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Operator Matrices on the Bergman Space

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Abstract. In this article, we characterize the sufficient and necessary conditions for positiveness of operator matrices with Toeplitz and little Hankel operators on the Bergman space. Further, we explore some conditions for operator matrices to be normal and unitary.

1. Introduction

Let $\mathbb{D}$ be the open unit disk on the complex plane $\mathbb{C}$. Let $dA(z) = \frac{1}{\pi}dxdy$ be the normalized area measure. Let $L^2(\mathbb{D}, dA)$ be the space of complex valued, square integrable, measuring functions on $\mathbb{C}$ with respect to the area measure. Let $A^2(\mathbb{D})$ be the closed subspace of $L^2(\mathbb{D}, dA)$ consisting of those functions in $L^2(\mathbb{D}, dA)$ that are analytic. The space $A^2(\mathbb{D})$ is referred to as the Bergman space of the unit disk $\mathbb{D}$.

For $z \in \mathbb{D}$, K_z denote the reproducing kernel on $A^2(\mathbb{D})$. This function satisfies $f(z) = \langle f, K_z \rangle$ for all $f \in A^2(\mathbb{D})$. Let $k_z = \frac{K_z}{\|K_z\|_2}$ be the normalised reproducing kernel on $A^2(\mathbb{D})$. For any integer $n \geq 0$, let $e_n(z) = \sqrt{n+1}z^n$. Then, $\{e_n\}_{n=0}^{\infty}$ forms an orthonormal basis for $A^2(\mathbb{D})$. Let $L^\infty(\mathbb{D})$ be the Banach space consisting of essentially bounded Lebesgue measurable functions on $\mathbb{D}$ with $\|f\|_\infty = ess \ \ sup\{|f(z)| : z \in \mathbb{D}\}$. The Toeplitz operator T_ϕ is defined on $A^2(\mathbb{D})$ by

$$T_\phi h = P(\phi h), \ \ \phi \in L^\infty(\mathbb{D}).$$

Thus we have

$$(T_\phi h)(w) = \int_{\mathbb{D}} \frac{\phi(z)h(z)}{(1-\bar{z}w)^2} dA(z),$$

for $h \in A^2(\mathbb{D})$ and $w \in \mathbb{D}$.

Similarly one can define the little Hankel operator S_ϕ is the operator defined on $A^2(\mathbb{D})$ by $S_\phi f = PJ(\phi f)$ where $J : L^2(\mathbb{D}, dA) \to L^2(\mathbb{D}, dA)$, is defined as $Jf(z) = f(\bar{z})$ and P is the orthogonal projection from $L^2(\mathbb{D}, dA)$ onto $A^2(\mathbb{D})$.

For $a \in \mathbb{D}$, define $U_a f(w) = k_a(w) f(\phi_a(w))$, where f is the measurable function on $\mathbb{D}$. Let U_a be a bounded linear operator on $L^2(\mathbb{D}, dA)$ and also in $A^2(\mathbb{D})$ for all $a \in \mathbb{D}$ [2]. Further, $U_a^2 = I$, the identity operator,

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which is easily verified and $U_a^* = U_a, U_a(A^2(\mathbb{D})) \subset A^2(\mathbb{D})$ and $U_a((A^2(\mathbb{D}))^\perp) \subset (A^2(\mathbb{D}))^\perp$ for all $a \in \mathbb{D}$. Thus $U_aP = PU_a$ for all $a \in \mathbb{D}$. Similarly for $t \in \mathbb{T}$, where $\mathbb{T}$ is the unit circle, there is another unitary operator R_t on $A^2(\mathbb{D})$ defined as $R_tf(z) = f(tz)$ and $R_t^{-1} = R_t^* = R_{\bar{t}}$, f be any function on $\mathbb{D}$. To know details see [5]. We can define $E_{n,\phi} = \langle T_\phi \sqrt{n+1}z^n, \sqrt{n+1}z^n\rangle$. To know Bergman space in detail see [2].

Let H denote the separable infinite dimensional complex Hilbert space and the algebra of bounded linear operators on H is denoted by $\mathcal{L}(H)$. Let H_1, H_2, ... H_n be the complex Hilbert spaces. An operator $A \in \mathcal{L}(\bigoplus_{i=1}^n H_i)$ may be expressed as an $n \times n$ operator matrix is of the form $A = [A_{ij}]$, where A_{ij} is a bounded linear operator from H_j into H_i. If $(Ax, x) \geq 0$ for all $x \in \bigoplus_{i=1}^n H_i$, then A is called positive and denoted by $A \geq 0$. The Berezin transform of a bounded linear operator S on $A^2(\mathbb{D})$ denoted by $\widetilde{S}$ and is defined by

$$\widetilde{S}(w) = \langle Sk_w, k_w\rangle, \text{ for } w \in \mathbb{D}.$$

Let $\widetilde{\phi}(w) = \langle T_\phi k_w, k_w\rangle$ for $w \in \mathbb{D}$. That is, $\widetilde{\phi} = \widetilde{T_\phi}$. An operator A in $\mathcal{L}(H)$ has a polar decomposition $A = V|A|$, where V is the partial isometry (with $ker(V) = ker(A)$ and $ker(V^*) = ker(A^*)$) and $|A| = (A^*A)^{\frac{1}{2}}$. The Aluthge transformation of an operator was first introduced by Aluthge [3] defined as $\Delta(A) = |A|^{\frac{1}{2}}V|A|^{\frac{1}{2}}$. An operator A is said to be normal if $A^*A = AA^*$ and unitary if $A^*A = I = AA^*$. For operators A and B we can define $[A, B] = AB - BA$.

The organization of the paper is as follows: In section-2, we survey some well known lemmas and theorems relating to the positive operator matrices as well as positive operators. In section-3, we obtained some sufficient and necessary conditions for operator matrices to be positive and in the last section of the paper, we discussed some sufficient conditions for operator matrices to be normal as well as unitary.

2. Preliminaries

Let $\mathcal{M}_n$ be the matrix algebra of all $n \times n$ matrices with entries in the complex field $\mathbb{C}$. We can write $A \geq 0$ if A is positive, that is $\langle Ax, x\rangle \geq 0$ for all $x \in H$. To design our main results, we survey some well known lemmas and theorems relating to the positive operator matrices as well as positive operators which can be found in [1, 4, 7, 9, 11, 12].

Lemma 2.1. *[1], (Corollary I.3.3) Let $R \in \mathcal{M}_n$. Then R is positive iff the block matrix* $\begin{pmatrix} R & R \\ R & R \end{pmatrix}$ *is positive.*

Theorem 2.2. *[1], (Theorem IX.5.9) The block matrix* $\begin{pmatrix} P & X \\ X^* & Q \end{pmatrix}$ *is positive iff $X = P^{\frac{1}{2}}KQ^{\frac{1}{2}}$ for some contraction K and $P, Q \in \mathcal{M}_n$ are positive.*

Let us assume that $A = [A_{ij}]$ be a operator matrix.

Lemma 2.3. *[9] Let A, B are positive and $C = D^*$, $\exists$ a contraction S such that $C = A^{\frac{1}{2}}SB^{\frac{1}{2}}$ iff the operator matrix* $T = \begin{pmatrix} A & C \\ D & B \end{pmatrix} \geq 0$.

