formulation with the Suitable Hilbert Space

In this section we will derive an inner product with the Hilbert space $\mathfrak{H} := \mathcal{H} \oplus \mathbb{C}^{m+2}$,

$$\mathcal{H} := \left\{ y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}, y_1(x), y_2(x) \in \bigoplus_{i=1}^{m+1} L^2(d_{i-1}, d_i) \right\}, \tag{10}$$

$$\mathbb{C}^{m+2} := \overbrace{\mathbb{C} \oplus \mathbb{C} \cdots \oplus \mathbb{C}}^{m+2 \text{ times}}$$

and a symmetric linear operator $\mathcal{T}$ defined on this Hilbert space such a way that the problem (1)–(5) can be considered as the eigenvalue problem corresponding this operator.

Let us introduce a new equivalent inner product on Hilbert space $\mathfrak{H} = \mathcal{H} \oplus \mathbb{C}^{m+2}$ by

$$\langle U(\cdot), V(\cdot)\rangle_{\mathfrak{H}} := \sum_{j=0}^{m}\left(\frac{\prod_{i=0}^{j}\Delta_i}{p_{j+1}}\right)\int_{d_j}^{d_{j+1}} u^{\top}(x)\overline{v}(x)\,dx + \frac{1}{\omega}\tilde{u}_1\overline{\tilde{v}}_1 + \frac{\prod_{i=0}^{m}\Delta_i}{\nu}\tilde{u}_2\overline{\tilde{v}}_2 + \sum_{j=1}^{m}\frac{\prod_{i=0}^{j-1}\Delta_i u_j\overline{v}_j}{\delta_{2,j}^2} \tag{11}$$

where $\Delta_i = \dfrac{\delta'_{1,i}\,\delta'_{2,i}}{\delta_{1,i}\,\delta_{2,i}} > 0$, $i = \overline{1,m}$, $\Delta_0 = 1$ and $\top$ denotes the matrix transpose,

$$U(x) = \begin{pmatrix} u(x)\\ \tilde{u}_1\\ \tilde{u}_2\\ u_1\\ \vdots\\ u_m\end{pmatrix},\ V(x) = \begin{pmatrix} v(x)\\ \tilde{v}_1\\ \tilde{v}_2\\ v_1\\ \vdots\\ v_m\end{pmatrix} \in \mathfrak{H},\quad u(x) = \begin{pmatrix} u_1(x)\\ u_2(x)\end{pmatrix},\ v(x) = \begin{pmatrix} v_1(x)\\ v_2(x)\end{pmatrix} \in \mathcal{H},$$

$$u_k(\cdot),\ v_k(\cdot) \in \bigoplus_{i=1}^{m+1} L^2(d_{i-1}, d_i),\quad \tilde{u}_k,\ \tilde{v}_k, u_j,\ v_j \in \mathbb{C},\quad j = \overline{1,m},\quad k = 1,2.$$

In the Hilbert space $\mathfrak{H}$ we define a linear operator $\mathcal{T} : \mathfrak{H} \to \mathfrak{H}$ with domain $\mathcal{D}(\mathcal{T})$ as following:

$$\mathcal{D}(\mathcal{T}) := \left\{ U(x) = \begin{pmatrix} u(x)\\ \tilde{u}_1\\ \tilde{u}_2\\ u_1\\ \vdots\\ u_m\end{pmatrix} \in \mathfrak{H} : u(x) = \begin{pmatrix} u_1(x)\\ u_2(x)\end{pmatrix},\ u_k(\cdot) \in \bigoplus_{i=1}^{m+1} L^2(d_{i-1}, d_i),\ \tilde{u}_k, u_j \in \mathbb{C},\ j = \overline{1,m},\ k = 1,2,\ ;\right.$$

1. $u_1(\cdot)$ and $u_2(\cdot)$ are absolutely continuous in $[d_0, d_1)\bigcup_{i=2}^{m}(d_{i-1}, d_i) \cup (d_m, d_{m+1}]$ and have finite limits $u_1(d_i^{\pm})$ and $u_2(d_i^{\pm})$, $i = \overline{1,m}$, respectively;
2. $\tau(u) \in \mathfrak{H}$;
3. $\tilde{u}_1 = \mathcal{L}_{\vartheta_1}(u)(a)$, $\tilde{u}_2 = \mathcal{L}_{\vartheta_2}(u)(b)$, $u_{i-2} = \mathcal{L}_{2,i-2}(u)(d_{i-2})$, $i = \overline{3, m+2}$;
4. $\tau_{1,i}(u) = 0$, $i = \overline{1,m}\Big\}$,

(12)

and the linear operator $\mathcal{T} : \mathcal{D}(\mathcal{T}) \longrightarrow \mathfrak{H}$ is defined by

$$\mathcal{T}\begin{pmatrix} u(x)\\ \mathcal{L}_{\vartheta_1}(u(a))\\ \mathcal{L}_{\vartheta_2}(u(b))\\ \mathcal{L}_{2,1}(u(d_1))\\ \vdots\\ \mathcal{L}_{2,m}(u(d_m))\end{pmatrix} = \begin{pmatrix} \tau(u)\\ -\mathcal{L}_{\alpha}(u(a))\\ -\mathcal{L}_{\beta}(u(b))\\ -\mathcal{L}_{1,1}(u(d_1))\\ \vdots\\ -\mathcal{L}_{1,m}(u(d_m))\end{pmatrix},\quad \begin{pmatrix} u(x)\\ \mathcal{L}_{\vartheta_1}(u(a))\\ \mathcal{L}_{\vartheta_2}(u(b))\\ \mathcal{L}_{2,1}(u(d_1))\\ \vdots\\ \mathcal{L}_{2,m}(u(d_m))\end{pmatrix} \in \mathcal{D}(\mathcal{T}). \tag{13}$$

Then, from the above definition of the operator and its domain, the multi-point discontinuous Dirac system (1)–(5) can be written in the form

$$\mathcal{T}U(x) = \lambda U(x), \quad U(x) = \begin{pmatrix} u(x) \\ \mathcal{L}_{\vartheta_1}(u(a)) \\ \mathcal{L}_{\vartheta_2}(u(b)) \\ \mathcal{L}_{2,1}(u(d_1)) \\ \vdots \\ \mathcal{L}_{2,m}(u(d_m)) \end{pmatrix} \in \mathcal{D}(\mathcal{T}).$$

Consequently, by the eigenvalues and the corresponding eigen-vector-functions of the problem (1)-(5) we mean eigenvalues and first components of corresponding eigenelements of the linear operator $\mathcal{T}$, respectively. For convenience we denote $\mathcal{W}(u,v;\cdot)$ the Wronskian of $u(\cdot)$ and $v(\cdot)$, defined in [12, p. 194], i.e.,

$$\mathcal{W}(u,v;x) := \det\begin{pmatrix} u_1(x) & u_2(x) \\ v_1(x) & v_2(x) \end{pmatrix}. \tag{14}$$

**Theorem 2.1.** *The linear operator $\mathcal{T}$ is symmetric operator.*

*Proof.* Let $U, V \in \mathcal{D}(\mathcal{T})$. Then, from the definition of the operator $\mathcal{T}$ and its domain $\mathcal{D}(\mathcal{T})$, we get

$$\begin{aligned} \langle \mathcal{T}U(\cdot), V(\cdot)\rangle_{\mathfrak{H}} = \sum_{j=0}^{m}\left(\frac{\prod_{i=0}^{j}\Delta_i}{p_{j+1}}\right)\int_{d_j}^{d_{j+1}}(\tau u(x))^{\top}\overline{v}(x)\,dx - \frac{1}{\omega}\mathcal{L}_{\alpha}(u(a))\mathcal{L}_{\vartheta_1}(\overline{v}(a)) - \frac{\prod_{i=0}^{m}\Delta_i}{\nu}\mathcal{L}_{\beta}(u(b))\mathcal{L}_{\vartheta_2}(\overline{v}(b)) \\ - \sum_{j=1}^{m}\frac{\prod_{i=0}^{j-1}\Delta_i\mathcal{L}_{1,j}(u(d_j))\mathcal{L}_{2,j}(\overline{v}(d_j))}{\delta_{2,j}^2}. \end{aligned} \tag{15}$$

Since equation (1) can be rewritten as

$$\left.\begin{aligned} p(x)u_2'(x) - q_1(x)u_1(x) = \lambda u_1(x), \\ p(x)u_1'(x) + q_2(x)u_2(x) = -\lambda u_2(x), \end{aligned}\right\} x \in [a,d_1)\bigcup_{i=2}^{m}(d_{i-1},\, d_i)\bigcup(d_m,b], \tag{16}$$

then, by partial integration and a short calculation, we have

$$\begin{aligned} \langle \mathcal{T}U(\cdot), V(\cdot)\rangle_{\mathfrak{H}} = \langle U(\cdot), \mathcal{T}V(\cdot)\rangle_{\mathfrak{H}} - \mathcal{W}(u,\overline{v};d_1^-) + \mathcal{W}(u,\overline{v};a) - \prod_{i=0}^{m}\Delta_i\mathcal{W}(u,\overline{v};b) + \prod_{i=0}^{m}\Delta_i\mathcal{W}(u,\overline{v};d_m^+) \\ - \sum_{j=1}^{m-1}\left(\prod_{i=0}^{j}\Delta_i\right)\left[\mathcal{W}(u,\overline{v};d_{j+1}^-) - \mathcal{W}(u,\overline{v};d_j^+)\right] - \frac{1}{\omega}\left[\mathcal{L}_{\alpha}(u(a))\mathcal{L}_{\vartheta_1}(\overline{v}(a)) - \mathcal{L}_{\vartheta_1}(u(a))\mathcal{L}_{\alpha}(\overline{v}(a))\right] \\ - \frac{\prod_{i=0}^{m}\Delta_i}{\nu}\left[\mathcal{L}_{\beta}(u(b))\mathcal{L}_{\vartheta_2}(\overline{v}(b)) - \mathcal{L}_{\vartheta_2}(u(b))\mathcal{L}_{\beta}(\overline{v}(b))\right] \\ - \sum_{j=1}^{m}\frac{\prod_{i=0}^{j-1}\Delta_i\left[\mathcal{L}_{1,j}(u(d_j))\mathcal{L}_{2,j}(\overline{v}(d_j)) - \mathcal{L}_{2,j}(u(d_j))\mathcal{L}_{1,j}(\overline{v}(d_j))\right]}{\delta_{2,j}^2}. \end{aligned} \tag{17}$$

Since $u(x)$ and $\overline{v}(x)$ are satisfied (2)-(5), consequently we can obtain the following three equations

$$\mathcal{L}_{\alpha}(u(a))\mathcal{L}_{\vartheta_1}(\overline{v}(a)) - \mathcal{L}_{\vartheta_1}(u(a))\mathcal{L}_{\alpha}(\overline{v}(a)) = \omega\mathcal{W}(u,\overline{v};a), \tag{18}$$

$$\mathcal{L}_{\beta}(u(b))\mathcal{L}_{\vartheta_2}(\overline{v}(b)) - \mathcal{L}_{\vartheta_2}(u(b))\mathcal{L}_{\beta}(\overline{v}(b)) = -\nu\mathcal{W}(u,\overline{v};b), \tag{19}$$

$$\frac{1}{\delta_{2,j}^2}\left[\mathcal{L}_{1,j}(u(d_j))\mathcal{L}_{2,j}(\overline{v}(d_j)) - \mathcal{L}_{2,j}(u(d_j))\mathcal{L}_{1,j}(\overline{v}(d_j))\right] = \Delta_j\mathcal{W}(u,\overline{v};d_j^+) - \mathcal{W}(u,\overline{v};d_j^-), \quad j=\overline{1,m}. \tag{20}$$

Substituting equations (18)-(20) into (17), we have

$$\langle \mathcal{T}U(\cdot), V(\cdot)\rangle_{\mathfrak{H}} = \langle U(\cdot), \mathcal{T}V(\cdot)\rangle_{\mathfrak{H}}$$

so the linear operator $\mathcal{T}$ is symmetric.

□

The following lemma explain that all eigenvalues of the problem (1)–(5) are real.

**Lemma 2.2.** *All the eigenvalues of the multi-point discontinuous Dirac system* (1)–(5) *are real.*

*Proof.* Let $\begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix}$ be a non-trivial eigen-vector-function corresponding the eigenvalue $\mu$ of the problem (1)–(5). Thus integrating by parts we get

$$\begin{aligned}\sum_{j=0}^{m}\left(\frac{\prod_{i=0}^{j}\Delta_i}{p_{j+1}}\right)\int_{d_j}^{d_{j+1}}(\tau u(x))^{\top}\overline{u}(x)\,dx &= \sum_{j=0}^{m}\left(\frac{\prod_{i=0}^{j}\Delta_i}{p_{j+1}}\right)\int_{d_j}^{d_{j+1}}(u(x))^{\top}\overline{\tau u(x)}\,dx \\ &- \mathcal{W}(u,\overline{u};d_1^-) + \mathcal{W}(u,\overline{u};a) - \prod_{i=0}^{m}\Delta_i\mathcal{W}(u,\overline{u};b) + \prod_{i=0}^{m}\Delta_i\mathcal{W}(u,\overline{u};d_m^+) \\ &- \sum_{j=1}^{m-1}\left(\prod_{i=0}^{j}\Delta_i\right)\left[\mathcal{W}(u,\overline{u};d_{j+1}^-) - \mathcal{W}(u,\overline{u};d_j^+)\right]\end{aligned} \tag{21}$$

From boundary conditions (2) and (3), and transmission conditions (4) and (5), we can obtain

$$\mathcal{W}(u,\overline{u};a) = \frac{\omega(\overline{\mu}-\mu)}{\left|\alpha+\mu\sin\vartheta_1\right|^2}\left|u_2(a)\right|^2, \tag{22}$$

$$\mathcal{W}(u,\overline{u};b) = -\frac{\nu(\overline{\mu}-\mu)}{\left|\beta+\mu\sin\vartheta_2\right|^2}\left|u_2(b)\right|^2, \tag{23}$$

$$\Delta_j\mathcal{W}(u,\overline{u};d_j^+) - \mathcal{W}(u,\overline{u};d_j^-) = (\overline{\mu}-\mu)\left|u_1(d_j^-)\right|^2, \quad j=\overline{1,m}. \tag{24}$$