Another interesting result was given by Choi.

Lemma 2.4. *[4] For operators P, Q and $R \in \mathcal{L}(H)$ with R being positive and invertible. The block matrix* $\begin{pmatrix} R & Q \\ Q^* & P \end{pmatrix} \geq 0$ *if and only if $R \geq Q^*P^{-1}Q$.*

Let H_1 and H_2 are Hilbert C^* modules. Suppose $\mathcal{L}(H_1, H_2)$ is the set of all bounded linear operators $T : H_1 \to H_2$, which are adjointable. Fang derived one interesting result to show the positivity of an operator matrix on Hilbert $A-$module.

Proposition 2.5. *[7] Let H_1 and H_2 are Hilbert A−modules. Let $A \in \mathcal{L}(H_1)$, $C \in \mathcal{L}(H_2, H_1)$ and $B \in \mathcal{L}(H_2)$. Then $\begin{pmatrix} A & C \\ C^* & B \end{pmatrix} \geq 0$ iff $A \geq 0$, $B \geq 0$ and $|\Phi(\langle Cy, x\rangle)|^2 \leq \phi(\langle Ax, x\rangle)\phi(\langle By, y\rangle)$ for all $x \in \mathcal{L}(H_1)$, $y \in \mathcal{L}(H_2)$ and $\phi \in S(A)$, where $S(A)$ is the state space of A.*

Let A and B be two positive operators. Then, $A\sharp B$ is defined as

$$A\sharp B = max\left\{C \geq 0 | \begin{pmatrix} A & C \\ C & B \end{pmatrix} \geq 0\right\}.$$

If the linear map $\phi^n : M_n(A) \to M_n(B)$ defined by $\phi^n([a_{i,j}]) = [\phi(a_{i,j})]$, where A and B are in C^* algebra, then ϕ is called completely positive. In 2017, Najafi [11] discussed on the positivity of block operator matrices.

Theorem 2.6. *[11] Let $R \geq 0$, $S \geq 0$ and $T \geq 0$ such that $T \leq R\sharp S$. Then, $\exists$ a unique map $\phi : \mathcal{L}(H) \to \mathcal{L}(H)$ such that $\begin{pmatrix} \phi(R) & T \\ T & \phi(S) \end{pmatrix} \geq 0$ and ϕ is completely positive. Furthermore, ϕ is trace preserving if dimension of H is finite.*

If A and B are two bounded linear operators on the Hilbert sapce satisfying $AB \geq 0$, $A^2B \geq 0$ and $AB^2 \geq 0$ then we have the following results about the positivity of A and B.

Lemma 2.7. *[12] If $\overline{Ran(B)} = H$, then $A \geq 0$. Similarly, if $\overline{Ran(A^*)} = H$, then $B \geq 0$.*

Proposition 2.8. *[12] If the operator AB has its bounded inverse, then A , B are positive.*

Theorem 2.9. *[12] If A, B are semi-Fredholm operators and $ker(AB) = 0$, then A, B are positive.*

3. Positive operator matrices

In this section, we obtained the necessary and sufficient conditions for operator matrices to be positive. Here, we used $S_{\psi^+} = S^*_\psi$ and $\psi^+(z) = \overline{\psi(\overline{z})}$, where S_ψ is a littile Hankel operator and for a Toeplitz operator T_ϕ, $T^*_\phi = T_{\overline{\phi}}$.

Theorem 3.1. *Let $\phi, \xi \in L^\infty(\mathbb{D})$ with $T_\phi \geq S^*_\xi S_\xi$. Assume that $p = \inf_{z\in\mathbb{D}} |\widetilde{\phi(z)}| > 0$ and $\exists$ a sequence $\eta = \{\xi_n\}_{n=0}^\infty \subset \mathbb{D}$ such that*

$$\lambda^\eta_\phi \;=\; (\Sigma_{n=0}^\infty(1 - 2Re(\widetilde{\phi(\overline{\xi_n})}E_{n,\phi})) + |\widetilde{\phi(\xi_n)}|^2)^{\frac{1}{2}} < \infty.$$

If $p > \lambda^n_\phi$, then

$$\begin{pmatrix} T_\phi - S_\xi S^*_\xi & (T_\phi - S_\xi S^*_\xi)S_\xi \\ S^*_\xi(T_\phi - S_\xi S^*_\xi) & T_\phi - S^*_\xi S_\xi + S^*_\xi(T_\phi - S_\xi S^*_\xi)S_\xi \end{pmatrix} \geq 0.$$

Proof. By [8], the Toeplitz operator T_ϕ is invertible. Let $W = S_\xi T_\phi^{-\frac{1}{2}} = T_\phi^{-\frac{1}{2}} S_\xi$. Then,

$$\begin{aligned} T_\phi \geq S^*_\xi S_\xi &\Rightarrow I \geq T_\phi^{-\frac{1}{2}} S^*_\xi S_\xi T_\phi^{-\frac{1}{2}} = W^*W \\ &\Rightarrow W \text{ is contraction} \\ &\Rightarrow I \geq WW^* = T_\phi^{-\frac{1}{2}} S_\xi S^*_\xi T_\phi^{-\frac{1}{2}} \\ &\Rightarrow T_\phi \geq S_\xi S^*_\xi. \end{aligned}$$

Since $\begin{pmatrix} T_\phi - S_\xi S_\xi^* & 0 \\ 0 & T_\phi - S_\xi^* S_\xi \end{pmatrix} =$

$$\begin{pmatrix} 1 & 0 \\ -S_\xi^* & 1 \end{pmatrix}\begin{pmatrix} T_\phi - S_\xi S_\xi^* & (T_\phi - S_\xi S_\xi^*)S_\xi \\ S_\xi^*(T_\phi - S_\xi S_\xi^*) & T_\phi - S_\xi^* S_\xi + S_\xi^*(T_\phi - S_\xi S_\xi^*)S_\xi \end{pmatrix}\begin{pmatrix} 1 & -S_\xi^* \\ 0 & 1 \end{pmatrix}.$$

Then,

$$\begin{pmatrix} T_\phi - S_\xi S_\xi^* & 0 \\ 0 & T_\phi - S_\xi^* S_\xi \end{pmatrix} \text{ and } \begin{pmatrix} T_\phi - S_\xi S_\xi^* & (T_\phi - S_\xi S_\xi^*)S_\xi \\ S_\xi^*(T_\phi - S_\xi S_\xi^*) & T_\phi - S_\xi^* S_\xi + S_\xi^*(T_\phi - S_\xi S_\xi^*)S_\xi \end{pmatrix}$$

are congruent to each other. Thus,

$$\begin{pmatrix} T_\phi - S_\xi S_\xi^* & 0 \\ 0 & T_\phi - S_\xi^* S_\xi \end{pmatrix} \geq 0 \ \ iff \ \ \begin{pmatrix} T_\phi - S_\xi S_\xi^* & (T_\phi - S_\xi S_\xi^*)S_\xi \\ S_\xi^*(T_\phi - S_\xi S_\xi^*) & T_\phi - S_\xi^* S_\xi + S_\xi^*(T_\phi - S_\xi S_\xi^*)S_\xi \end{pmatrix} \geq 0.$$

Therefore, $T_\phi \geq S_\xi S_\xi^*$ and $T_\phi \geq S_\xi^* S_\xi$ combiningly implies,

$$\begin{pmatrix} T_\phi - S_\xi S_\xi^* & 0 \\ 0 & T_\phi - S_\xi^* S_\xi \end{pmatrix} \geq 0.$$

Hence the result follows. □

Corollary 3.2. *Let $\phi, \xi \in L^\infty(\mathbb{D})$. Then, $T_\phi \geq S_\xi^* S_\xi$, $T_\phi - S_\xi S_\xi^* = I$ and $p > \lambda_\phi^n$, defined as in theorem 3.1 implies* $\begin{pmatrix} I & S_\xi \\ S_\xi^\star & T_\phi \end{pmatrix} \geq 0$

Proof. Since $\begin{pmatrix} I & 0 \\ 0 & T_\phi - S_\xi^\star S_\xi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -S_\xi^\star & 1 \end{pmatrix}\begin{pmatrix} 1 & S_\xi \\ S_\xi^\star & T_\phi \end{pmatrix}\begin{pmatrix} 1 & -S_\xi \\ 0 & 1 \end{pmatrix}$. It follows that $\begin{pmatrix} I & S_\xi \\ S_\xi^\star & T_\phi \end{pmatrix} \geq 0$ if and only if $T_\phi \geq S_\xi^\star S_\xi$ □

Theorem 3.3. *Let $\phi, \xi \in L^\infty(\mathbb{D})$. Then,*

$$\begin{pmatrix} T_{\overline{\phi}}(T_\phi - S_\xi S_\xi^*)T_\phi & T_{\overline{\phi}}(T_\phi - S_\xi S_\xi^*)U_a \\ U_a^*(T_\phi - S_\xi S_\xi^*)T_\phi & U_a^*(T_\phi - S_\xi S_\xi^*)U_a + T_\phi T_{\overline{\phi}} \end{pmatrix} \geq 0 \ \ iff \ \ T_\phi \geq S_\xi S_\xi^*.$$

Proof. Since $\begin{pmatrix} T_{\overline{\phi}}(T_\phi - S_\xi S_\xi^*)T_\phi & T_{\overline{\phi}}(T_\phi - S_\xi S_\xi^*)U_a \\ U_a^*(T_\phi - S_\xi S_\xi^*)T_\phi & U_a^*(T_\phi - S_\xi S_\xi^*)U_a + T_\phi T_{\overline{\phi}} \end{pmatrix} =$

$$\begin{pmatrix} T_{\overline{\phi}} & 0 \\ U_a^* & -T_\phi \end{pmatrix}\begin{pmatrix} T_\phi - S_\xi S_\xi^* & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} T_\phi & U_a \\ 0 & -T_{\overline{\phi}} \end{pmatrix}$$

then, $T_\phi \geq S_\xi S_\xi^* \Leftrightarrow \begin{pmatrix} T_\phi - S_\xi S_\xi^* & 0 \\ 0 & I \end{pmatrix} \geq 0$. Hence proved. □

Theorem 3.4. *Let $\phi, \psi \in L^\infty(\mathbb{D})$. Then, $\begin{pmatrix} T_{|\phi|} & S_{\psi^+} \\ S_\psi & T_{|\psi|} \end{pmatrix} \geq 0$ for $\psi^+(z) = \overline{\psi(\overline{z})}$ if and only if $|\langle S_\psi K_x, K_y\rangle|^2 \leq \langle T_{|\phi|}K_x, K_x\rangle\langle T_{|\psi|}K_y, K_y\rangle$ for all $x, y \in \mathbb{D}$.*

Proof. Suppose $\begin{pmatrix} T_{|\phi|} & S_{\psi^+} \\ S_\psi & T_{|\psi|} \end{pmatrix} \geq 0$. Since for any positive operator $A \in \mathcal{L}(A^2(\mathbb{D}))$, it follows from [10] that $|\langle AK_x, K_y\rangle|^2 \leq \langle AK_x, K_x\rangle\langle AK_y, K_y\rangle$ for all $x, y \in \mathbb{D}$. Then we obtain,

$$\left|\left\langle\begin{pmatrix} T_{|\phi|} & S_{\psi^+} \\ S_\psi & T_{|\psi|}\end{pmatrix}\begin{pmatrix} K_x \\ 0\end{pmatrix},\begin{pmatrix} 0 \\ K_y\end{pmatrix}\right\rangle\right|^2$$

$\leq \left\langle\begin{pmatrix} T_{|\phi|} & S_{\psi^+} \\ S_\psi & T_{|\psi|}\end{pmatrix}\begin{pmatrix} K_x \\ 0\end{pmatrix},\begin{pmatrix} K_x \\ 0\end{pmatrix}\right\rangle\left\langle\begin{pmatrix} T_{|\phi|} & S_{\psi^+} \\ S_\psi & T_{|\psi|}\end{pmatrix}\begin{pmatrix} 0 \\ K_y\end{pmatrix},\begin{pmatrix} 0 \\ K_y\end{pmatrix}\right\rangle$ for all $x, y \in \mathbb{D}$. Hence,

$$\left|\left\langle\begin{pmatrix} T_{|\phi|}K_x \\ S_\psi K_x\end{pmatrix},\begin{pmatrix} 0 \\ K_y\end{pmatrix}\right\rangle\right|^2 \leq \left\langle\begin{pmatrix} T_{|\phi|}K_x \\ S_\psi K_x\end{pmatrix},\begin{pmatrix} K_x \\ 0\end{pmatrix}\right\rangle\left\langle\begin{pmatrix} S_{\psi^+}K_y \\ T_{|\psi|}K_y\end{pmatrix},\begin{pmatrix} 0 \\ K_y\end{pmatrix}\right\rangle.$$

Therefore,

$$\left|\int_{\mathbb{D}}\begin{pmatrix} T_\phi K_x \\ S_\psi K_x\end{pmatrix}\begin{pmatrix} 0 \\ \overline{K_y}\end{pmatrix}dA(z)\right|^2$$

$$\leq \left(\int_{\mathbb{D}}\begin{pmatrix} T_{|\phi|}K_x \\ S_\psi K_x\end{pmatrix}\begin{pmatrix} \overline{K_x} \\ 0\end{pmatrix}dA(z)\right)\left(\int_{\mathbb{D}}\begin{pmatrix} S_{\psi^+}K_y \\ T_{|\psi|}K_y\end{pmatrix}\begin{pmatrix} 0 \\ \overline{K_y}\end{pmatrix}dA(z)\right).$$

That is,

$$\left|\int_{\mathbb{D}}\begin{pmatrix} 0 \\ S_\psi K_x\overline{K_y}\end{pmatrix}dA(z)\right|^2 \leq \left(\int_{\mathbb{D}} T_{|\phi|}K_x\overline{K_x}dA(z)\right)\left(\int_{\mathbb{D}} T_{|\psi|}K_y\overline{K_y}dA(z)\right).$$

Thus,

$$|\langle S_\psi K_x, K_y\rangle|^2 \leq \langle T_{|\phi|}K_x, K_x\rangle\langle T_{|\psi|}K_y, K_y\rangle$$

for all $x, y \in \mathbb{D}$.