Substituting equations (1), (22), (23) and (24) in (21) and by using a short calculation, we get

$$\begin{aligned}\sum_{j=0}^{m}\left(\frac{\prod_{i=0}^{j}\Delta_i}{p_{j+1}}\right)\int_{d_j}^{d_{j+1}}\mu(u(x))^{\top}\overline{u}(x)\,dx &= \sum_{j=0}^{m}\left(\frac{\prod_{i=0}^{j}\Delta_i}{p_{j+1}}\right)\int_{d_j}^{d_{j+1}}(u(x))^{\top}\overline{\mu u}(x)\,dx \\ &+ \frac{\omega(\overline{\mu}-\mu)}{\left|\alpha+\mu\sin\vartheta_1\right|^2}\left|u_2(a)\right|^2 + \prod_{i=0}^{m}\Delta_i\frac{\nu(\overline{\mu}-\mu)}{\left|\beta+\mu\sin\vartheta_2\right|^2}\left|u_2(b)\right|^2 \\ &+ (\overline{\mu}-\mu)\sum_{j=1}^{m}\left(\prod_{i=0}^{j-1}\Delta_i\right)\left|u_1(d_j^-)\right|^2,\end{aligned} \tag{25}$$

which leads to

$$(\mu-\overline{\mu})\left[\sum_{j=0}^{m}\left(\frac{\prod_{i=0}^{j}\Delta_i}{p_{j+1}}\right)\int_{d_j}^{d_{j+1}}\left(|u_1(x)|^2+|u_2(x)|^2\right)dx+\frac{\omega\,|u_2(a)|^2}{\left|\alpha+\mu\sin\vartheta_1\right|^2}+\prod_{i=0}^{m}\Delta_i\frac{\nu\,|u_2(b)|^2}{\left|\beta+\mu\sin\vartheta_2\right|^2}\right.$$
$$\left.+\sum_{j=1}^{m}\left(\prod_{i=0}^{j-1}\Delta_i\right)\left|u_1(d_j^-)\right|^2\right]=0. \tag{26}$$

Since $\omega,\nu>0$, $\Delta_i>0$, $i=\overline{1,m}$ and $p_{j+1}>0$, $j=\overline{0,m}$, then we get $\mu=\overline{\mu}$. Consequently all eigenvalues of the discontinuous Dirac system (1)–(5) are real. □

The next lemma follows immediately from $U\perp V$ in the Hilbert space $\mathfrak{H}$,

$$U(x,\mu_1)=\begin{pmatrix}u(x,\mu_1)\\ \mathcal{L}_{\vartheta_1}(u(a,\mu_1))\\ \mathcal{L}_{\vartheta_2}(u(b,\mu_1))\\ \mathcal{L}_{2,1}(u(d_1,\mu_1))\\ \vdots\\ \mathcal{L}_{2,m}(u(d_m,\mu_1))\end{pmatrix},\quad V(x,\mu_2)=\begin{pmatrix}v(x,\mu_2)\\ \mathcal{L}_{\vartheta_1}(v(a,\mu_2))\\ \mathcal{L}_{\vartheta_2}(v(b,\mu_2))\\ \mathcal{L}_{2,1}(v(d_1,\mu_2))\\ \vdots\\ \mathcal{L}_{2,m}(v(d_m,\mu_2))\end{pmatrix},\quad \mu_1\neq\mu_2.$$

**Lemma 2.3.** *Two eigen-vector-functions* $u(x,\mu_1)=\begin{pmatrix}u_1(x,\mu_1)\\ u_2(x,\mu_1)\end{pmatrix}$ *and* $v(x,\mu_2)=\begin{pmatrix}v_1(x,\mu_2)\\ v_2(x,\mu_2)\end{pmatrix}$ *corresponding to different eigenvalues* $\mu_1$ *and* $\mu_2$ *of the problem* (1)–(5), *respectively, are orthogonal in the sense of*

$$\sum_{j=0}^{m}\left(\frac{\prod_{i=0}^{j}\Delta_i}{p_{j+1}}\right)\int_{d_j}^{d_{j+1}}\left[\{u_1(x,\mu_1)v_1(x,\mu_2)+u_2(x,\mu_1)v_2(x,\mu_2)\}\right]dx+\frac{1}{\omega}\mathcal{L}_{\vartheta_1}(u(a,\mu_1))\mathcal{L}_{\vartheta_1}(v(a,\mu_2))$$
$$+\frac{\prod_{i=0}^{m}\Delta_i}{\nu}\mathcal{L}_{\vartheta_2}(u(b,\mu_1))\mathcal{L}_{\vartheta_2}(v(b,\mu_2))+\sum_{j=1}^{m}\frac{\prod_{i=0}^{j-1}\Delta_i\mathcal{L}_{2,j}(u(d_j,\mu_1))\mathcal{L}_{2,j}(v(d_j,\mu_2))}{\delta_{2,j}^2}=0. \tag{27}$$

## 3. Construction of the Fundamental System of Solutions and the Characteristic Function

With a view to constructing the characteristic function $\Lambda(\lambda)$ we define two fundamental system of solutions of (1)

$$\mathfrak{y}(\cdot,\lambda)=\begin{pmatrix}\mathfrak{y}_1(\cdot,\lambda)\\ \mathfrak{y}_2(\cdot,\lambda)\end{pmatrix},\quad \mathfrak{z}(\cdot,\lambda)=\begin{pmatrix}\mathfrak{z}_1(\cdot,\lambda)\\ \mathfrak{z}_2(\cdot,\lambda)\end{pmatrix}, \tag{28}$$

where

$$\mathfrak{y}_1(x,\lambda)=\begin{cases}\mathfrak{y}_{1,1}(x,\lambda), & x\in[a,d_1),\\ \mathfrak{y}_{1,i}(x,\lambda), & x\in(d_{i-1},d_i),\ i=\overline{2,m}\\ \mathfrak{y}_{1,m+1}(x,\lambda), & x\in(d_m,b],\end{cases}\qquad \mathfrak{y}_2(x,\lambda)=\begin{cases}\mathfrak{y}_{2,1}(x,\lambda), & x\in[a,d_1),\\ \mathfrak{y}_{2,i}(x,\lambda), & x\in(d_{i-1},d_i),\ i=\overline{2,m}\\ \mathfrak{y}_{2,m+1}(x,\lambda), & x\in(d_m,b],\end{cases} \tag{29}$$

$$\mathfrak{z}_1(x,\lambda)=\begin{cases}\mathfrak{z}_{1,1}(x,\lambda), & x\in[a,d_1),\\ \mathfrak{z}_{1,i}(x,\lambda), & x\in(d_{i-1},d_i),\ i=\overline{2,m}\\ \mathfrak{z}_{1,m+1}(x,\lambda), & x\in(d_m,b],\end{cases}\qquad \mathfrak{z}_2(x,\lambda)=\begin{cases}\mathfrak{z}_{2,1}(x,\lambda), & x\in[a,d_1),\\ \mathfrak{z}_{2,i}(x,\lambda), & x\in(d_{i-1},d_i),\ i=\overline{2,m}\\ \mathfrak{z}_{2,m+1}(x,\lambda), & x\in(d_m,b].\end{cases} \tag{30}$$

For the next consideration, we need the well-known Theorem 1.1.1, cf. [11, p. 3]. By using the same method as in proof of Theorem 1.1.1 in [11, p. 3], we have the following results:

1. The problem, which consists of the equation (1) and the initial conditions,
$$y_1(a,\lambda)=\alpha'+\lambda\cos\vartheta_1,\quad y_2(a,\lambda)=\alpha+\lambda\sin\vartheta_1, \tag{31}$$
has the solution $y(x,\lambda)=\begin{pmatrix}\mathfrak{y}_{1,1}(x,\lambda)\\ \mathfrak{y}_{2,1}(x,\lambda)\end{pmatrix}$ on $[a,\, d_1)$. This solution is a unique solution and an entire function of $\lambda\in\mathbb{C}$ for each fixed $x\in[a,\, d_1]$.
2. The solutions $y(x,\lambda)=\begin{pmatrix}\mathfrak{y}_{1,i+1}(x,\lambda)\\ \mathfrak{y}_{2,i+1}(x,\lambda)\end{pmatrix}$, $i=\overline{1,m}$, of (1) on $(d_i,\, d_{i+1})$, which satisfy the initial conditions
$$y_1(d_i,\lambda)=\frac{\delta_{1,i}}{\delta'_{1,i}}\mathfrak{y}_{1,i}(d_i^-,\lambda),\quad y_2(d_i,\lambda)=\frac{\delta_{2,i}}{\delta'_{2,i}}\left[\mathfrak{y}_{2,i}(d_i^-,\lambda)+\lambda\mathfrak{y}_{1,i}(d_i^-,\lambda)\right],\quad i=\overline{1,m}, \tag{32}$$
are the unique solution and the entire functions of $\lambda\in\mathbb{C}$ for each fixed $x\in[d_i,\, d_{i+1}]$, $i=\overline{1,m}$.
3. Again, the differential equation of (1) on $(d_{i-1},\, d_i)$, $i=\overline{1,m}$, have unique solutions $y(x,\lambda)=\begin{pmatrix}\mathfrak{z}_{1,i}(x,\lambda)\\ \mathfrak{z}_{2,i}(x,\lambda)\end{pmatrix}$, $i=\overline{1,m}$, satisfying the initial conditions
$$y_1(d_i,\lambda)=\frac{\delta'_{1,i}}{\delta_{1,i}}\mathfrak{z}_{1,i+1}(d_i^+,\lambda),\quad y_2(d_i,\lambda)=\frac{\delta'_{2,i}}{\delta_{2,i}}\mathfrak{z}_{2,i+1}(d_i^+,\lambda)-\lambda\frac{\delta'_{1,i}}{\delta_{1,i}}\mathfrak{z}_{1,i+1}(d_i^+,\lambda),\quad i=\overline{1,m}, \tag{33}$$
which are entire functions of $\lambda\in\mathbb{C}$ for each fixed $x\in[d_{i-1},\, d_i]$, $i=\overline{1,m}$.
4. The differential equation (1) on $(d_m,\, b]$ together with initial conditions
$$y_1(b,\lambda)=\beta'+\lambda\cos\vartheta_2,\quad y_2(b,\lambda)=\beta+\lambda\sin\vartheta_2, \tag{34}$$
has an unique solution $y(x,\lambda)=\begin{pmatrix}\mathfrak{z}_{1,m+1}(x,\lambda)\\ \mathfrak{z}_{2,m+1}(x,\lambda)\end{pmatrix}$, which also is an entire function of parameter $\lambda\in\mathbb{C}$ for each fixed $x\in[d_m,\, b]$.

Then, from the initial conditions (31) and (34) we can verify that
$$\mathcal{L}_{\vartheta_1}(\mathfrak{y}(a,\lambda))=\omega,\ \text{ and }\ \mathcal{L}_{\vartheta_2}(\mathfrak{z}(b,\lambda))=-\nu, \tag{35}$$
respectively.

Let us consider the Wronskians
$$\Lambda_i(\lambda):=\mathcal{W}_\lambda(\mathfrak{y},\mathfrak{z};x)=\det\begin{pmatrix}\mathfrak{y}_{1,i}(x,\lambda) & \mathfrak{y}_{2,i}(x,\lambda)\\ \mathfrak{z}_{1,i}(x,\lambda) & \mathfrak{z}_{2,i}(x,\lambda)\end{pmatrix},\quad x\in[d_{i-1},\, d_i],\quad i=\overline{1,m+1}, \tag{36}$$
which are independent of $x\in[d_{i-1},\, d_i]$. Since the functions $\mathfrak{y}_{k,i}(x,\lambda)$ and $\mathfrak{z}_{k,i}(x,\lambda)$, $k=1,2$, $i=\overline{1,m+1}$, are entire of the parameter $\lambda$ for all $x\in[d_{i-1},\, d_i]$, $i=\overline{1,m+1}$, then the functions $\Lambda_i(\lambda)$, $i=\overline{1,m+1}$, are entire of the parameter $\lambda$. From the initial conditions (32) and (33), a short calculation gives
$$\Lambda_{i+1}(\lambda)=\frac{1}{\prod_{j=1}^{i}\Delta_j}\Lambda_1(\lambda),\quad i=\overline{1,m},\quad \lambda\in\mathbb{C}.$$
Consequently, the zeros of the functions $\Lambda_i(\lambda)$, $i=\overline{1,m+1}$, coincide. Then, it is convenient to define the characteristic function $\Lambda(\lambda)$ of problem (1)–(5) as
$$\Lambda(\lambda):=\prod_{j=1}^{i}\Delta_j\,\Lambda_{i+1}(\lambda)=\Lambda_1(\lambda),\quad i=\overline{1,m}. \tag{37}$$

**Lemma 3.1.** *The eigenvalues of the problem of finite number of transmission conditions* (1)–(5) *consist of the zeros of the characteristic function* $\Lambda(\lambda)$ *and form an at most countable set without finite limit points.*

*Proof.* From the initial conditions (31) and (32), the vector-valued function $\begin{pmatrix} \mathfrak{y}_1(x,\lambda) \\ \mathfrak{y}_2(x,\lambda) \end{pmatrix}$ satisfies the boundary condition (2) and the transmission conditions (4)–(5), respectively. Consequently, to find the eigenvalues of the boundary value problem (1)–(5) we mean that to find the zeros of the following equation

$$(\beta + \lambda \sin \vartheta_2)\mathfrak{y}_{1,m+1}(b,\lambda) - (\beta' + \lambda \cos \vartheta_2)\mathfrak{y}_{2,m+1}(b,\lambda) = 0,$$

which leads to

$$\Lambda(\lambda) = \prod_{i=1}^{m} \Delta_i \left[(\beta + \lambda \sin \vartheta_2)\mathfrak{y}_{1,m+1}(b,\lambda) - (\beta' + \lambda \cos \vartheta_2)\mathfrak{y}_{2,m+1}(b,\lambda)\right] = 0. \tag{38}$$

From Lemma 2.2, the function $\Lambda(\lambda)$, define in (38), can have only real roots, and so it does not vanish identically. Therefore, the zeros of $\Lambda(\lambda)$ form an at most countable set without finite limit points. □

The simplicity of the eigenvalues of the problem (1)–(5) is given from the following lemma.

**Lemma 3.2.** *Every zero of the function* $\Lambda(\lambda)$ *has multiplicity one.*

*Proof.* Let $\lambda, \mu \in \mathbb{C}$, $\lambda \neq \mu$. Then from (16), a short calculation gives

$$(\mu - \lambda)\left(\mathfrak{y}_1(x,\lambda)\mathfrak{y}_1(x,\mu) + \mathfrak{y}_2(x,\lambda)\mathfrak{y}_2(x,\mu)\right) = \frac{d}{dx}\left(\mathfrak{y}_1(x,\lambda)\mathfrak{y}_2(x,\mu) - \mathfrak{y}_1(x,\mu)\mathfrak{y}_2(x,\lambda)\right). \tag{39}$$

Applying the above equation and the definition of the solution $\begin{pmatrix} \mathfrak{y}_1(x,\lambda) \\ \mathfrak{y}_2(x,\lambda) \end{pmatrix}$ in (29), we can conclude that

$$\begin{aligned}
(\mu - \lambda)\sum_{j=0}^{m}\left(\frac{\prod_{i=0}^{j}\Delta_i}{p_{j+1}}\right)\int_{d_j}^{d_{j+1}} &\Big(\mathfrak{y}_{1,j+1}(x,\lambda)\mathfrak{y}_{1,j+1}(x,\mu) + \mathfrak{y}_{2,j+1}(x,\lambda)\mathfrak{y}_{2,j+1}(x,\mu)\Big)\,dx = \\
&\mathfrak{y}_{1,1}(d_1^-,\lambda)\mathfrak{y}_{2,1}(d_1^-,\mu) - \mathfrak{y}_{1,1}(d_1^-,\mu)\mathfrak{y}_{2,1}(d_1^-,\lambda) - \left(\mathfrak{y}_{1,1}(a,\lambda)\mathfrak{y}_{2,1}(a,\mu) - \mathfrak{y}_{1,1}(a,\mu)\mathfrak{y}_{2,1}(a,\lambda)\right) \\
&+ \prod_{i=0}^{m}\Delta_i\left(\mathfrak{y}_{1,m+1}(b,\lambda)\mathfrak{y}_{2,m+1}(b,\mu) - \mathfrak{y}_{1,m+1}(b,\mu)\mathfrak{y}_{2,m+1}(b,\lambda)\right) \\
&- \prod_{i=0}^{m}\Delta_i\left(\mathfrak{y}_{1,m+1}(d_m^+,\lambda)\mathfrak{y}_{2,m+1}(d_m^+,\mu) - \mathfrak{y}_{1,m+1}(d_m^+,\mu)\mathfrak{y}_{2,m+1}(d_m^+,\lambda)\right) \\
&+ \sum_{j=1}^{m-1}\left(\prod_{i=0}^{j}\Delta_i\right)\Big(\mathfrak{y}_{1,j+1}(d_{j+1}^-,\lambda)\mathfrak{y}_{2,j+1}(d_{j+1}^-,\mu) - \mathfrak{y}_{1,j+1}(d_{j+1}^-,\mu)\mathfrak{y}_{2,j+1}(d_{j+1}^-,\lambda)\Big) \\
&- \sum_{j=1}^{m-1}\left(\prod_{i=0}^{j}\Delta_i\right)\Big(\mathfrak{y}_{1,j+1}(d_j^+,\lambda)\mathfrak{y}_{2,j+1}(d_j^+,\mu) - \mathfrak{y}_{1,j+1}(d_j^+,\mu)\mathfrak{y}_{2,j+1}(d_j^+,\lambda)\Big),
\end{aligned}$$

which leads to

$$\begin{aligned}
&(\mu-\lambda)\sum_{j=0}^{m}\left(\frac{\prod_{i=0}^{j}\Delta_i}{p_{j+1}}\right)\int_{d_j}^{d_{j+1}}\Big(\mathfrak{y}_{1,j+1}(x,\lambda)\mathfrak{y}_{1,j+1}(x,\mu)+\mathfrak{y}_{2,j+1}(x,\lambda)\mathfrak{y}_{2,j+1}(x,\mu)\Big)\,dx=\\
&\prod_{i=0}^{m}\Delta_i\left(\mathfrak{y}_{1,m+1}(b,\lambda)\mathfrak{y}_{2,m+1}(b,\mu)-\mathfrak{y}_{1,m+1}(b,\mu)\mathfrak{y}_{2,m+1}(b,\lambda)\right)-\left(\mathfrak{y}_{1,1}(a,\lambda)\mathfrak{y}_{2,1}(a,\mu)-\mathfrak{y}_{1,1}(a,\mu)\mathfrak{y}_{2,1}(a,\lambda)\right)\\
&+\sum_{j=0}^{m-1}\left(\prod_{i=0}^{j}\Delta_i\right)\Big(\mathfrak{y}_{1,j+1}(d_{j+1}^{-},\lambda)\mathfrak{y}_{2,j+1}(d_{j+1}^{-},\mu)-\mathfrak{y}_{1,j+1}(d_{j+1}^{-},\mu)\mathfrak{y}_{2,j+1}(d_{j+1}^{-},\lambda)\Big)\\
&-\sum_{j=1}^{m}\left(\prod_{i=0}^{j}\Delta_i\right)\Big(\mathfrak{y}_{1,j+1}(d_{j}^{+},\lambda)\mathfrak{y}_{2,j+1}(d_{j}^{+},\mu)-\mathfrak{y}_{1,j+1}(d_{j}^{+},\mu)\mathfrak{y}_{2,j+1}(d_{j}^{+},\lambda)\Big).
\end{aligned}\tag{40}$$

From the initial conditions (31) and (32) we have the order respectively

$$\mathfrak{y}_{1,1}(a,\lambda)\mathfrak{y}_{2,1}(a,\mu)-\mathfrak{y}_{1,1}(a,\mu)\mathfrak{y}_{2,1}(a,\lambda)=(\mu-\lambda)\omega \tag{41}$$

$$\begin{aligned}
&\sum_{j=0}^{m-1}\left(\prod_{i=0}^{j}\Delta_i\right)\Big(\mathfrak{y}_{1,j+1}(d_{j+1}^{-},\lambda)\mathfrak{y}_{2,j+1}(d_{j+1}^{-},\mu)-\mathfrak{y}_{1,j+1}(d_{j+1}^{-},\mu)\mathfrak{y}_{2,j+1}(d_{j+1}^{-},\lambda)\Big)\\
&-\sum_{j=1}^{m}\left(\prod_{i=0}^{j}\Delta_i\right)\Big(\mathfrak{y}_{1,j+1}(d_{j}^{+},\lambda)\mathfrak{y}_{2,j+1}(d_{j}^{+},\mu)-\mathfrak{y}_{1,j+1}(d_{j}^{+},\mu)\mathfrak{y}_{2,j+1}(d_{j}^{+},\lambda)\Big)\\
&=(\lambda-\mu)\sum_{j=0}^{m-1}\left(\prod_{i=0}^{j}\Delta_i\right)\mathfrak{y}_{1,j+1}(d_{j+1}^{-},\lambda)\mathfrak{y}_{1,j+1}(d_{j+1}^{-},\mu).
\end{aligned}\tag{42}$$

Substituting equations (41) and (42) in (40), we obtain

$$\begin{aligned}
(\mu-\lambda)\sum_{j=0}^{m}\left(\frac{\prod_{i=0}^{j}\Delta_i}{p_{j+1}}\right)\int_{d_j}^{d_{j+1}}\Big(\mathfrak{y}_{1,j+1}(x,\lambda)\mathfrak{y}_{1,j+1}(x,\mu)+\mathfrak{y}_{2,j+1}(x,\lambda)\mathfrak{y}_{2,j+1}(x,\mu)\Big)\,dx&=(\lambda-\mu)\omega\\
+\prod_{i=0}^{m}\Delta_i\left(\mathfrak{y}_{1,m+1}(b,\lambda)\mathfrak{y}_{2,m+1}(b,\mu)-\mathfrak{y}_{1,m+1}(b,\mu)\mathfrak{y}_{2,m+1}(b,\lambda)\right)&\\
+(\lambda-\mu)\sum_{j=0}^{m-1}\left(\prod_{i=0}^{j}\Delta_i\right)\mathfrak{y}_{1,j+1}(d_{j+1}^{-},\lambda)\mathfrak{y}_{1,j+1}(d_{j+1}^{-},\mu).&
\end{aligned}\tag{43}$$

Dividing both sides of (44) by $(\lambda-\mu)$ and by letting $\mu\longrightarrow\lambda$, we have

$$\begin{aligned}
&-\sum_{j=0}^{m}\left(\frac{\prod_{i=0}^{j}\Delta_i}{p_{j+1}}\right)\int_{d_j}^{d_{j+1}}\Big(\big(\mathfrak{y}_{1,j+1}(x,\lambda)\big)^2+\big(\mathfrak{y}_{2,j+1}(x,\lambda)\big)^2\Big)\,dx=\omega+\sum_{j=0}^{m-1}\left(\prod_{i=0}^{j}\Delta_i\right)\Big(\mathfrak{y}_{1,j+1}(d_{j+1}^{-},\lambda)\Big)^2\\
&+\prod_{i=0}^{m}\Delta_i\left(\mathfrak{y}_{2,m+1}(b,\lambda)\frac{\partial\mathfrak{y}_{1,m+1}(b,\lambda)}{\partial\lambda}-\mathfrak{y}_{1,m+1}(b,\lambda)\frac{\partial\mathfrak{y}_{2,m+1}(b,\lambda)}{\partial\lambda}\right).
\end{aligned}\tag{44}$$

To prove the lemma, we need to prove that the equation (38) has only simple zeros. To prove this, we suppose the conversely, that is the equation (38) has a zero, say $\lambda^*$, of multiplicity two. Then the next two equations hold

$$(\beta+\lambda^*\sin\vartheta_2)\mathfrak{y}_{1,m+1}(b,\lambda^*)-(\beta'+\lambda^*\cos\vartheta_2)\mathfrak{y}_{2,m+1}(b,\lambda^*)=0, \tag{45}$$

$$(\beta+\lambda^*\sin\vartheta_2)\frac{\partial \mathfrak{y}_{1,m+1}(b,\lambda^*)}{\partial\lambda}+\sin\vartheta_2\mathfrak{y}_{1,m+1}(b,\lambda^*)-(\beta'+\lambda^*\cos\vartheta_2)\frac{\partial \mathfrak{y}_{2,m+1}(b,\lambda^*)}{\partial\lambda}-\cos\vartheta_2\mathfrak{y}_{2,m+1}(b,\lambda^*)=0. \quad (46)$$

Because of $\nu \neq 0$, $\beta - \lambda^* \sin\vartheta_2 \neq 0$, consequently the above two equations can be rewritten as

$$\mathfrak{y}_{1,m+1}(b,\lambda^*)=\frac{\beta'+\lambda^*\cos\vartheta_2}{\beta+\lambda^*\sin\vartheta_2}\mathfrak{y}_{2,m+1}(b,\lambda^*), \quad (47)$$

$$\frac{\partial \mathfrak{y}_{1,m+1}(b,\lambda^*)}{\partial\lambda}=\frac{\beta'+\lambda^*\cos\vartheta_2}{\beta+\lambda^*\sin\vartheta_2}\frac{\partial \mathfrak{y}_{2,m+1}(b,\lambda^*)}{\partial\lambda}+\frac{\nu}{(\beta+\lambda^*\sin\vartheta_2)^2}\mathfrak{y}_{2,m+1}(b,\lambda^*). \quad (48)$$

Then, from equations (47) and (48) in (44),with $\lambda = \lambda^*$, we get

$$\begin{aligned}\sum_{j=0}^{m}\left(\frac{\prod_{i=0}^{j}\Delta_i}{p_{j+1}}\right)\int_{d_j}^{d_{j+1}}\left(\left(\mathfrak{y}_{1,j+1}(x,\lambda^*)\right)^2+\left(\mathfrak{y}_{2,j+1}(x,\lambda^*)\right)^2\right)dx+\omega+\frac{\nu\left(\mathfrak{y}_{2,m+1}(b,\lambda^*)\right)^2}{(\beta+\lambda^*\sin\vartheta_2)^2}\\ +\sum_{j=0}^{m-1}\left(\prod_{i=0}^{j}\Delta_i\right)\left(\mathfrak{y}_{1,j+1}(d_{j+1}^-,\lambda^*)\right)^2=0.\end{aligned} \quad (49)$$

Since $\omega, \nu > 0$, $\Delta_i > 0$, $i = \overline{1,m}$ and $p_j > 0$, $j = \overline{1,m+1}$, consequently, $\mathfrak{y}_{1,j}(x,\lambda^*) = \mathfrak{y}_{2,j}(x,\lambda^*) = 0$, $j = \overline{1,m+1}$, which is impossible. Therefore, all zeros of $\Lambda(\lambda)$ are simple.. □

Here, the sequence $\{\lambda_n\}_{n\in\mathbb{Z}}$ will be a sequence of eigenvalues of the problem (1)–(5) corresponding to a sequence of eigen-vector-functions $\{\mathfrak{y}(\cdot,\lambda_n)\}_{n\in\mathbb{Z}}$. From equation (37), we have

$$(\alpha'+\lambda\cos\vartheta_1)\mathfrak{z}_{2,1}(a,\lambda)-(\alpha+\lambda\sin\vartheta_1)\mathfrak{z}_{1,1}(a,\lambda)=-\Lambda(\lambda) \quad (50)$$

and since $\mathfrak{z}(\cdot,\lambda)$, defined in (30), satisfies (3)–(5), then $\{\mathfrak{z}(\cdot,\lambda_n)\}_{n\in\mathbb{Z}}$ is another sequence of eigen-vector-functions. Since all eigenvalues of the problem (1)–(5) are of multiplicity one, then there exist non-zero constants $c_n$ such that

$$\mathfrak{z}(x,\lambda_n)=c_n\mathfrak{y}(x,\lambda_n),\quad x\in[d_0,d_1)\bigcup_{i=2}^{m}(d_{i-1},d_i)\bigcup(d_m,d_{m+1}],\ n\in\mathbb{Z}. \quad (51)$$

Since all eigenvalues of our problem (1)–(5) are real, consequently, we can now assume that all eigen-vector-functions are real-valued functions.

## 4. Asymptotic Behavior of Eigenvalues and Eigen-Vector-Functions

Our main in this section is to derive the asymptotic formulas of the eigenvalues $\{\lambda_n\}_{n\in\mathbb{Z}}$ of the problem (1)–(5) and the corresponding eigen-vector-functions $\{\mathfrak{y}(\cdot,\lambda_n)\}_{n\in\mathbb{Z}}$. To obtain these asymptotics, we need the following lemma, see [12].