Conversely, assume that

$|\langle S_\psi K_x, K_y\rangle|^2 \leq \langle T_{|\phi|}K_x, K_x\rangle\langle T_{|\psi|}K_y, K_y\rangle$ for all $x, y \in \mathbb{D}$. Then,

$$\begin{aligned}\left\langle\begin{pmatrix} T_{|\phi|} & S_{\psi^+} \\ S_\psi & T_{|\psi|}\end{pmatrix}\begin{pmatrix} K_x \\ K_y\end{pmatrix},\begin{pmatrix} K_x \\ K_y\end{pmatrix}\right\rangle &= \langle T_{|\phi|}K_x, K_x\rangle + \langle S_{\psi^+}K_y, K_x\rangle + \langle S_\psi K_x, K_y\rangle + \langle T_{|\psi|}K_y, K_y\rangle \\ &= \langle T_{|\phi|}K_x, K_x\rangle + 2\mathrm{Re}\langle S_\psi K_x, K_y\rangle + \langle T_{|\psi|}K_y, K_y\rangle \\ &\geq 2\langle T_{|\phi|}K_x, K_x\rangle^{\frac{1}{2}}\langle T_{|\psi|}K_y, K_y\rangle^{\frac{1}{2}} + 2\mathrm{Re}\langle S_\psi K_x, K_y\rangle \\ &\geq 2|\langle S_\psi K_x, K_y\rangle| + 2\mathrm{Re}\langle S_\psi K_x, K_y\rangle \\ &\geq 2|\langle S_\psi K_x, K_y\rangle| - 2|\langle S_\psi K_x, K_y\rangle| = 0.\end{aligned}$$

Hence, $\begin{pmatrix} T_{|\phi|} & S_{\psi^+} \\ S_\psi & T_{|\psi|}\end{pmatrix} \geq 0.$ □

Theorem 3.5. *Let $\phi, \psi \in L^\infty(\mathbb{D})$ where $\phi, \psi \geq 0$. Assume that T_ϕ and T_ψ are invertible and $\widetilde{T_{\psi o \psi_a}}(z) = \widetilde{T_\phi}(z), \forall z \in \mathbb{D}$. If there exist an operator M with $\|M\| \leq 1$ such that $T_\psi^{\frac{1}{2}} M T_\phi^{\frac{1}{2}} = U_a, a \in \mathbb{D}$ and h, j are two non negative functions defined by $h(x) = x^t$ and $j(x) = x^{1-t}$, $0 < t \leq \frac{1}{2}$ and $0 \leq x < \infty$. Then, $\begin{pmatrix} h(T_\psi)^2 & U_a \\ U_a & j(T_\phi)^2\end{pmatrix} \geq 0$.*

Proof. Since $\phi, \psi \geq 0$, and T_ϕ, T_ψ are invertible and $\widetilde{T_{\psi o \psi_a}}(z) = \widetilde{T_\phi}(z), \forall z \in \mathbb{D}$, then $\langle T_{\psi o \psi_a}k_z, k_z\rangle = \langle T_\phi k_z, k_z\rangle$. Thus, $\langle U_a T_\psi U_a k_z, k_z\rangle = \langle T_\phi k_z, k_z\rangle, \forall z \in \mathbb{D}$. Therefore, $T_\psi U_a = U_a T_\phi$ ($\because$ U_a is self adjoint) that implies $h(T_\psi)U_a = U_a h(T_\phi)$, since h is a continuous function on $[0, \infty)$. As $h(T_\phi) = T_\phi^t$ and $j(T_\phi) = T_\phi^{1-t}$, therefore $h(T_\phi)j(T_\phi) = T_\phi$. Now

$$\begin{aligned} h(T_\psi)U_a = U_a h(T_\phi) &\Rightarrow h(T_\psi)U_a j(T_\phi) = U_a h(T_\phi) j(T_\phi) \\ &\Rightarrow h(T_\psi)U_a j(T_\phi) = U_a T_\phi \\ &\Rightarrow h(T_\psi)U_a j(T_\phi) = U_a T_\phi^{\frac{1}{2}} T_\phi^{\frac{1}{2}} \\ &\Rightarrow T_\psi^{\frac{-1}{2}} h(T_\psi)U_a j(T_\phi)T_\phi^{\frac{-1}{2}} = T_\psi^{\frac{-1}{2}} U_a T_\phi^{\frac{1}{2}} \qquad (\because T_\psi \text{ and } T_\phi \text{ are invertible}) \\ &\Rightarrow h(T_\psi)T_\psi^{\frac{-1}{2}} U_a j(T_\phi)T_\phi^{\frac{-1}{2}} = U_a \qquad (\because U_a T_\psi^{\frac{1}{2}} = T_\phi^{\frac{1}{2}} U_a).\end{aligned}$$

Then,

$$\begin{pmatrix} h(T_\psi)^2 & U_a \\ U_a & j(T_\phi)^2 \end{pmatrix} = \begin{pmatrix} h(T_\psi)T_\psi^{\frac{-1}{2}} & 0 \\ 0 & j(T_\phi)T_\phi^{\frac{-1}{2}} \end{pmatrix} \begin{pmatrix} T_\psi & U_a \\ U_a & T_\phi \end{pmatrix} \begin{pmatrix} h(T_\psi)T_\psi^{\frac{-1}{2}} & 0 \\ 0 & j(T_\phi)T_\phi^{\frac{-1}{2}} \end{pmatrix}.$$

Since $T_\psi^{\frac{1}{2}} M T_\phi^{\frac{1}{2}} = U_a, a \in \mathbb{D}$ with M as contraction, then by using [9], $\begin{pmatrix} T_\psi & U_a \\ U_a & T_\phi \end{pmatrix} \geq 0$, which completes the proof of the theorem. □

Theorem 3.6. *Let $\phi \in L^\infty(\mathbb{D})$. $\begin{pmatrix} I & T_\phi \\ T_{\overline{\phi}} & I \end{pmatrix}$ is positive iff T_ϕ is contraction.*

Proof. Suppose $\begin{pmatrix} I & T_\phi \\ T_{\overline{\phi}} & I \end{pmatrix} \geq 0$, so

$$\begin{aligned} \left\langle \begin{pmatrix} I & T_\phi \\ T_{\overline{\phi}} & I \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}, \begin{pmatrix} f \\ g \end{pmatrix} \right\rangle &= \left\langle \begin{pmatrix} f + T_\phi g \\ T_{\overline{\phi}} f + g \end{pmatrix}, \begin{pmatrix} f \\ g \end{pmatrix} \right\rangle \\ &= \langle f, f\rangle + \langle T_\phi g, f\rangle + \langle T_{\overline{\phi}} f, g\rangle + \langle g, g\rangle \\ &= \|f\|^2 + \|g\|^2 + 2Re(\langle T_\phi g, f\rangle) \\ &\geq 0 \end{aligned}$$