**Lemma 4.1.** *The solution $\mathfrak{y}(\cdot,\lambda)$, defined in* (29), *satisfies the following integral equations, for $x \in [a, d_1)$,*

$$\mathfrak{y}_{1,1}(x,\lambda)=-(\alpha+\lambda\sin\vartheta_1)\sin\left[\frac{\lambda(x-a)}{p_1}\right]+(\alpha'+\lambda\cos\vartheta_1)\cos\left[\frac{\lambda(x-a)}{p_1}\right]-\mathfrak{T}_{1,1}[\mathfrak{y}_{1,1}](x,\lambda)-\widetilde{\mathfrak{T}}_{2,1}[\mathfrak{y}_{2,1}](x,\lambda), \quad (52)$$

$$\mathfrak{y}_{2,1}(x,\lambda)=(\alpha+\lambda\sin\vartheta_1)\cos\left[\frac{\lambda(x-a)}{p_1}\right]+(\alpha'+\lambda\cos\vartheta_1)\sin\left[\frac{\lambda(x-a)}{p_1}\right]+\widetilde{\mathfrak{T}}_{1,1}[\mathfrak{y}_{1,1}](x,\lambda)-\mathfrak{T}_{2,1}[\mathfrak{y}_{2,1}](x,\lambda), \quad (53)$$

*and for $x \in (d_i, d_{i+1})$, $i = \overline{1,m}$,*

$$\begin{aligned}\mathfrak{y}_{1,i+1}(x,\lambda)=&\frac{\delta_{1,i}}{\delta'_{1,i}}\mathfrak{y}_{1,i}(d_i^-,\lambda)\cos\left[\frac{\lambda(x-d_i)}{p_{i+1}}\right]-\frac{\delta_{2,i}}{\delta'_{2,i}}\left[\mathfrak{y}_{2,i}(d_i^-,\lambda)+\lambda\mathfrak{y}_{1,i}(d_i^-,\lambda)\right]\sin\left[\frac{\lambda(x-d_i)}{p_{i+1}}\right]\\ &-\mathfrak{T}_{1,i+1}[\mathfrak{y}_{1,i+1}](x,\lambda)-\widetilde{\mathfrak{T}}_{2,i+1}[\mathfrak{y}_{2,i+1}](x,\lambda),\quad i=\overline{1,m},\end{aligned} \quad (54)$$

$$\mathfrak{y}_{2,i+1}(x,\lambda) = \frac{\delta_{1,i}}{\delta'_{1,i}}\mathfrak{y}_{1,i}(d_i^-,\lambda)\sin\left[\frac{\lambda(x-d_i)}{p_{i+1}}\right] + \frac{\delta_{2,i}}{\delta'_{2,i}}\left[\mathfrak{y}_{2,i}(d_i^-,\lambda) + \lambda\mathfrak{y}_{1,i}(d_i^-,\lambda)\right]\cos\left[\frac{\lambda(x-d_i)}{p_{i+1}}\right] + \widetilde{\mathfrak{T}}_{1,i+1}[\mathfrak{y}_{1,i+1}](x,\lambda) - \mathfrak{T}_{2,i+1}[\mathfrak{y}_{2,i+1}](x,\lambda), \quad i=\overline{1,m}, \tag{55}$$

*where $\mathfrak{T}_{k,i+1}$ and $\widetilde{\mathfrak{T}}_{k,i+1}$ ($k = 1,2$, $i = \overline{0,m}$) are the Volterra integral operators defined by*

$$\mathfrak{T}_{k,i+1}[\mathfrak{y}_{k,i+1}](x,\lambda) := \frac{1}{p_{i+1}}\int_{d_i}^{x}\sin\left[\frac{\lambda(x-t)}{p_{i+1}}\right]q_k(t)\mathfrak{y}_{k,i+1}(t,\lambda)\,dt, \tag{56}$$

$$\widetilde{\mathfrak{T}}_{k,i+1}[\mathfrak{y}_{k,i+1}](x,\lambda) := \frac{1}{p_{i+1}}\int_{d_i}^{x}\cos\left[\frac{\lambda(x-t)}{p_{i+1}}\right]q_k(t)\mathfrak{y}_{k,i+1}(t,\lambda)\,dt. \tag{57}$$

*Proof.* Taking into account (29), (31) and (32), for proving this lemma we need only substitute $p_{i+1}\mathfrak{y}'_{2,i+1}(t,\lambda) - \lambda\mathfrak{y}_{1,i+1}(t,\lambda)$ and $-p_{1+i}\mathfrak{y}'_{1,i+1}(t,\lambda) - \lambda\mathfrak{y}_{2,i+1}(t,\lambda)$ instead of $q_1(t)\mathfrak{y}_{1,i+1}(t,\lambda)$ and $q_2(t)\mathfrak{y}_{2,i+1}(t,\lambda)$, respectively, in the integrals $-\mathfrak{T}_{1,i+1}[\mathfrak{y}_{1,i+1}](x,\lambda) - \widetilde{\mathfrak{T}}_{2,i+1}[\mathfrak{y}_{2,i+1}](x,\lambda)$ and $\widetilde{\mathfrak{T}}_{1,i+1}[\mathfrak{y}_{1,i+1}](x,\lambda) - \mathfrak{T}_{2,i+1}[\mathfrak{y}_{2,i+1}](x,\lambda)$, $i=\overline{0,m}$, and integrate by parts. □

Similarly one can establish the following lemma for the solution $\mathfrak{z}(\cdot,\lambda)$ defined in Section 3.

**Lemma 4.2.** *The solution $\mathfrak{z}(\cdot,\lambda)$, defined in* (30)*, satisfies the following integral equations:*

$$\mathfrak{z}_{1,m+1}(x,\lambda) = (\beta' + \lambda\cos\vartheta_2)\cos\left[\frac{\lambda(b-x)}{p_{m+1}}\right] + (\beta + \lambda\sin\vartheta_2)\sin\left[\frac{\lambda(b-x)}{p_{m+1}}\right] - \mathfrak{S}_{1,m+1}[\mathfrak{z}_{1,m+1}](x,\lambda) - \widetilde{\mathfrak{S}}_{2,m+1}[\mathfrak{z}_{2,m+1}](x,\lambda), \quad x\in(d_m,b], \tag{58}$$

$$\mathfrak{z}_{2,m+1}(x,\lambda) = -(\beta' + \lambda\cos\vartheta_2)\sin\left[\frac{\lambda(b-x)}{p_{m+1}}\right] + (\beta + \lambda\sin\vartheta_2)\cos\left[\frac{\lambda(b-x)}{p_{m+1}}\right] - \widetilde{\mathfrak{S}}_{1,m+1}[\mathfrak{z}_{1,m+1}](x,\lambda) - \mathfrak{S}_{2,m+1}[\mathfrak{y}_{2,m+1}](x,\lambda), \quad x\in(d_m,b], \tag{59}$$

$$\mathfrak{z}_{1,i}(x,\lambda) = \frac{\delta'_{1,i}}{\delta_{1,i}}\mathfrak{z}_{1,i+1}(d_i^+,\lambda)\cos\left[\frac{\lambda(d_i-x)}{p_i}\right] + \left[\frac{\delta'_{2,i}}{\delta_{2,i}}\mathfrak{z}_{2,i+1}(d_i^+,\lambda) - \lambda\frac{\delta'_{1,i}}{\delta_{1,i}}\mathfrak{z}_{1,i+1}(d_i^+,\lambda)\right]\sin\left[\frac{\lambda(d_i-x)}{p_i}\right] - \mathfrak{S}_{1,i}[\mathfrak{z}_{1,i}](x,\lambda) + \widetilde{\mathfrak{S}}_{2,i}[\mathfrak{z}_{2,i}](x,\lambda), \quad x\in(d_{i-1},d_i),\ i=\overline{1,m}, \tag{60}$$

$$\mathfrak{z}_{2,i}(x,\lambda) = -\frac{\delta'_{1,i}}{\delta_{1,i}}\mathfrak{z}_{1,i+1}(d_i^+,\lambda)\sin\left[\frac{\lambda(d_i-x)}{p_i}\right] + \left[\frac{\delta'_{2,i}}{\delta_{2,i}}\mathfrak{z}_{2,i+1}(d_i^+,\lambda) - \lambda\frac{\delta'_{1,i}}{\delta_{1,i}}\mathfrak{z}_{1,i+1}(d_i^+,\lambda)\right]\cos\left[\frac{\lambda(d_i-x)}{p_i}\right] - \widetilde{\mathfrak{S}}_{1,i}[\mathfrak{z}_{1,i}](x,\lambda) - \mathfrak{S}_{2,i}[\mathfrak{z}_{2,i}](x,\lambda), \quad x\in(d_{i-1},d_i),\ i=\overline{1,m}, \tag{61}$$

*where $\mathfrak{S}_{k,i}$ and $\widetilde{\mathfrak{S}}_{k,i}$ ($k = 1,2$, $i = \overline{1,m+1}$) are the Volterra integral operators defined by*

$$\mathfrak{S}_{k,i}[\mathfrak{z}_{k,i}](x,\lambda) := \frac{1}{p_i}\int_{x}^{d_i}\sin\left[\frac{\lambda(t-x)}{p_i}\right]q_k(t)\mathfrak{z}_{k,i}(t,\lambda)\,dt, \tag{62}$$

$$\widetilde{\mathfrak{S}}_{k,i}[\mathfrak{z}_{k,i}](x,\lambda) := \frac{1}{p_i}\int_{x}^{d_i}\cos\left[\frac{\lambda(t-x)}{p_i}\right]q_k(t)\mathfrak{z}_{k,i}(t,\lambda)\,dt. \tag{63}$$

Let $\tau = |\Im\lambda|$. In the next lemma, the solution $\begin{pmatrix}\mathfrak{y}_{1,1}(x,\lambda)\\ \mathfrak{y}_{2,1}(x,\lambda)\end{pmatrix}$ has the asymptotic representation uniformly with respect to $x$, $x\in[a,d_1)$, $|\lambda|\longrightarrow\infty$, cf. [11, p. 55], see also [4, 19, 20, 23].

**Lemma 4.3.** *Let $\tau = |\Im\lambda|$. Then, for $|\lambda| \longrightarrow \infty$ and $x \in [a, d_1)$, we have*

$$\mathfrak{y}_{1,1}(x,\lambda) = \lambda\cos\left[\frac{\lambda(x-a)}{p_1} + \vartheta_1\right] + O\left(\exp\left[\tau\frac{(x-a)}{p_1}\right]\right), \tag{64}$$

$$\mathfrak{y}_{2,1}(x,\lambda) = \lambda\sin\left[\frac{\lambda(x-a)}{p_1} + \vartheta_1\right] + O\left(\exp\left[\tau\frac{(x-a)}{p_1}\right]\right). \tag{65}$$

*Proof.* By the same technique as in the proof of Lemma 1.11.1 in [11, p. 55], we can prove (52) and (53). □

Consequently, by using the results of the above lemma and equations (54)–(55), the asymptotic formulas of the solution $\begin{pmatrix} \mathfrak{y}_{1,i+1}(x,\lambda) \\ \mathfrak{y}_{2,i+1}(x,\lambda) \end{pmatrix}$, with respect to $x \in (d_i, d_{i+1})$, $i = \overline{1,m}$, are given from the following lemma.

**Lemma 4.4.** *Let $\tau = |\Im\lambda|$. Then, for $|\lambda| \longrightarrow \infty$, we get*

$$\mathfrak{y}_{1,2}(x,\lambda) = -\lambda^2\frac{\delta_{2,1}}{\delta'_{2,1}}\cos\left[\frac{\lambda(d_1-a)}{p_1} + \vartheta_1\right]\sin\left[\frac{\lambda(x-d_1)}{p_2}\right] + O\left(\lambda\exp\left[\tau\left(\frac{x-d_1}{p_2} + \frac{d_1-a}{p_1}\right)\right]\right), \tag{66}$$

$$\mathfrak{y}_{2,2}(x,\lambda) = \lambda^2\frac{\delta_{2,1}}{\delta'_{2,1}}\cos\left[\frac{\lambda(d_1-a)}{p_1} + \vartheta_1\right]\cos\left[\frac{\lambda(x-d_1)}{p_2}\right] + O\left(\lambda\exp\left[\tau\left(\frac{x-d_1}{p_2} + \frac{d_1-a}{p_1}\right)\right]\right), \tag{67}$$

*where $x \in (d_1, d_2)$ and for $x \in (d_i, d_{i+1})$, $i = \overline{2,m}$, we have*

$$\begin{aligned}\mathfrak{y}_{1,i+1}(x,\lambda) =&(-1)^i\lambda^{i+1}\prod_{j=1}^{i}\frac{\delta_{2,j}}{\delta'_{2,j}}\times\cos\left[\frac{\lambda(d_1-a)}{p_1} + \vartheta_1\right]\times\prod_{j=2}^{i}\sin\left[\frac{\lambda\left(d_j-d_{j-1}\right)}{p_j}\right]\times\sin\left[\frac{\lambda(x-d_i)}{p_{i+1}}\right]\\ &+O\left(\lambda^i\exp\left[\tau\left(\frac{x-d_i}{p_{i+1}} + \sum_{j=1}^{i}\frac{d_j-d_{j-1}}{p_j}\right)\right]\right),\end{aligned} \tag{68}$$

$$\begin{aligned}\mathfrak{y}_{2,i+1}(x,\lambda) =&(-1)^{i-1}\lambda^{i+1}\prod_{j=1}^{i}\frac{\delta_{2,j}}{\delta'_{2,j}}\times\cos\left[\frac{\lambda(d_1-a)}{p_1} + \vartheta_1\right]\times\prod_{j=2}^{i}\sin\left[\frac{\lambda\left(d_j-d_{j-1}\right)}{p_j}\right]\times\cos\left[\frac{\lambda(x-d_i)}{p_{i+1}}\right]\\ &+O\left(\lambda^i\exp\left[\tau\left(\frac{x-d_i}{p_{i+1}} + \sum_{j=1}^{i}\frac{d_j-d_{j-1}}{p_j}\right)\right]\right).\end{aligned} \tag{69}$$

As in Lemma 4.3 and Lemma 4.4, we have proven the following lemma.

**Lemma 4.5.** *Let $\tau = |\Im\lambda|$. Then, for $|\lambda| \longrightarrow \infty$, we obtain*

$$\mathfrak{z}_{1,m+1}(x,\lambda) = \lambda\cos\left[\frac{\lambda(b-x)}{p_{m+1}} - \vartheta_2\right] + O\left(\exp\left[\tau\frac{(b-x)}{p_{m+1}}\right]\right), \quad x \in (d_m, b], \tag{70}$$

$$\mathfrak{z}_{2,m+1}(x,\lambda) = -\lambda\sin\left[\frac{\lambda(b-x)}{p_{m+1}} - \vartheta_2\right] + O\left(\exp\left[\tau\frac{(b-x)}{p_{m+1}}\right]\right), \quad x \in (d_m, b], \tag{71}$$

$$\begin{aligned}\mathfrak{z}_{1,m}(x,\lambda) =& -\lambda^2\frac{\delta'_{1,m}}{\delta_{1,m}}\cos\left[\frac{\lambda(b-d_m)}{p_{m+1}} - \vartheta_2\right]\sin\left[\frac{\lambda(d_m-x)}{p_m}\right]\\ &+O\left(\lambda\exp\left[\tau\left(\frac{d_m-x}{p_m} + \frac{b-d_m}{p_{m+1}}\right)\right]\right), \quad x \in (d_{m-1}, d_m),\end{aligned} \tag{72}$$

$$\mathfrak{z}_{2,m}(x,\lambda) = -\lambda^2 \frac{\delta'_{1,m}}{\delta_{1,m}} \cos\left[\frac{\lambda(b-d_m)}{p_{m+1}} - \vartheta_2\right] \cos\left[\frac{\lambda(d_m - x)}{p_m}\right] + O\left(\lambda \exp\left[\tau\left(\frac{d_m - x}{p_m} + \frac{b-d_m}{p_{m+1}}\right)\right]\right), \quad x \in (d_{m-1}, d_m), \tag{73}$$

$$\mathfrak{z}_{1,m-i}(x,\lambda) = (-1)^{i+1}\lambda^{i+2} \prod_{j=0}^{i} \frac{\delta'_{1,m-j}}{\delta_{1,m-j}} \times \cos\left[\frac{\lambda(b-d_m)}{p_{m+1}} - \vartheta_2\right] \times \prod_{j=1}^{i} \sin\left[\frac{\lambda\left(d_{m+1-j} - d_{m-j}\right)}{p_{m+1-j}}\right] \times \sin\left[\frac{\lambda(d_{m-i} - x)}{p_{m-i}}\right] + O\left(\lambda^{i+1} \exp\left[\tau\left(\frac{d_{m-i} - x}{p_{m-i}} + \sum_{j=0}^{i} \frac{d_{m+1-j} - d_{m-j}}{p_{m+1-j}}\right)\right]\right), \ x \in (d_{m-i-1}, d_{m-i}),\ i = \overline{1, m-1}, \tag{74}$$

$$\mathfrak{z}_{2,m-i}(x,\lambda) = (-1)^{i+1}\lambda^{i+2} \prod_{j=0}^{i} \frac{\delta'_{1,m-j}}{\delta_{1,m-j}} \times \cos\left[\frac{\lambda(b-d_m)}{p_{m+1}} - \vartheta_2\right] \times \prod_{j=1}^{i} \sin\left[\frac{\lambda\left(d_{m+1-j} - d_{m-j}\right)}{p_{m+1-j}}\right] \times \cos\left[\frac{\lambda(d_{m-i} - x)}{p_{m-i}}\right] + O\left(\lambda^{i+1} \exp\left[\tau\left(\frac{d_{m-i} - x}{p_{m-i}} + \sum_{j=0}^{i} \frac{d_{m+1-j} - d_{m-j}}{p_{m+1-j}}\right)\right]\right), \ x \in (d_{m-i-1}, d_{m-i}),\ i = \overline{1, m-1}. \tag{75}$$

Now we are ready to derived the needed asymptotic formulas for eigenvalues and eigen-vector-functions of the boundary-value-transmission problem (1)–(5).

**Theorem 4.6.** *The eigenvalues of the consider problem* (1)–(5)*, whose behavior may be expressed by the sequence* $\left\{\lambda_n^{(\ell)}\right\}_{\ell=1}^{m+1}$*, have the following asymptotic representation for* $n \to \infty$*:*

$$\lambda_n^{(1)} := \frac{p_1}{d_1 - a}\left((n + \frac{1}{2})\pi - \vartheta_1\right) + O\left(\frac{1}{n}\right), \tag{76}$$

$$\lambda_n^{(\ell)} := \frac{p_\ell n\pi}{d_\ell - d_{\ell-1}} + O\left(\frac{1}{n}\right), \quad \ell = \overline{2, m}, \tag{77}$$

$$\lambda_n^{(m+1)} := \frac{p_{m+1}}{b - d_m}\left((n + \frac{1}{2})\pi + \vartheta_2\right) + O\left(\frac{1}{n}\right). \tag{78}$$

*Proof.* By substituting (68) and (69), with $i = m$, in equation (38), we obtain the asymptotic representation of the characteristic function $\Lambda(\lambda)$

$$\Lambda(\lambda) = (-1)^m \prod_{j=1}^{m} \frac{\delta'_{1,j}}{\delta_{1,j}} \times \lambda^{m+2} \cos\left[\frac{\lambda(d_1 - a)}{p_1} + \vartheta_1\right] \times \prod_{j=2}^{m} \sin\left[\frac{\lambda\left(d_j - d_{j-1}\right)}{p_j}\right] \times \cos\left[\frac{\lambda(b-d_m)}{p_{m+1}} - \vartheta_2\right] + O\left(\lambda^{m+1} \exp\left[\tau\left(\frac{b-d_m}{p_{m+1}} + \sum_{j=1}^{m} \frac{d_j - d_{j-1}}{p_j}\right)\right]\right). \tag{79}$$

The equation (79) can be rewritten as

$$\Lambda(\lambda) = \Xi_1(\lambda) + \Xi_2(\lambda),$$

where

$$\Xi_1(\lambda) := (-1)^m \prod_{j=1}^{m} \frac{\delta'_{1,j}}{\delta_{1,j}} \times \lambda^{m+2} \cos\left[\frac{\lambda\,(d_1-a)}{p_1} + \vartheta_1\right] \times \prod_{j=2}^{m} \sin\left[\frac{\lambda\left(d_j - d_{j-1}\right)}{p_j}\right] \times \cos\left[\frac{\lambda\,(b-d_m)}{p_{m+1}} - \vartheta_2\right], \quad (80)$$

$$\Xi_2(\lambda) := O\left(\lambda^{m+1} \exp\left[\tau\left(\frac{b-d_m}{p_{m+1}} + \sum_{j=1}^{m} \frac{d_j - d_{j-1}}{p_j}\right)\right]\right). \quad (81)$$

Now we apply the Rouche's Theorem, for $\Xi_1(\lambda)$ and $\Xi_2(\lambda)$, which states that: For any two analytic functions $f_1(\lambda)$ and $f_2(\lambda)$ inside and on a closed contour $\Gamma$, if $f_1(\lambda) > f_2(\lambda)$ on $\Gamma$, then $f_1(\lambda)$ and $f_1(\lambda) + f_2(\lambda)$ have the same number of zeros inside $\Gamma$, provided that each zero is counted according to its multiplicity. Therefore, the characteristic function $\Lambda(\lambda)$ of our problem has the same number of zeros inside the suitable contour as $\Xi_1(\lambda)$. Consequently, with a short calculation, the asymptotics formulas of the zeros of $\Lambda(\lambda)$ are given by (76), (77) and (78). □

By putting the asymptotic expressions of the eigenvalues $\lambda_n^{(\ell)}$, $\ell = \overline{1, m+1}$ in (64)–(69) we can obtain the corresponding asymptotic expressions for eigen-vector-functions of the boundary-value-transmission problem (1)–(5):

$$\mathfrak{y}(x, \lambda_n^{(\ell)}) = \begin{pmatrix} \mathfrak{y}_1(x, \lambda_n^{(\ell)}) \\ \mathfrak{y}_2(x, \lambda_n^{(\ell)}) \end{pmatrix},$$

$$\mathfrak{y}_k(x, \lambda_n^{(\ell)}) = \begin{cases} \mathfrak{y}_{k,1}(x, \lambda_n^{(\ell)}), & x \in [a, d_1), \\ \mathfrak{y}_{k,i}(x, \lambda_n^{(\ell)}), & x \in (d_{i-1}, d_i),\ (i = \overline{2, m}), \quad \ell = \overline{1, m+1},\ k = 1, 2, \\ \mathfrak{y}_{k,m+1}(x, \lambda_n^{(\ell)}), & x \in (d_m, b], \end{cases} \quad (82)$$

where

$$\mathfrak{y}_1(x, \lambda_n^{(1)}) = \begin{cases} \frac{p_1}{d_1 - a}\left((n+\tfrac{1}{2})\pi - \vartheta_1\right) \cos\left[\left((n+\tfrac{1}{2})\pi - \vartheta_1\right)\frac{x-a}{d_1-a} + \vartheta_1\right] + O(1), & x \in [a, d_1), \\ O\left(n^i\right), \quad x \in (d_i, d_{i+1}),\ (i = \overline{1, m}), \end{cases} \quad (83)$$

$$\mathfrak{y}_2(x, \lambda_n^{(1)}) = \begin{cases} \frac{p_1}{d_1 - a}\left((n+\tfrac{1}{2})\pi - \vartheta_1\right) \sin\left[\left((n+\tfrac{1}{2})\pi - \vartheta_1\right)\frac{x-a}{d_1-a} + \vartheta_1\right] + O(1), & x \in [a, d_1), \\ O\left(n^i\right), \quad x \in (d_i, d_{i+1}),\ (i = \overline{1, m}), \end{cases} \quad (84)$$

for $\ell = \overline{2, m}$,

$$\mathfrak{y}_1(x, \lambda_n^{(\ell)}) = \begin{cases} \frac{p_\ell\, n\pi}{d_\ell - d_{\ell-1}} \cos\left[\frac{p_\ell\, n\pi(x-a)}{p_1(d_\ell - d_{\ell-1})} + \vartheta_1\right] + O(1), \quad x \in [a, d_1), \\ -\frac{\delta_{2,1}}{\delta'_{2,1}}\left(\frac{p_\ell\, n\pi}{d_\ell - d_{\ell-1}}\right)^2 \cos\left[\frac{p_\ell\, n\pi(d_1-a)}{p_1(d_\ell - d_{\ell-1})} + \vartheta_1\right] \times \sin\left[\frac{p_\ell\, n\pi(x-d_1)}{p_2(d_\ell - d_{\ell-1})}\right] + O(n),\ x \in (d_1, d_2), \\ (-1)^i \prod_{j=1}^{i} \frac{\delta_{2,j}}{\delta'_{2,j}} \times \left(\frac{p_\ell\, n\pi}{d_\ell - d_{\ell-1}}\right)^{i+1} \cos\left[\frac{p_\ell\, n\pi(d_1-a)}{p_1(d_\ell - d_{\ell-1})} + \vartheta_1\right] \times \prod_{j=2}^{i} \sin\left[\frac{p_\ell\, n\pi(d_j - d_{j-1})}{p_j(d_\ell - d_{\ell-1})}\right] \\ \times \sin\left[\frac{p_\ell\, n\pi(x-d_i)}{p_{i+1}(d_\ell - d_{\ell-1})}\right] + O\left(n^i\right), \quad x \in (d_i, d_{i+1}),\ (i = \overline{2, m}), \end{cases} \quad (85)$$

$$\mathfrak{y}_2(x,\lambda_n^{(\ell)}) = \begin{cases} \dfrac{p_\ell n\pi}{d_\ell - d_{\ell-1}} \sin\left[\dfrac{p_\ell n\pi(x-a)}{p_1(d_\ell - d_{\ell-1})} + \vartheta_1\right] + O(1), \quad x \in [a,d_1), \\ \dfrac{\delta_{2,1}}{\delta'_{2,1}} \left(\dfrac{p_\ell n\pi}{d_\ell - d_{\ell-1}}\right)^2 \cos\left[\dfrac{p_\ell n\pi(d_1-a)}{p_1(d_\ell - d_{\ell-1})} + \vartheta_1\right] \times \cos\left[\dfrac{p_\ell n\pi(x-d_1)}{p_2(d_\ell - d_{\ell-1})}\right] + O(n),\ x \in (d_1,d_2), \\ (-1)^{i-1} \prod_{j=1}^{i} \dfrac{\delta_{2,j}}{\delta'_{2,j}} \times \left(\dfrac{p_\ell n\pi}{d_\ell - d_{\ell-1}}\right)^{i+1} \cos\left[\dfrac{p_\ell n\pi(d_1-a)}{p_1(d_\ell - d_{\ell-1})} + \vartheta_1\right] \times \prod_{j=2}^{i} \sin\left[\dfrac{p_\ell n\pi(d_j - d_{j-1})}{p_j(d_\ell - d_{\ell-1})}\right] \\ \times \cos\left[\dfrac{p_\ell n\pi(x-d_i)}{p_{i+1}(d_\ell - d_{\ell-1})}\right] + O\left(n^i\right), \qquad x \in (d_i,d_{i+1}),\ (i=\overline{2,m}), \end{cases} \tag{86}$$

$$\mathfrak{y}_1(x,\lambda_n^{(m+1)}) = \begin{cases} \dfrac{p_{m+1}}{b-d_m}\left((n+\tfrac{1}{2})\pi + \vartheta_2\right) \cos\left[\left((n+\tfrac{1}{2})\pi + \vartheta_2\right) \dfrac{p_{m+1}(x-a)}{p_1(b-d_m)} + \vartheta_1\right] + O(1), \qquad x \in [a,d_1), \\ -\dfrac{\delta_{2,1}}{\delta'_{2,1}} \left(\dfrac{p_{m+1}}{b-d_m}\left((n+\tfrac{1}{2})\pi + \vartheta_2\right)\right)^2 \cos\left[\left((n+\tfrac{1}{2})\pi + \vartheta_2\right) \dfrac{p_{m+1}(d_1-a)}{p_1(b-d_m)} + \vartheta_1\right] \times \\ \times \sin\left[\left((n+\tfrac{1}{2})\pi + \vartheta_2\right) \dfrac{p_{m+1}(x-d_1)}{p_2(b-d_m)}\right] + O(n), \qquad x \in (d_1,d_2), \\ (-1)^i \prod_{j=1}^{i} \dfrac{\delta_{2,j}}{\delta'_{2,j}} \times \left(\dfrac{p_{m+1}}{b-d_m}\left((n+\dfrac{1}{2})\pi + \vartheta_2\right)\right)^{i+1} \cos\left[\left((n+\dfrac{1}{2})\pi + \vartheta_2\right) \dfrac{p_{m+1}(d_1-a)}{p_1(b-d_m)} + \vartheta_1\right] \times \\ \times \prod_{j=2}^{i} \sin\left[\left((n+\dfrac{1}{2})\pi + \vartheta_2\right) \dfrac{p_{m+1}(d_j-d_{j-1})}{p_j(b-d_m)}\right] \times \sin\left[\left((n+\dfrac{1}{2})\pi + \vartheta_2\right) \dfrac{p_{m+1}(x-d_i)}{p_{i+1}(b-d_m)}\right] \\ +O\left(n^i\right), \qquad x \in (d_i,d_{i+1}),\ (i=\overline{2,m}), \end{cases} \tag{87}$$

$$\mathfrak{y}_2(x,\lambda_n^{(m+1)}) = \begin{cases} \frac{p_{m+1}}{b-d_m}\left((n+\frac{1}{2})\pi+\vartheta_2\right)\sin\left[\left((n+\frac{1}{2})\pi+\vartheta_2\right)\frac{p_{m+1}(x-a)}{p_1(b-d_m)}+\vartheta_1\right]+O(1), & x\in[a,d_1), \\ \frac{\delta_{2,1}}{\delta'_{2,1}}\left(\frac{p_{m+1}}{b-d_m}\left((n+\frac{1}{2})\pi+\vartheta_2\right)\right)^2\cos\left[\left((n+\frac{1}{2})\pi+\vartheta_2\right)\frac{p_{m+1}(d_1-a)}{p_1(b-d_m)}+\vartheta_1\right]\times & \\ \times\cos\left[\left((n+\frac{1}{2})\pi+\vartheta_2\right)\frac{p_{m+1}(x-d_1)}{p_2(b-d_m)}\right]+O(n), \quad x\in(d_1,d_2), & \\ (-1)^{i-1}\prod_{j=1}^{i}\frac{\delta_{2,j}}{\delta'_{2,j}}\times\left(\frac{p_{m+1}}{b-d_m}\left((n+\frac{1}{2})\pi+\vartheta_2\right)\right)^{i+1}\cos\left[\left((n+\frac{1}{2})\pi+\vartheta_2\right)\frac{p_{m+1}(d_1-a)}{p_1(b-d_m)}+\vartheta_1\right]\times & \\ \times\prod_{j=2}^{i}\sin\left[\left((n+\frac{1}{2})\pi+\vartheta_2\right)\frac{p_{m+1}(d_j-d_{j-1})}{p_j(b-d_m)}\right]\times\cos\left[\left((n+\frac{1}{2})\pi+\vartheta_2\right)\frac{p_{m+1}(x-d_i)}{p_{i+1}(b-d_m)}\right] & \\ +O\left(n^i\right), \quad x\in(d_i,d_{i+1}),\ (i=\overline{2,m}), & \end{cases} \tag{88}$$

where the above asymptotic approximations of eigen-vector-functions hold uniformly for $x\in[d_0,d_1)\bigcup_{i=2}^{m}(d_{i-1},d_i)\bigcup(d_m,d_{m+1}]$.