By letting $f = -T_\phi g$, we have $\|T_\phi g\|^2 + \|g\|^2 - 2Re(\langle T_\phi g, T_\phi g\rangle) \geq 0$. We recall that

$$Re(\langle T_\phi g, T_\phi g\rangle) \leq |\langle T_\phi g, T_\phi g\rangle|.$$

So $\|T_\phi g\|^2 + \|g\|^2 - 2\|T_\phi g\|^2 \geq 0$. That implies $\|T_\phi g\|^2 \leq \|g\|^2 \Rightarrow \langle T_\phi g, T_\phi g\rangle \leq \langle g, g\rangle$.
Therefore $T_\phi^* T_\phi \leq I$. Conversely, let $T_\phi^* T_\phi \leq I$. Since

$$\begin{pmatrix} I & 0 \\ 0 & I - T_{\overline{\phi}} T_\phi \end{pmatrix} = \begin{pmatrix} I & 0 \\ T_{\overline{\phi}} & I \end{pmatrix} \begin{pmatrix} I & T_\phi \\ T_{\overline{\phi}} & I \end{pmatrix} \begin{pmatrix} I & -T_\phi \\ 0 & I \end{pmatrix}.$$

Then $\begin{pmatrix} I & T_\phi \\ T_{\overline{\phi}} & I \end{pmatrix} \geq 0$. Hence proved. □

Theorem 3.7. *Let $\phi \in L^\infty(\mathbb{D})$. Then,* $\begin{pmatrix} |T_\phi| & T_{\overline{\phi}} \\ T_\phi & |T_{\overline{\phi}}| \end{pmatrix} \geq 0$.

Proof. Let $Q = \begin{pmatrix} 0 & T_{\overline{\phi}} \\ T_\phi & 0 \end{pmatrix}$. Thus Q is self-adjoint. Therefore, $Q^2 = \begin{pmatrix} T_{\overline{\phi}} T_\phi & 0 \\ 0 & T_\phi T_{\overline{\phi}} \end{pmatrix}$. Since the square root of a positive operator is unique, then $|Q| = \begin{pmatrix} |T_\phi| & 0 \\ 0 & |T_{\overline{\phi}}| \end{pmatrix}$. Again since Q is self-adjoint, therefore by using the spectral theory, $Q + |Q|$ is positive. Hence, $\begin{pmatrix} |T_\phi| & T_{\overline{\phi}} \\ T_\phi & |T_{\overline{\phi}}| \end{pmatrix} \geq 0$. □

Theorem 3.8. *Let $\phi \in L^\infty(\mathbb{D})$. Then,*

$$\begin{pmatrix} \Delta(T_\phi) & |T_\phi|^{\frac{1}{2}} |S_\phi|^{\frac{1}{2}} \\ |S_\phi|^{\frac{1}{2}} |T_\phi|^{\frac{1}{2}} & \Delta(S_\phi) \end{pmatrix} \geq 0 \;\; iff \;\; |\langle k_z, k_w\rangle|^2 \leq \langle U k_z, k_z\rangle \langle V k_w, k_w\rangle, \; \forall k_z, k_w \in A^2(\mathbb{D})$$

and $T_\phi = U|T_\phi|$, $S_\phi = V|S_\phi|$ be the polar decompositions of T_ϕ and S_ϕ respectively.

Proof. Since

$$\begin{pmatrix} \Delta(T_\phi) & |T_\phi|^{\frac{1}{2}}|S_\phi|^{\frac{1}{2}} \\ |S_\phi|^{\frac{1}{2}}|T_\phi|^{\frac{1}{2}} & \Delta(S_\phi) \end{pmatrix} = \begin{pmatrix} |T_\phi|^{\frac{1}{2}}U|T_\phi|^{\frac{1}{2}} & |T_\phi|^{\frac{1}{2}}|S_\phi|^{\frac{1}{2}} \\ |S_\phi|^{\frac{1}{2}}|T_\phi|^{\frac{1}{2}} & |S_\phi|^{\frac{1}{2}}V|S_\phi|^{\frac{1}{2}} \end{pmatrix}$$
$$= \begin{pmatrix} |T_\phi|^{\frac{1}{2}} & 0 \\ 0 & |S_\phi|^{\frac{1}{2}} \end{pmatrix}\begin{pmatrix} U & I \\ I & V \end{pmatrix}\begin{pmatrix} |T_\phi|^{\frac{1}{2}} & 0 \\ 0 & |S_\phi|^{\frac{1}{2}} \end{pmatrix}$$

Assume that $T = S^*WS$, where $T = \begin{pmatrix} \Delta(T_\phi) & |T_\phi|^{\frac{1}{2}}|S_\phi|^{\frac{1}{2}} \\ |S_\phi|^{\frac{1}{2}}|T_\phi|^{\frac{1}{2}} & \Delta(S_\phi) \end{pmatrix}$, $S = \begin{pmatrix} |T_\phi|^{\frac{1}{2}} & 0 \\ 0 & |S_\phi|^{\frac{1}{2}} \end{pmatrix} = S^*$ and $W = \begin{pmatrix} U & I \\ I & V \end{pmatrix}$. Since the operator matrices T and W are congruent between each other, so $T \geq 0$ iff $W \geq 0$. Now

$$\left\langle \begin{pmatrix} U & I \\ I & V \end{pmatrix}\begin{pmatrix} k_z \\ k_w \end{pmatrix}, \begin{pmatrix} k_z \\ k_w \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} Uk_z + k_w \\ k_z + Vk_w \end{pmatrix}, \begin{pmatrix} k_z \\ k_w \end{pmatrix} \right\rangle$$
$$= \langle Uk_z, k_z\rangle + \langle k_z, k_w\rangle + \langle k_w, k_z\rangle + \langle Vk_w, k_w\rangle$$
$$= \langle Uk_z, k_z\rangle + \langle Vk_w, k_w\rangle + 2Re\langle k_z, k_w\rangle$$
$$\geq 2\sqrt{(\langle Uk_z, k_z\rangle\langle Vk_w, k_w\rangle)} - 2|\langle k_z, k_w\rangle|$$
$$\geq 0.$$

Therefore, $W \geq 0$ if $|\langle k_z, k_w\rangle|^2 \leq \langle Uk_z, k_z\rangle\langle Vk_w, k_w\rangle$.
Conversely, suppose $W \geq 0$. Then $|\langle Wk_z, k_w\rangle|^2 \leq \langle Wk_z, k_z\rangle\langle Wk_w, k_w\rangle \quad \forall k_z, k_w \in A^2(\mathbb{D})$. That is

$$\left|\left\langle \begin{pmatrix} U & I \\ I & V \end{pmatrix}\begin{pmatrix} k_z \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ k_w \end{pmatrix} \right\rangle\right|^2 \leq \left\langle \begin{pmatrix} U & I \\ I & V \end{pmatrix}\begin{pmatrix} k_z \\ 0 \end{pmatrix}, \begin{pmatrix} k_z \\ 0 \end{pmatrix} \right\rangle \left\langle \begin{pmatrix} U & I \\ I & V \end{pmatrix}\begin{pmatrix} 0 \\ k_w \end{pmatrix}, \begin{pmatrix} 0 \\ k_w \end{pmatrix} \right\rangle.$$