**Remark 4.7.** *If $\lambda^*$ is an eigenvalue, $\mathfrak{y}(x,\lambda^*)$ and $\mathfrak{z}(x,\lambda^*)$, see Section 3, are linearly dependent. Thus, the asymptotic approximations of $\mathfrak{z}(x,\lambda_n^{\ell})$, $\ell=\overline{1,m+1}$, do not require individual consideration.*

## 5. The Asymptotic Formulae for Eigenvectors Norms

Let $\{\lambda_n\}_{n\in\mathbb{Z}}$ be the sequence of zeros of $\Lambda(\lambda)$ corresponding the eigenvectors $\Phi(\cdot,\lambda_n)$ of the operator $\mathcal{T}$, where

$$\Phi(x,\lambda_n) := \begin{pmatrix} \mathfrak{y}(x,\lambda_n) \\ \mathcal{L}_{\vartheta_1}(\mathfrak{y}(a,\lambda_n)) \\ \mathcal{L}_{\vartheta_2}(\mathfrak{y}(b,\lambda_n)) \\ \mathcal{L}_{2,1}(\mathfrak{y}(d_1,\lambda_n)) \\ \vdots \\ \mathcal{L}_{2,m}(\mathfrak{y}(d_m,\lambda_n)) \end{pmatrix}, \quad n\in\mathbb{Z}. \tag{89}$$

In this section, we will examine the asymptotic behaviour for $\|\Phi(\cdot,\lambda_n)\|_{\mathfrak{H}}$, where $\|\cdot\|_{\mathfrak{H}}$ is the norm corresponding to the space $\mathfrak{H}$.

For $n\neq k$, since $\mathcal{T}$ is symmetric, the following orthogonality relation

$$\langle\Phi(\cdot,\lambda_n),\Phi(\cdot,\lambda_k)\rangle_{\mathfrak{H}} = 0 \tag{90}$$

holds. Let

$$\psi(x,\lambda_n) := \frac{\mathfrak{y}(x,\lambda_n)}{\|\Phi(\cdot,\lambda_n)\|_{\mathfrak{H}}}. \tag{91}$$

With short calculations, it is easy to prove that the eigenvectors

$$\Psi(x,\lambda_n) := \begin{pmatrix} \psi(x,\lambda_n) \\ \mathcal{L}_{\vartheta_1}(\psi(a,\lambda_n)) \\ \mathcal{L}_{\vartheta_2}(\psi(b,\lambda_n)) \\ \mathcal{L}_{2,1}(\psi(d_1,\lambda_n)) \\ \vdots \\ \mathcal{L}_{2,m}(\psi(d_m,\lambda_n)) \end{pmatrix}, \quad n \in \mathbb{Z}. \tag{92}$$

are orthonormal, that is

$$\mathcal{T}\Psi(x,\lambda_n) = \lambda_n \Psi(x,\lambda_n), \quad \langle \Psi(\cdot,\lambda_n), \Psi(\cdot,\lambda_k)\rangle_{\mathfrak{H}} = \delta_n^k, \tag{93}$$

where $\delta_n^k$ is the Kronecker delta.

In what follows we will obtain the asymptotic formulae of $\|\Phi(\cdot,\lambda_n)\|_{\mathfrak{H}}$ in case $\lambda_n = \lambda_n^{(m+1)}$. The other cases may be considered analogically. To obtain the asymptotic formulae of $\|\Phi(\cdot,\lambda_n)\|_{\mathfrak{H}}$, we need the next lemma.

**Lemma 5.1.** *Let $\lambda_n = \lambda_n^{(m+1)}$. The following asymptotic formulaes hold:*

$$\mathcal{L}_{\vartheta_1}(\mathfrak{y}(a,\lambda_n^{(m+1)})) = O(1), \tag{94}$$

$$\mathcal{L}_{2,1}(\mathfrak{y}(d_1,\lambda_n^{(m+1)})) = \frac{\delta_{2,1}p_{m+1}}{b-d_m}\left((n+\frac{1}{2})\pi+\vartheta_2\right)\cos\left[\left((n+\frac{1}{2})\pi+\vartheta_2\right)\frac{p_{m+1}(d_1-a)}{p_1(b-d_m)}+\vartheta_1\right]+O(1), \tag{95}$$

$$\begin{aligned} \mathcal{L}_{2,i}(\mathfrak{y}(d_i,\lambda_n^{(m+1)})) =& (-1)^{i-1}\delta_{2,i}\prod_{j=1}^{i-1}\frac{\delta_{2,j}}{\delta'_{2,j}}\times\left(\frac{p_{m+1}}{b-d_m}\left((n+\frac{1}{2})\pi+\vartheta_2\right)\right)^i\times \\ &\times\cos\left[\left((n+\frac{1}{2})\pi+\vartheta_2\right)\frac{p_{m+1}(d_1-a)}{p_1(b-d_m)}+\vartheta_1\right]\times \\ &\times\prod_{j=1}^{i-1}\sin\left[\left((n+\frac{1}{2})\pi+\vartheta_2\right)\frac{p_{m+1}(d_{j+1}-d_j)}{p_{j+1}(b-d_m)}\right]+O\left(n^{i-1}\right), \quad i=\overline{2,m}, \end{aligned} \tag{96}$$

$$\mathcal{L}_{\vartheta_2}(\mathfrak{y}(b,\lambda_n^{(m+1)})) = O(n^m). \tag{97}$$

*Proof.* From (8) and (29), we can get

$$\mathcal{L}_{\vartheta_1}(\mathfrak{y}(a,\lambda_n^{(m+1)})) = \sin\vartheta_1\mathfrak{y}_{1,1}(a,\lambda_n^{(m+1)}) - \cos\vartheta_1\mathfrak{y}_{2,1}(a,\lambda_n^{(m+1)})$$

$$\mathcal{L}_{2,i}(\mathfrak{y}(d_i,\lambda_n^{(m+1)})) = \delta_{2,i}\mathfrak{y}_{1,i}(d_i,\lambda_n^{(m+1)}), \quad i=\overline{1,m},$$

$$\mathcal{L}_{\vartheta_2}(\mathfrak{y}(b,\lambda_n^{(m+1)})) = \sin\vartheta_2\mathfrak{y}_{1,m+1}(b,\lambda_n^{(m+1)}) - \cos\vartheta_2\mathfrak{y}_{2,m+1}(b,\lambda_n^{(m+1)}).$$

By a short calculation, the above three equalities and the formulaes in (87)–(88) lead to the asymptotic equalities in (94)–(97). □

**Lemma 5.2.** *Let $\Phi(\cdot,\lambda_n^{(m+1)})$ define by (89). The following asymptotic formulae of the norms $\|\Phi(\cdot,\lambda_n^{(m+1)})\|_{\mathfrak{H}}$ hold:*

$$\begin{aligned} \|\Phi(\cdot,\lambda_n^{(m+1)})\|_{\mathfrak{H}} = \sqrt{\frac{1}{p_{m+1}}\prod_{i=1}^{m}\frac{\delta'_{1,i}\delta_{2,i}}{\delta_{1,i}\delta'_{2,i}}}&\times\left(\frac{p_{m+1}}{b-d_m}\left((n+\frac{1}{2})\pi+\vartheta_2\right)\right)^{m+1} \\ &\times\cos\left[\left((n+\frac{1}{2})\pi+\vartheta_2\right)\frac{p_{m+1}(d_1-a)}{p_1(b-d_m)}+\vartheta_1\right]\times \\ &\times\prod_{i=2}^{m}\sin\left[\left((n+\frac{1}{2})\pi+\vartheta_2\right)\frac{p_{m+1}(d_i-d_{i-1})}{p_i(b-d_m)}\right]\sqrt{d_{m+1}-d_m}+O(n^m). \end{aligned} \tag{98}$$

*Proof.* From the definition of the inner product space defined in (11) and the definition of the eigenvectors $\Phi(\cdot,\lambda_n^{(m+1)})$, we have

$$\begin{aligned}\left\langle \Phi(\cdot,\lambda_n^{(m+1)}), \Phi(\cdot,\lambda_n^{(m+1)})\right\rangle_{\mathfrak{H}} =& \sum_{j=0}^{m}\left(\frac{\prod_{i=0}^{j}\Delta_i}{p_{j+1}}\right)\int_{d_j}^{d_{j+1}}\left[\left(\mathfrak{y}_1(x,\lambda_n^{(m+1)})\right)^2+\left(\mathfrak{y}_2(x,\lambda_n^{(m+1)})\right)^2\right]dx\\ &+\frac{1}{\omega}\mathcal{L}_{\vartheta_1}(\mathfrak{y}(a,\lambda_n^{(m+1)}))\mathcal{L}_{\vartheta_1}(\mathfrak{y}(a,\lambda_n^{(m+1)}))\\ &+\frac{\prod_{i=0}^{m}\Delta_i}{\nu}\mathcal{L}_{\vartheta_2}(\mathfrak{y}(b,\lambda_n^{(m+1)}))\mathcal{L}_{\vartheta_2}(\mathfrak{y}(b,\lambda_n^{(m+1)}))\\ &+\sum_{j=1}^{m}\frac{\prod_{i=0}^{j-1}\Delta_i\mathcal{L}_{2,j}(\mathfrak{y}(d_j,\lambda_n^{(m+1)}))\mathcal{L}_{2,j}(\mathfrak{y}(d_j,\lambda_n^{(m+1)}))}{\delta_{2,j}^2}.\end{aligned}\tag{99}$$

By a short calculation, the formulaes in (87)–(88) lead to the following equalities:

$$\frac{1}{p_1}\int_{d_0}^{d_1}\left[\left(\mathfrak{y}_1(x,\lambda_n^{(m+1)})\right)^2+\left(\mathfrak{y}_2(x,\lambda_n^{(m+1)})\right)^2\right]dx=\frac{1}{p_1}\left(\frac{p_{m+1}}{b-d_m}\left((n+\frac{1}{2})\pi+\vartheta_2\right)\right)^2(d_1-d_0)+O(1),\tag{100}$$

$$\begin{aligned}\frac{\Delta_1}{p_2}\int_{d_1}^{d_2}\left[\left(\mathfrak{y}_1(x,\lambda_n^{(m+1)})\right)^2+\left(\mathfrak{y}_2(x,\lambda_n^{(m+1)})\right)^2\right]dx =& \frac{1}{p_2}\frac{\delta'_{1,1}}{\delta_{1,1}}\frac{\delta_{2,1}}{\delta'_{2,1}}\left(\frac{p_{m+1}}{b-d_m}\left((n+\frac{1}{2})\pi+\vartheta_2\right)\right)^4\\ \times\cos^2\left[\left((n+\frac{1}{2})\pi+\vartheta_2\right)\frac{p_{m+1}(d_1-a)}{p_1(b-d_m)}+\vartheta_1\right]&(d_2-d_1)+O\left(n^2\right),\end{aligned}\tag{101}$$

and, for $j=\overline{2,m}$,

$$\begin{aligned}&\frac{\prod_{i=0}^{j}\Delta_i}{p_{j+1}}\int_{d_j}^{d_{j+1}}\left[\left(\mathfrak{y}_1(x,\lambda_n^{(m+1)})\right)^2+\left(\mathfrak{y}_2(x,\lambda_n^{(m+1)})\right)^2\right]dx\\ &=\frac{1}{p_{j+1}}\prod_{i=1}^{j}\frac{\delta'_{1,i}\delta_{2,i}}{\delta_{1,i}\delta'_{2,i}}\times\left(\frac{p_{m+1}}{b-d_m}\left((n+\frac{1}{2})\pi+\vartheta_2\right)\right)^{2j+2}\times\cos^2\left[\left((n+\frac{1}{2})\pi+\vartheta_2\right)\frac{p_{m+1}(d_1-a)}{p_1(b-d_m)}+\vartheta_1\right]\\ &\times\left(\prod_{i=2}^{j}\sin\left[\left((n+\frac{1}{2})\pi+\vartheta_2\right)\frac{p_{m+1}(d_i-d_{i-1})}{p_i(b-d_m)}\right]\right)^2(d_{j+1}-d_j)+O\left(n^{2j}\right).\end{aligned}\tag{102}$$