Therefore, $W \geq 0$ that implies $|\langle k_z, k_w\rangle|^2 \leq \langle Uk_z, k_z\rangle\langle Vk_w, k_w\rangle$. Hence equivalently, $T \geq 0$ iff $|\langle k_z, k_w\rangle|^2 \leq \langle Uk_z, k_z\rangle\langle Vk_w, k_w\rangle$. This completes the proof. □

Corollary 3.9. *Let ϕ, $\psi \in L^\infty(\mathbb{D})$. Then,*

$$\begin{pmatrix} \Delta(T_\phi) & |T_\phi|^{\frac{1}{2}}U|S_\psi|^{\frac{1}{2}} \\ |S_\psi|^{\frac{1}{2}}U|T_\phi|^{\frac{1}{2}} & \Delta(S_\psi) + \Delta(T_\phi) \end{pmatrix} \geq 0 \ iff \ \langle Uf, f\rangle\langle Vg, g\rangle \geq 0 \quad \forall f,\ g \in A^2(\mathbb{D})$$

with $T_\phi = U|T_\phi|$ and $S_\psi = V|S_\psi|$ be the polar decompositions of T_ϕ and S_ψ respectively.

Proof. Since, $\begin{pmatrix} \Delta(T_\phi) & |T_\phi|^{\frac{1}{2}}U|S_\psi|^{\frac{1}{2}} \\ |S_\psi|^{\frac{1}{2}}U|T_\phi|^{\frac{1}{2}} & \Delta(S_\psi) + \Delta(T_\phi) \end{pmatrix} =$

$$\begin{pmatrix} |T_\phi|^{\frac{1}{2}} & 0 \\ |S_\psi|^{\frac{1}{2}} & -|T_\phi|^{\frac{1}{2}} \end{pmatrix}\begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}\begin{pmatrix} |T_\phi|^{\frac{1}{2}} & |S_\psi|^{\frac{1}{2}} \\ 0 & -|T_\phi|^{\frac{1}{2}} \end{pmatrix}$$

then, from Theorem- 3.8,

$$\begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} \geq 0 \ iff \ \langle Uf, f\rangle\langle Vg, g\rangle \geq 0.$$

Hence complete the assertion. □

Theorem 3.10. *Let ϕ, $\psi \in L^\infty(\mathbb{D})$. Then, $\begin{pmatrix} |S_\psi| & \Delta(T_\phi) \\ \Delta^*(T_\phi) & |T_\phi| \end{pmatrix} \geq 0$ iff $\exists$ a contraction M such that $|S_\psi|^{\frac{1}{2}}M = |T_\phi|^{\frac{1}{2}}U$, with $T_\phi = U|T_\phi|$ is the polar decomposition of T_ϕ, where U is the partial isometry.*

Proof. Since $\begin{pmatrix} |S_\psi| & \Delta(T_\phi) \\ \Delta^*(T_\phi) & |T_\phi| \end{pmatrix} =$

$$\begin{pmatrix} |S_\psi|^{\frac{1}{2}} & 0 \\ 0 & |T_\phi|^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} I & |S_\psi|^{-\frac{1}{2}}|T_\phi|^{\frac{1}{2}}U \\ U^*|T_\phi|^{\frac{1}{2}}|S_\psi|^{-\frac{1}{2}} & I \end{pmatrix} \begin{pmatrix} |S_\psi|^{\frac{1}{2}} & 0 \\ 0 & |T_\phi|^{\frac{1}{2}} \end{pmatrix}$$

Then,

$$\begin{pmatrix} |S_\psi| & \Delta(T_\phi) \\ \Delta^*(T_\phi) & |T_\phi| \end{pmatrix} \text{ and } \begin{pmatrix} I & |S_\psi|^{-\frac{1}{2}}|T_\phi|^{\frac{1}{2}}U \\ U^*|T_\phi|^{\frac{1}{2}}|S_\psi|^{-\frac{1}{2}} & I \end{pmatrix}$$

are congruent to each other. So by Theorem-3.6,

$$\begin{pmatrix} I & M \\ M^* & I \end{pmatrix} \geq 0 \ iff \ I \geq M^*M,$$

where $M = |S_\psi|^{-\frac{1}{2}}|T_\phi|^{\frac{1}{2}}U$. □

Theorem 3.11. *Let $\phi \in L^\infty(\mathbb{D})$. Assume that $T_\phi T_{\overline{\phi}} \geq 0$ and $T_\phi^2 T_{\overline{\phi}}^2 \geq 0$. Then*

$$W = \begin{pmatrix} T_{\overline{\phi}}T_\phi - T_\phi T_{\overline{\phi}} & T_{\overline{\phi}}^2 T_\phi - T_\phi T_{\overline{\phi}}^2 \\ T_{\overline{\phi}}T_\phi^2 - T_\phi^2 T_{\overline{\phi}} & T_{\overline{\phi}}^2 T_\phi^2 - T_\phi^2 T_{\overline{\phi}}^2 \end{pmatrix} \geq 0$$

if $\exists$ a contraction M such that

$$T_\phi T_{\overline{\phi}}^2 = |T_{\overline{\phi}}|M|T_{\overline{\phi}}^2|$$

Proof. Suppose $W \geq 0$. Then,

$$\begin{pmatrix} T_{\overline{\phi}}T_\phi & T_{\overline{\phi}}^2 T_\phi \\ T_{\overline{\phi}}T_\phi^2 & T_{\overline{\phi}}^2 T_\phi^2 \end{pmatrix} \geq \begin{pmatrix} T_\phi T_{\overline{\phi}} & T_\phi T_{\overline{\phi}}^2 \\ T_\phi^2 T_{\overline{\phi}} & T_\phi^2 T_{\overline{\phi}}^2 \end{pmatrix}$$

Then by [9], $\begin{pmatrix} T_\phi T_{\overline{\phi}} & T_\phi T_{\overline{\phi}}^2 \\ T_\phi^2 T_{\overline{\phi}} & T_\phi^2 T_{\overline{\phi}}^2 \end{pmatrix} \geq 0$ iff $\exists$ a M such that $\|M\| \leq 1$ and $T_\phi T_{\overline{\phi}}^2 = |T_{\overline{\phi}}|M|T_{\overline{\phi}}^2|$. Hence proved. □

4. Unitary and normal operator matrices

In this section, we discussed some sufficient conditions for operator matrices to be normal as well as unitary.

Theorem 4.1. *Let $\phi \in L^\infty(\mathbb{D})$ with $\|\phi\|_\infty \leq 1$. Then, $\begin{pmatrix} T_\phi & (I - T_\phi T_{\overline{\phi}})^{\frac{1}{2}} \\ (I - T_{\overline{\phi}}T_\phi)^{\frac{1}{2}} & -T_{\overline{\phi}} \end{pmatrix}$ is unitary.*

Proof. Since $\|\phi\|_\infty \leq 1$, so $\|T_\phi\| \leq \|\phi\|_\infty \leq 1$. Then, $I \geq T_{\overline{\phi}}T_\phi$ that implies T_ϕ is contraction, which implies $I \geq T_\phi T_{\overline{\phi}}$. Consider $S = \begin{pmatrix} T_\phi & (I - T_\phi T_{\overline{\phi}})^{\frac{1}{2}} \\ (I - T_{\overline{\phi}}T_\phi)^{\frac{1}{2}} & -T_{\overline{\phi}} \end{pmatrix}$

Now

$$S^*S = \begin{pmatrix} I & T_{\overline{\phi}}(I - T_\phi T_{\overline{\phi}})^{\frac{1}{2}} - (I - T_{\overline{\phi}}T_\phi)^{\frac{1}{2}}T_{\overline{\phi}} \\ (I - T_\phi T_{\overline{\phi}})^{\frac{1}{2}}T_\phi - T_\phi(I - T_{\overline{\phi}}T_\phi)^{\frac{1}{2}} & I \end{pmatrix}$$

and

$$SS^* = \begin{pmatrix} I & T_\phi(I - T_{\overline{\phi}}T_\phi)^{\frac{1}{2}} - (I - T_\phi T_{\overline{\phi}})^{\frac{1}{2}}T_\phi \\ (I - T_{\overline{\phi}}T_\phi)^{\frac{1}{2}}T_{\overline{\phi}} - T_{\overline{\phi}}(I - T_\phi T_{\overline{\phi}})^{\frac{1}{2}} & I \end{pmatrix}.$$

So $S^*S = SS^* = I$ when $T_\phi(I - T_{\overline{\phi}}T_\phi)^{\frac{1}{2}} = (I - T_\phi T_{\overline{\phi}})^{\frac{1}{2}}T_\phi$.

To prove this we use elementary concepts of operator theory. Put $W = \begin{pmatrix} 0 & T_{\overline{\phi}} \\ T_\phi & 0 \end{pmatrix}, P = \begin{pmatrix} I - T_{\overline{\phi}}T_\phi & 0 \\ 0 & I - T_\phi T_{\overline{\phi}} \end{pmatrix}$.

It is clear that $P \geq 0$. Since $WP = PW$ for $P \geq 0$. That implies $WP^{\frac{1}{2}} = P^{\frac{1}{2}}W$.

So $WP^{\frac{1}{2}} = \begin{pmatrix} 0 & T_{\overline{\phi}} \\ T_\phi & 0 \end{pmatrix}\begin{pmatrix} (I - T_{\overline{\phi}}T_\phi)^{\frac{1}{2}} & 0 \\ 0 & (I - T_\phi T_{\overline{\phi}})^{\frac{1}{2}} \end{pmatrix} = \begin{pmatrix} 0 & T_{\overline{\phi}}(I - T_\phi T_{\overline{\phi}})^{\frac{1}{2}} \\ T_\phi(I - T_{\overline{\phi}}T_\phi)^{\frac{1}{2}} & 0 \end{pmatrix}$.

Similarly, $P^{\frac{1}{2}}W = \begin{pmatrix} (I - T_{\overline{\phi}}T_\phi)^{\frac{1}{2}} & 0 \\ 0 & (I - T_\phi T_{\overline{\phi}})^{\frac{1}{2}} \end{pmatrix}\begin{pmatrix} 0 & T_{\overline{\phi}} \\ T_\phi & 0 \end{pmatrix} = \begin{pmatrix} 0 & (I - T_{\overline{\phi}}T_\phi)^{\frac{1}{2}}T_{\overline{\phi}} \\ (I - T_\phi T_{\overline{\phi}})^{\frac{1}{2}}T_\phi & 0 \end{pmatrix}$.

Since $WP^{\frac{1}{2}} = P^{\frac{1}{2}}W$, then $T_\phi(I - T_{\overline{\phi}}T_\phi)^{\frac{1}{2}} = (I - T_\phi T_{\overline{\phi}})^{\frac{1}{2}}T_\phi$. Hence S is unitary. □

Theorem 4.2. *Let $\phi, \psi \in L^\infty(\mathbb{D})$. Then, $\begin{pmatrix} T_{\phi\circ\phi_a} & I \\ 0 & S_\psi \end{pmatrix}$ is normal iff $U_aT_\phi = S_\psi U_a$ and $[T_{\overline{\phi}}, T_\phi] = [S_\psi, S_{\psi^+}]$.*

Proof. It is easy to verify that $\begin{pmatrix} U_a & 0 \\ 0 & I \end{pmatrix}$ is unitary. Since,

$$\begin{pmatrix} T_{\phi\circ\phi_a} & I \\ 0 & S_\psi \end{pmatrix} = \begin{pmatrix} U_a^* & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} T_\phi & U_a \\ 0 & S_\psi \end{pmatrix}\begin{pmatrix} U_a & 0 \\ 0 & I \end{pmatrix}$$

then, $\begin{pmatrix} T_{\phi\circ\phi_a} & I \\ 0 & S_\psi \end{pmatrix}$ and $\begin{pmatrix} T_\phi & U_a \\ 0 & S_\psi \end{pmatrix}$ are unitarily equivalent. Therefore, $\begin{pmatrix} T_{\phi\circ\phi_a} & I \\ 0 & S_\psi \end{pmatrix}$ is normal iff $\begin{pmatrix} T_\phi & U_a \\ 0 & S_\psi \end{pmatrix}$ is normal. Hence, $\begin{pmatrix} T_\phi & U_a \\ 0 & S_\psi \end{pmatrix}$ is normal iff $U_aT_\phi = S_\psi U_a$ and $T_{\overline{\phi}}T_\phi + S_{\psi^+}S_\psi = T_\phi T_{\overline{\phi}} + S_\psi S_{\psi^+}$. □

Theorem 4.3. *Let $\phi, \psi \in L^\infty(\mathbb{D})$ and $e^{i\theta}$ be any complex number in $\mathbb{C}$. Then, $\begin{pmatrix} U_a & 0 & 0 \\ 0 & T_\psi & e^{i\theta}S_\psi \\ 0 & e^{-(i\theta)}S_\psi^* & T_\phi \end{pmatrix}$ is*

(i) *normal iff T_ϕ, T_ψ are normal and S_ψ intertwines with T_ϕ and T_ψ as well as $T_{\overline{\phi}}$ and $T_{\overline{\psi}}$ respectively.*

(ii) *unitary iff T_ϕ, T_ψ are unitary and $T_\psi S_\psi - S_\psi T_\phi = I = T_{\overline{\psi}}S_\psi - S_\psi T_{\overline{\phi}}$.*

Proof. Since

$$\begin{pmatrix} U_a & 0 & 0 \\ 0 & T_\psi & e^{i\theta}S_\psi \\ 0 & e^{-(i\theta)}S_\psi^* & T_\phi \end{pmatrix} = \begin{pmatrix} U_a^* & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & e^{-(i\theta)}I \end{pmatrix}\begin{pmatrix} U_a & 0 & 0 \\ 0 & T_\psi & S_\psi \\ 0 & S_\psi^* & T_\phi \end{pmatrix}\begin{pmatrix} U_a & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & e^{i\theta}I \end{pmatrix}.$$