Using (100), (101), (102) and Lemma 5.1 in (99) we obtain

$$\begin{aligned}\|\Phi(\cdot,\lambda_n^{(m+1)})\|_{\mathfrak{H}}^2=&\frac{1}{p_{m+1}}\prod_{i=1}^{m}\frac{\delta'_{1,i}\delta_{2,i}}{\delta_{1,i}\delta'_{2,i}}\times\left(\frac{p_{m+1}}{b-d_m}\left((n+\frac{1}{2})\pi+\vartheta_2\right)\right)^{2m+2}\\ &\times\cos^2\left[\left((n+\frac{1}{2})\pi+\vartheta_2\right)\frac{p_{m+1}(d_1-a)}{p_1(b-d_m)}+\vartheta_1\right]\\ &\times\left(\prod_{i=2}^{m}\sin\left[\left((n+\frac{1}{2})\pi+\vartheta_2\right)\frac{p_{m+1}(d_i-d_{i-1})}{p_i(b-d_m)}\right]\right)^2(d_{m+1}-d_m)+O\left(n^{2m}\right),\end{aligned}\tag{103}$$

and the proof of the equality (98) is complete. □

## 6. Green's Matrix, Resolvent Operator and Self-Adjointness of the Problem

Let $\mathcal{T}$ be the operator defined in Section 2 and let $\lambda$ be not an eigenvalue of the operator $\mathcal{T}$. To conclude the the resolvent operator $\mathcal{R}(\mathcal{T},\lambda) = (\mathcal{T} - \lambda I)^{-1}$, we consider the following operator equation

$$(\mathcal{T} - \lambda I)U(x) = F(x), \quad U(x) = \begin{pmatrix} u(x) \\ \mathcal{L}_{\vartheta_1}(u(a)) \\ \mathcal{L}_{\vartheta_2}(u(b)) \\ \mathcal{L}_{2,1}(u(d_1)) \\ \vdots \\ \mathcal{L}_{2,m}(u(d_m)) \end{pmatrix}, \ u(x) = \begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix}, \ F(x) = \begin{pmatrix} f(x) \\ \tilde{z}_1 \\ \tilde{z}_2 \\ z_1 \\ \vdots \\ z_m \end{pmatrix}, \ f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}, \tag{104}$$

Consequently, from the definition of the operator $\mathcal{T}$ ((12) and (13)), the operator equation (104) is equivalent to the nonhomogeneous differential equation

$$\left.\begin{aligned} p(x)u_2'(x) - (\lambda + q_1(x))u_1(x) &= f_1(x), \\ p(x)u_1'(x) + (\lambda + q_2(x))u_2(x) &= -f_2(x), \end{aligned}\right\} x \in [a, d_1) \bigcup_{i=2}^{m} (d_{i-1}, d_i) \bigcup (d_m, b], \tag{105}$$

nonhomogeneous boundary conditions

$$\tilde{z}_1 = -\mathcal{L}_\alpha(u(a)) - \lambda \mathcal{L}_{\vartheta_1}(u(a)), \tag{106}$$

$$\tilde{z}_2 = -\mathcal{L}_\beta(u(b)) - \lambda \mathcal{L}_{\vartheta_2}(u(b)), \tag{107}$$

nonhomogeneous transmission conditions

$$z_i = -\mathcal{L}_{1,i}(u(d_i)) - \lambda \mathcal{L}_{2,i}(u(d_i)), \quad i = \overline{1,m}, \tag{108}$$

and homogeneous transmission conditions (4). The general solution of the following homogenous differential equation

$$\left.\begin{aligned} p(x)u_2'(x,\lambda) - q_1(x)u_1(x,\lambda) &= \lambda u_1(x,\lambda), \\ p(x)u_1'(x,\lambda) + q_2(x)u_2(x,\lambda) &= -\lambda u_2(x,\lambda), \end{aligned}\right\} x \in [a, d_1) \bigcup_{i=2}^{m} (d_{i-1}, d_i) \bigcup (d_m, b],$$

has the next form

$$u(x,\lambda) = \begin{cases} C_1 \begin{pmatrix} \mathfrak{y}_{1,1}(x,\lambda) \\ \mathfrak{y}_{2,1}(x,\lambda) \end{pmatrix} + \widetilde{C}_1 \begin{pmatrix} \mathfrak{z}_{1,1}(x,\lambda) \\ \mathfrak{z}_{2,1}(x,\lambda) \end{pmatrix}, & x \in [a, d_1), \\ C_i \begin{pmatrix} \mathfrak{y}_{1,i}(x,\lambda) \\ \mathfrak{y}_{2,i}(x,\lambda) \end{pmatrix} + \widetilde{C}_i \begin{pmatrix} \mathfrak{z}_{1,i}(x,\lambda) \\ \mathfrak{z}_{2,i}(x,\lambda) \end{pmatrix}, & x \in (d_{i-1}, d_i)\ (i = \overline{2,m}), \\ C_{m+1} \begin{pmatrix} \mathfrak{y}_{1,m+1}(x,\lambda) \\ \mathfrak{y}_{2,m+1}(x,\lambda) \end{pmatrix} + \widetilde{C}_{m+1} \begin{pmatrix} \mathfrak{z}_{1,m+1}(x,\lambda) \\ \mathfrak{z}_{2,m+1}(x,\lambda) \end{pmatrix}, & x \in (d_m, b], \end{cases} \tag{109}$$

where $C_i$ and $\widetilde{C}_i$, $i = \overline{1, m+1}$, are arbitrary constants. Using the method of variation of parameters to the nonhomogeneous linear differential equation (105), we can conclude the general solution of (105) in the form

$$u(x,\lambda)=\begin{cases} C_1(x,\lambda)\begin{pmatrix}\mathfrak{y}_{1,1}(x,\lambda)\\ \mathfrak{y}_{2,1}(x,\lambda)\end{pmatrix}+\widetilde{C}_1(x,\lambda)\begin{pmatrix}\mathfrak{z}_{1,1}(x,\lambda)\\ \mathfrak{z}_{2,1}(x,\lambda)\end{pmatrix}, & x\in[a,d_1),\\ C_i(x,\lambda)\begin{pmatrix}\mathfrak{y}_{1,i}(x,\lambda)\\ \mathfrak{y}_{2,i}(x,\lambda)\end{pmatrix}+\widetilde{C}_i(x,\lambda)\begin{pmatrix}\mathfrak{z}_{1,i}(x,\lambda)\\ \mathfrak{z}_{2,i}(x,\lambda)\end{pmatrix}, & x\in(d_{i-1},\,d_i)\,(i=\overline{2,m}),\\ C_{m+1}(x,\lambda)\begin{pmatrix}\mathfrak{y}_{1,m+1}(x,\lambda)\\ \mathfrak{y}_{2,m+1}(x,\lambda)\end{pmatrix}+\widetilde{C}_{m+1}(x,\lambda)\begin{pmatrix}\mathfrak{z}_{1,m+1}(x,\lambda)\\ \mathfrak{z}_{2,m+1}(x,\lambda)\end{pmatrix}, & x\in(d_m,b],\end{cases} \tag{110}$$

where the functions $C_i(x,\lambda)$ and $\widetilde{C}_i(x,\lambda)$ $(i=\overline{1,m+1})$ satisfy the next system of equations

$$\left.\begin{aligned} C_1'(x,\lambda)\mathfrak{y}_{2,1}(x,\lambda)+\widetilde{C}_1'(x,\lambda)\mathfrak{z}_{2,1}(x,\lambda)&=\frac{f_1(x)}{p_1},\\ C_1'(x,\lambda)\mathfrak{y}_{1,1}(x,\lambda)+\widetilde{C}_1'(x,\lambda)\mathfrak{z}_{1,1}(x,\lambda)&=-\frac{f_2(x)}{p_1},\end{aligned}\right\} x\in[a,d_1), \tag{111}$$

$$\left.\begin{aligned} C_i'(x,\lambda)\mathfrak{y}_{2,k}(x,\lambda)+\widetilde{C}_i'(x,\lambda)\mathfrak{z}_{2,k}(x,\lambda)&=\frac{f_1(x)}{p_i},\\ C_i'(x,\lambda)\mathfrak{y}_{1,k}(x,\lambda)+\widetilde{C}_i'(x,\lambda)\mathfrak{z}_{1,k}(x,\lambda)&=-\frac{f_2(x)}{p_i},\end{aligned}\right\} x\in(d_{i-1},d_i),\ (i=\overline{2,m}), \tag{112}$$

and

$$\left.\begin{aligned} C_{m+1}'(x,\lambda)\mathfrak{y}_{2,m+1}(x,\lambda)+\widetilde{C}_{m+1}'(x,\lambda)\mathfrak{z}_{2,m+1}(x,\lambda)&=\frac{f_1(x)}{p_{m+1}},\\ C_{m+1}'(x,\lambda)\mathfrak{y}_{1,m+1}(x,\lambda)+\widetilde{C}_{m+1}'(x,\lambda)\mathfrak{z}_{1,m+1}(x,\lambda)&=-\frac{f_2(x)}{p_{m+1}},\end{aligned}\right\} x\in(d_m,b]. \tag{113}$$

Thus the equations (111), (112) and (113) have the order respectively unique solution, with taking into account $\lambda$ is not an eigenvalue and $\Lambda_i(\lambda)\neq 0$ $(i=\overline{1,m+1})$,

$$\left.\begin{aligned} C_1(x,\lambda)&=\frac{1}{p_1\Lambda_1(\lambda)}\int_x^{d_1}\mathfrak{z}^\top(t,\lambda)f(t)\,dt+C_1,\\ \widetilde{C}_1(x,\lambda)&=\frac{1}{p_1\Lambda_1(\lambda)}\int_a^{x}\mathfrak{y}^\top(t,\lambda)f(t)\,dt+\widetilde{C}_1,\end{aligned}\right\} x\in[a,\,d_1), \tag{114}$$

$$\left.\begin{aligned} C_i(x,\lambda)&=\frac{1}{p_i\Lambda_i(\lambda)}\int_x^{d_i}\mathfrak{z}^\top(t,\lambda)f(t)\,dt+C_i,\\ \widetilde{C}_i(x,\lambda)&=\frac{1}{p_i\Lambda_i(\lambda)}\int_{d_{i-1}}^{x}\mathfrak{y}^\top(t,\lambda)f(t)\,dt+\widetilde{C}_i,\end{aligned}\right\} x\in(d_{i-1},\,d_i),\ (i=\overline{2,m}), \tag{115}$$

$$\left.\begin{aligned} C_{m+1}(x,\lambda)&=\frac{1}{p_{m+1}\Lambda_{m+1}(\lambda)}\int_x^{b}\mathfrak{z}^\top(t,\lambda)f(t)\,dt+C_{m+1},\\ \widetilde{C}_{m+1}(x,\lambda)&=\frac{1}{p_{m+1}\Lambda_{m+1}(\lambda)}\int_{d_m}^{x}\mathfrak{y}^\top(t,\lambda)f(t)\,dt+\widetilde{C}_{m+1},\end{aligned}\right\} x\in(c_m,\,b], \tag{116}$$

where

$$\mathfrak{y}(t,\lambda)=\begin{cases}\begin{pmatrix}\mathfrak{y}_{1,1}(t,\lambda)\\ \mathfrak{y}_{2,1}(t,\lambda)\end{pmatrix}, & t\in[a,d_1),\\ \begin{pmatrix}\mathfrak{y}_{1,i}(t,\lambda)\\ \mathfrak{y}_{2,i}(t,\lambda)\end{pmatrix}, & t\in(d_{i-1},d_i),\ (i=\overline{2,m}),\\ \begin{pmatrix}\mathfrak{y}_{1,m+1}(t,\lambda)\\ \mathfrak{y}_{2,m+1}(t,\lambda)\end{pmatrix}, & t\in(d_m,b],\end{cases}$$

$$\mathfrak{z}(t,\lambda)=\begin{cases}\begin{pmatrix}\mathfrak{z}_{1,1}(t,\lambda)\\ \mathfrak{z}_{2,1}(t,\lambda)\end{pmatrix}, & t\in[a,d_1),\\ \begin{pmatrix}\mathfrak{z}_{1,i}(t,\lambda)\\ \mathfrak{z}_{2,i}(t,\lambda)\end{pmatrix}, & t\in(d_{i-1},d_i),\ (i=\overline{2,m}),\\ \begin{pmatrix}\mathfrak{z}_{1,m+1}(t,\lambda)\\ \mathfrak{z}_{2,m+1}(t,\lambda)\end{pmatrix}, & t\in(d_m,b].\end{cases}$$

Substituting equations (114), (115) and (116) into (110), the general solution of (105) has the following form

$$u(x,\lambda)=\begin{cases}\dfrac{\mathfrak{y}(x,\lambda)}{p_1\Lambda_1(\lambda)}\displaystyle\int_x^{d_1}\mathfrak{z}^\top(t,\lambda)f(t)\,dt+\dfrac{\mathfrak{z}(x,\lambda)}{p_1\Lambda_1(\lambda)}\displaystyle\int_a^x\mathfrak{y}^\top(t,\lambda)f(t)\,dt+C_1\mathfrak{y}(x,\lambda)+\widetilde{C}_1\mathfrak{z}(x,\lambda),\\ x\in[a,d_1),\\ \dfrac{\mathfrak{y}(x,\lambda)}{p_i\Lambda_i(\lambda)}\displaystyle\int_x^{d_i}\mathfrak{z}^\top(t,\lambda)f(t)\,dt+\dfrac{\mathfrak{z}(x,\lambda)}{p_i\Lambda_i(\lambda)}\displaystyle\int_{d_{i-1}}^x\mathfrak{y}^\top(t,\lambda)f(t)\,dt+C_i\mathfrak{y}(x,\lambda)+\widetilde{C}_i\mathfrak{z}(x,\lambda),\\ x\in(d_{i-1},\,d_i),\ (i=\overline{2,m}),\\ \dfrac{\mathfrak{y}(x,\lambda)}{p_{m+1}\Lambda_{m+1}(\lambda)}\displaystyle\int_x^{b}\mathfrak{z}^\top(t,\lambda)f(t)\,dt+\dfrac{\mathfrak{z}(x,\lambda)}{p_{m+1}\Lambda_{m+1}(\lambda)}\displaystyle\int_{d_m}^x\mathfrak{y}^\top(t,\lambda)f(t)\,dt+C_{m+1}\mathfrak{y}(x,\lambda)\\ +\widetilde{C}_{m+1}\mathfrak{z}(x,\lambda),\quad x\in(d_m,\,b].\end{cases}\tag{117}$$