Then, $\begin{pmatrix} U_a & 0 & 0 \\ 0 & T_\psi & e^{i\theta}S_\psi \\ 0 & e^{-(i\theta)}S_\psi^* & T_\phi \end{pmatrix}$ is normal iff $\begin{pmatrix} U_a & 0 & 0 \\ 0 & T_\psi & S_\psi \\ 0 & S_\psi^* & T_\phi \end{pmatrix}$ is normal. Therefore, from the direct computation $\begin{pmatrix} U_a & 0 & 0 \\ 0 & T_\psi & S_\psi \\ 0 & S_\psi^* & T_\phi \end{pmatrix}$ is normal iff $T_\phi T_{\overline{\phi}} = T_{\overline{\phi}}T_\phi$, $T_\psi T_{\overline{\psi}} = T_{\overline{\psi}}T_\psi$, $T_\phi S_\psi = S_\psi T_\psi$, $T_{\overline{\phi}}S_\psi = S_\psi T_{\overline{\psi}}$. Similarly, the proof of (**ii**) is same as the proof of (**i**). □

Theorem 4.4. *Let $\phi, \psi \in L^\infty(\mathbb{D})$. Then, $\begin{pmatrix} R_t & 0 & 0 \\ 0 & T_{\phi\circ\phi_a} & R_t^*S_\psi \\ 0 & 0 & T_\psi \end{pmatrix}$ is*

(i) *normal iff* S_ψ *is normal and* $T_{\overline{\phi}}S_\psi = S_\psi T_{\overline{\psi}}$, $[T_{\overline{\phi}}, T_\phi] = [T_\psi, T_{\overline{\psi}}] = S_\psi^* S_\psi$.

(ii) *unitary iff* S_ψ *is unitary,* $[T_{\overline{\phi}}, T_\phi] = [T_\psi, T_{\overline{\psi}}] = S_\psi^* S_\psi = I$ *and* $T_{\overline{\phi}}S_\psi = S_\psi T_{\overline{\psi}} = 0$.

Proof. Since

$$\begin{pmatrix} R_t & 0 & 0 \\ 0 & T_{\phi\circ\phi_a} & R_t^* S_\psi \\ 0 & 0 & T_\psi \end{pmatrix} = \begin{pmatrix} R_t^* & 0 & 0 \\ 0 & R_t^* & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} R_t & 0 & 0 \\ 0 & T_\phi & S_\psi \\ 0 & 0 & T_\psi \end{pmatrix} \begin{pmatrix} R_t & 0 & 0 \\ 0 & R_t & 0 \\ 0 & 0 & I \end{pmatrix},$$

then,

$$\begin{pmatrix} R_t & 0 & 0 \\ 0 & T_{\phi\circ\phi_a} & R_t^* S_\psi \\ 0 & 0 & T_\psi \end{pmatrix} \text{ and } \begin{pmatrix} R_t & 0 & 0 \\ 0 & T_\phi & S_\psi \\ 0 & 0 & T_\psi \end{pmatrix}$$

are unitarily equivalent. Again since,

$$\begin{pmatrix} R_t & 0 & 0 \\ 0 & T_\phi & S_\psi \\ 0 & 0 & T_\psi \end{pmatrix} \begin{pmatrix} R_t^* & 0 & 0 \\ 0 & T_{\overline{\phi}} & 0 \\ 0 & S_\psi^* & T_{\overline{\psi}} \end{pmatrix} = \begin{pmatrix} R_t R_t^* & 0 & 0 \\ 0 & T_\phi T_{\overline{\phi}} + S_\psi S_\psi^* & S_\psi T_{\overline{\psi}} \\ 0 & T_\psi S_\psi^* & T_\psi T_{\overline{\psi}} \end{pmatrix}.$$

Again since

$$\begin{pmatrix} R_t^* & 0 & 0 \\ 0 & T_{\overline{\phi}} & 0 \\ 0 & S_\psi^* & T_{\overline{\psi}} \end{pmatrix} \begin{pmatrix} R_t & 0 & 0 \\ 0 & T_\phi & S_\psi \\ 0 & 0 & T_\psi \end{pmatrix} = \begin{pmatrix} R_t R_t^* & 0 & 0 \\ 0 & T_{\overline{\phi}} T_\phi & T_{\overline{\phi}} S_\psi \\ 0 & S_\psi^* T_\psi & S_\psi^* S_\psi + T_{\overline{\psi}} T_\psi \end{pmatrix}.$$

It is easy to prove that $\begin{pmatrix} R_t & 0 & 0 \\ 0 & T_{\phi\circ\phi_a} & R_t^* S_\psi \\ 0 & 0 & T_\psi \end{pmatrix}$ is normal iff S_ψ is normal and $T_{\overline{\phi}}S_\psi = S_\psi T_{\overline{\psi}}$, $[T_{\overline{\phi}}, T_\phi] = [T_\psi, T_{\overline{\psi}}] = S_\psi^* S_\psi$. Similarly, one can easily prove $\begin{pmatrix} R_t & 0 & 0 \\ 0 & T_{\phi\circ\phi_a} & R_t^* S_\psi \\ 0 & 0 & T_\psi \end{pmatrix}$ is unitary if and only if S_ψ is unitary, $[T_{\overline{\phi}}, T_\phi] = [T_\psi, T_{\overline{\psi}}] = S_\psi^* S_\psi = I$ and $T_{\overline{\phi}}S_\psi = S_\psi T_{\overline{\psi}} = 0$. □

References

[1] R. Bhatia, Matrix analysis, Springer Verlag, New York, 1997.
[2] K. Zhu, Operator Theory in Fuction Spaces, Monographs and textbooks in pure and applied Mathematics, **139**, Marcel Dekker, New York, 1990.
[3] A. Aluthge, On p-hyponormal operators for $0 < p < 1$, Integr. Equat. Oper. Th., **13**(1990), 307-315.
[4] M. D. Choi, Some assorted inequalities for positive linear maps on C*-algebras, J. Oper. Th., **4**(1980), 271-285.
[5] N. Das and P. K. Jena, Asymptotic properties of composition operators on the Bergman space, New Zealand J. Math., **39**(2009), 215-239.
[6] R. G. Douglas, On majorization, factorization and range inclusion of operators on Hilbert space, Proc. Amer. Math. Soc., **17**(1966), 413-415.
[7] X. Fang, J. Yu and H. Yao, Solutions to operator equations on Hilbert C^*-modules, Linear Algebra Appl., **431**(2009), 2142-2153.
[8] M. Gurdal and F. Sohret, Some results for Toeplitz operators on the Bergman space, Appl. Math. Comput., **218**(2011), 789-793.
[9] J. C. Hou and H. K. Du, Norm inequalities of positive operator matrices, Integr. Equat. Oper. Th., **22**(1995), 281-294.
[10] T. Kato, Notes on some inequalities for linear operators, Math. Ann., **125**(1952), 208-212.
[11] H. Najafi, Operator means and positivity of block operators, Math. Z., DOI:10.1007/s00209-017-1958-0, 2017, 1-10.
[12] S. Wada, On some operators whose products are positive, Nihonkai Math. J., **10**(1999), 111-115.