Substituting equation (117) into equations (106) and (107) as well as equations (108) and (4), we obtain the following equalities, by taking into account the initial conditions (31)–(34),

$$\begin{aligned}&\widetilde{C}_1=\frac{\tilde{z}_1}{\Lambda_1(\lambda)},\\ &\widetilde{C}_{i+1}=\widetilde{C}_i+\frac{1}{p_i\Lambda_i(\lambda)}\int_{d_{i-1}}^{d_i}\mathfrak{y}^\top(t,\lambda)f(t)\,dt+\frac{z_i}{\delta'_{2i}\Lambda_{i+1}}\mathfrak{y}_{1,i+1}(d_i^+,\lambda),\quad i=\overline{1,m},\\ &C_{m+1}=-\frac{\tilde{z}_2}{\Lambda_{m+1}(\lambda)},\\ &C_i=C_{i+1}+\frac{1}{p_{i+1}\Lambda_{i+1}(\lambda)}\int_{d_i}^{d_{i+1}}\mathfrak{z}^\top(t,\lambda)f(t)\,dt+\frac{z_i}{\delta'_{2i}\Lambda_{i+1}}\mathfrak{z}_{1,i+1}(d_i^+,\lambda),\quad i=\overline{1,m},\end{aligned}\tag{118}$$

which lead to

$$C_i := \begin{cases} \sum_{j=i}^{m}\left[\frac{1}{p_{j+1}\Lambda_{j+1}(\lambda)}\int_{d_j}^{d_{j+1}} \mathfrak{z}^{\top}(t,\lambda)f(t)\,dt + \frac{z_j}{\delta'_{2j}\Lambda_{j+1}}\mathfrak{z}_{1,j+1}(d_j^{+},\lambda)\right] - \frac{\tilde{z}_2}{\Lambda_{m+1}(\lambda)}, & i=\overline{1,m}, \\ -\frac{\tilde{z}_2}{\Lambda_{m+1}(\lambda)}, & i=m+1, \end{cases} \tag{119}$$

and

$$\widetilde{C}_i := \begin{cases} \frac{\tilde{z}_1}{\Lambda_1(\lambda)}, & i=1, \\ \sum_{j=1}^{i-1}\left[\frac{1}{p_j\Lambda_j(\lambda)}\int_{d_{j-1}}^{d_j} \mathfrak{y}^{\top}(t,\lambda)f(t)\,dt, + \frac{z_j}{\delta'_{2j}\Lambda_{j+1}}\mathfrak{y}_{1,j+1}(d_j^{+},\lambda)\right] + \frac{\tilde{z}_1}{\Lambda_1(\lambda)} & i=\overline{2,m+1}. \end{cases} \tag{120}$$

From (119) and (120), the general solution in (117) can be written as

$$u(x,\lambda) = \begin{cases} \frac{\mathfrak{y}(x,\lambda)}{p_1\Lambda_1(\lambda)}\int_x^{d_1}\mathfrak{z}^{\top}(t,\lambda)f(t)\,dt + \frac{\mathfrak{z}(x,\lambda)}{p_1\Lambda_1(\lambda)}\int_a^x \mathfrak{y}^{\top}(t,\lambda)f(t)\,dt \\ +\sum_{j=1}^{m}\left[\frac{\mathfrak{y}(x,\lambda)}{p_{j+1}\Lambda_{j+1}(\lambda)}\int_{d_j}^{d_{j+1}}\mathfrak{z}^{\top}(t,\lambda)f(t)\,dt + \frac{z_j\mathfrak{z}_{1,j+1}(d_j^{+},\lambda)}{\delta'_{2j}\Lambda_{j+1}}\mathfrak{y}(x,\lambda)\right] - \frac{\tilde{z}_2}{\Lambda_{m+1}(\lambda)}\mathfrak{y}(x,\lambda) \\ +\frac{\tilde{z}_1}{\Lambda_1(\lambda)}\mathfrak{z}(x,\lambda), \quad x\in[a,d_1), \\ \frac{\mathfrak{y}(x,\lambda)}{p_i\Lambda_i(\lambda)}\int_x^{d_i}\mathfrak{z}^{\top}(t,\lambda)f(t)\,dt + \frac{\mathfrak{z}(x,\lambda)}{p_i\Lambda_i(\lambda)}\int_{d_{i-1}}^x \mathfrak{y}^{\top}(t,\lambda)f(t)\,dt \\ +\sum_{j=i}^{m}\left[\frac{\mathfrak{y}(x,\lambda)}{p_{j+1}\Lambda_{j+1}(\lambda)}\int_{d_j}^{d_{j+1}}\mathfrak{z}^{\top}(t,\lambda)f(t)\,dt + \frac{z_j\mathfrak{z}_{1,j+1}(d_j^{+},\lambda)}{\delta'_{2j}\Lambda_{j+1}}\mathfrak{y}(x,\lambda)\right] - \frac{\tilde{z}_2}{\Lambda_{m+1}(\lambda)}\mathfrak{y}(x,\lambda) \\ +\sum_{j=1}^{i-1}\left[\frac{\mathfrak{z}(x,\lambda)}{p_j\Lambda_j(\lambda)}\int_{d_{j-1}}^{d_j}\mathfrak{y}^{\top}(t,\lambda)f(t)\,dt, + \frac{z_j\mathfrak{y}_{1,j+1}(d_j^{+},\lambda)}{\delta'_{2j}\Lambda_{j+1}}\mathfrak{z}(x,\lambda)\right] + \frac{\tilde{z}_1}{\Lambda_1(\lambda)}\mathfrak{z}(x,\lambda), \\ x\in(d_{i-1},\,d_i),\ (i=\overline{2,m}), \\ \frac{\mathfrak{y}(x,\lambda)}{p_{m+1}\Lambda_{m+1}(\lambda)}\int_x^{b}\mathfrak{z}^{\top}(t,\lambda)f(t)\,dt + \frac{\mathfrak{z}(x,\lambda)}{p_{m+1}\Lambda_{m+1}(\lambda)}\int_{d_m}^x \mathfrak{y}^{\top}(t,\lambda)f(t)\,dt - \frac{\tilde{z}_2}{\Lambda_{m+1}(\lambda)}\mathfrak{y}(x,\lambda) \\ +\sum_{j=1}^{m}\left[\frac{\mathfrak{z}(x,\lambda)}{p_j\Lambda_j(\lambda)}\int_{d_{j-1}}^{d_j}\mathfrak{y}^{\top}(t,\lambda)f(t)\,dt, + \frac{z_j\mathfrak{y}_{1,j+1}(d_j^{+},\lambda)}{\delta'_{2j}\Lambda_{j+1}}\mathfrak{z}(x,\lambda)\right] + \frac{\tilde{z}_1}{\Lambda_1(\lambda)}\mathfrak{z}(x,\lambda), \quad x\in(d_m,\,b], \end{cases} \tag{121}$$

which can be represented in the next form, by taking into account (37),

$$u(x,\lambda) = \sum_{i=0}^{m}\frac{\prod_{j=0}^{i}\Delta_j}{p_{i+1}}\int_{d_i}^{d_{i+1}}\mathcal{G}(x,t,\lambda)f(t)\,dt + v(x,\lambda), \tag{122}$$

where

$$v(x,\lambda)=\begin{cases}\sum_{j=1}^{m}\frac{z_j\mathfrak{z}_{1,j+1}(d_j^+,\lambda)}{\delta'_{2j}\Lambda_{j+1}}\mathfrak{y}(x,\lambda)-\frac{\tilde{z}_2}{\Lambda_{m+1}(\lambda)}\mathfrak{y}(x,\lambda)+\frac{\tilde{z}_1}{\Lambda_1(\lambda)}\mathfrak{z}(x,\lambda), \quad x\in[a,d_1),\\ \sum_{j=i}^{m}\frac{z_j\mathfrak{z}_{1,j+1}(d_j^+,\lambda)}{\delta'_{2j}\Lambda_{j+1}}\mathfrak{y}(x,\lambda)+\sum_{j=1}^{i-1}\frac{z_j\mathfrak{y}_{1,j+1}(d_j^+,\lambda)}{\delta'_{2j}\Lambda_{j+1}}\mathfrak{z}(x,\lambda)-\frac{\tilde{z}_2}{\Lambda_{m+1}(\lambda)}\mathfrak{y}(x,\lambda)+\frac{\tilde{z}_1}{\Lambda_1(\lambda)}\mathfrak{z}(x,\lambda)\\ x\in(d_{i-1},d_i),\ (i=\overline{2,m}),\\ \sum_{j=1}^{m}\frac{z_j\mathfrak{y}_{1,j+1}(d_j^+,\lambda)}{\delta'_{2j}\Lambda_{j+1}}\mathfrak{z}(x,\lambda)-\frac{\tilde{z}_2}{\Lambda_{m+1}(\lambda)}\mathfrak{y}(x,\lambda)+\frac{\tilde{z}_1}{\Lambda_1(\lambda)}\mathfrak{z}(x,\lambda), \quad x\in(d_m,b],\end{cases} \tag{123}$$

and $\mathcal{G}(x,t,\lambda)$ is called Green's matrix of problem (1)–(5) and is given by, see [1, 3, 19, 20],

$$\mathcal{G}(x,t,\lambda):=\frac{1}{\Lambda(\lambda)}\begin{cases}\mathfrak{z}(x,\lambda)\mathfrak{y}^{\top}(t,\lambda), & a\le t\le x\le b, \quad x,t\neq d_i,\ i=\overline{1,m},\\ \mathfrak{y}(x,\lambda)\mathfrak{z}^{\top}(t,\lambda), & a\le x\le t\le b, \quad x,t\neq d_i,\ i=\overline{1,m}.\end{cases} \tag{124}$$

With a short calculation, using the equation (8) and the first equality in the initial conditions (32) and (33), the solution in (122) can also be written as

$$u(x,\lambda)=\sum_{i=0}^{m}\frac{\prod_{j=0}^{i}\Delta_j}{p_{i+1}}\int_{d_i}^{d_{i+1}}\mathcal{G}(x,t,\lambda)f(t)\,dt+\frac{1}{\omega}\mathcal{L}_{\vartheta_1}(\mathcal{G}(x,a,\lambda))\tilde{z}_1+\frac{\prod_{i=0}^{m}\Delta_i}{\nu}\mathcal{L}_{\vartheta_2}(\mathcal{G}(x,b,\lambda))\tilde{z}_2 \\ +\sum_{j=1}^{m}\frac{\prod_{i=0}^{j-1}\Delta_i\mathcal{L}_{2j}(\mathcal{G}(x,d_j,\lambda))z_j}{\delta_{2,j}^2}. \tag{125}$$

Consequently, the resolvent of the operator $\mathcal{R}(\mathcal{T},\lambda)=(\mathcal{T}-\lambda I)^{-1}$ can be expressed in the next form, see [2],

$$\mathcal{R}(\mathcal{T},\lambda)F(x)=\begin{pmatrix}\left\langle\widetilde{\mathcal{G}}(\cdot,\lambda),\overline{F}(\cdot)\right\rangle_{\mathfrak{H}}\\ \mathcal{L}_{\vartheta_1}(\left\langle\widetilde{\mathcal{G}}(\cdot,\lambda),\overline{F}(\cdot)\right\rangle_{\mathfrak{H}})(a)\\ \mathcal{L}_{\vartheta_2}(\left\langle\widetilde{\mathcal{G}}(\cdot,\lambda),\overline{F}(\cdot)\right\rangle_{\mathfrak{H}})(b)\\ \mathcal{L}_{2,1}(\left\langle\widetilde{\mathcal{G}}(\cdot,\lambda),\overline{F}(\cdot)\right\rangle_{\mathfrak{H}})(d_1)\\ \vdots\\ \mathcal{L}_{2,m}(\left\langle\widetilde{\mathcal{G}}(\cdot,\lambda),\overline{F}(\cdot)\right\rangle_{\mathfrak{H}})(d_m)\end{pmatrix},\ u(x,\lambda)=\left\langle\widetilde{\mathcal{G}}(\cdot,\lambda),\overline{F}(\cdot)\right\rangle_{\mathfrak{H}},\ \widetilde{\mathcal{G}}(x,\lambda)=\begin{pmatrix}\mathcal{G}^{\top}(x,\cdot,\lambda)\\ \mathcal{L}_{\vartheta_1}(\mathcal{G}(x,a,\lambda))\\ \mathcal{L}_{\vartheta_2}(\mathcal{G}(x,b,\lambda))\\ \mathcal{L}_{2,1}(\mathcal{G}(x,d_1,\lambda))\\ \vdots\\ \mathcal{L}_{2,m}(\mathcal{G}(x,d_m,\lambda))\end{pmatrix}. \tag{126}$$

Since the spectrum of the operator $\mathcal{T}$ consist of eigenvalues and all eigenvalues of the operator $\mathcal{T}$ are real, then the next lemma shows us that each $\lambda\in\mathbb{C}$, $\Im\lambda\neq 0$, is regular point of $\mathcal{T}$.

**Lemma 6.1.** *Let $\lambda\in\mathbb{C}$, $\Im\lambda\neq 0$. The resolvent operator $\mathcal{R}(\mathcal{T},\lambda)$ satisfies the following inequality*

$$\|\mathcal{R}(\mathcal{T},\lambda)F(\cdot)\|_{\mathfrak{H}}\le|\Im\lambda|^{-1}\|F(\cdot)\|_{\mathfrak{H}}, \quad F(\cdot)\in\mathfrak{H}. \tag{127}$$

*Proof.* Let $U(x) = \mathcal{R}(\mathcal{T}, \lambda)F(x)$,

$$U(\cdot) \in D(\mathcal{T}), \quad F(x) = \begin{pmatrix} f(x) \\ \tilde{z}_1 \\ \tilde{z}_2 \\ z_1 \\ \vdots \\ z_m \end{pmatrix} \in \mathfrak{H}, \quad f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}.$$

Since $\mathcal{T}$ is a symmetric operator and $\mathcal{T}U = \lambda U + F$, then

$$\begin{aligned} 0 &= \langle \mathcal{T}U(\cdot), U(\cdot)\rangle_{\mathfrak{H}} - \langle U(\cdot), \mathcal{T}U(\cdot)\rangle_{\mathfrak{H}} \\ &= \lambda \langle U(\cdot), U(\cdot)\rangle_{\mathfrak{H}} + \langle F(\cdot), U(\cdot)\rangle_{\mathfrak{H}} - \overline{\lambda} \langle U(\cdot), U(\cdot)\rangle_{\mathfrak{H}} - \langle U(\cdot), F(\cdot)\rangle_{\mathfrak{H}}. \end{aligned}$$

Consequently, from the last equality we conclude

$$\Im\lambda \times \|U(\cdot)\|^2_{\mathfrak{H}} = \Im \langle U(\cdot), F(\cdot)\rangle_{\mathfrak{H}}. \tag{128}$$

By using Cauchy–Schwartz inequality we obtain

$$|\Im \langle U(\cdot), F(\cdot)\rangle_{\mathfrak{H}}| \le |\langle U(\cdot), F(\cdot)\rangle_{\mathfrak{H}}| \le \|U(\cdot)\|_{\mathfrak{H}}\|F(\cdot)\|_{\mathfrak{H}}. \tag{129}$$

Hence, from (128) and (129) we get (127). □

The following lemma shows us that the operator $\mathcal{T}$, corresponding the problem (1)–(5), is self-adjoint.

**Lemma 6.2.** *The operator $\mathcal{T}$ is self-adjoint on the space $\mathfrak{H}$.*

*Proof.* By the same technique as in the proof of Lemma 3.1 in [20, p. 13], we can prove the deficiency spaces of the operator $\mathcal{T}$ are the null spaces and so $\mathcal{T} = \mathcal{T}^*$. □

## References


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