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A Self-Adaptive Method for Split Common Null Point Problems and Fixed Point Problems for Multivalued Bregman Quasi-Nonexpansive Mappings in Banach Spaces

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Abstract. In this paper, we propose a self-adaptive algorithm for solving the split common null point problem and the fixed point problem for multivalued Bregman quasi-nonexpansive mappings in Banach spaces. We prove that the sequence generated by our iterative scheme converges strongly to a common solution of the above-mentioned problems under some suitable conditions. We also apply our main result to split feasibility problems in Banach spaces. Finally, numerical examples are given to support our main theorem. The results presented in this paper improve and extend many recent results in the literature.

1. Introduction

Let E_1 and E_2 be two real Banach spaces. Let $B_1 : E_1 \multimap E_1^*$ and $B_2 : E_2 \multimap E_2^*$ be two set-valued maximal monotone operators and $A : E_1 \to E_2$ be a bounded linear operator with its adjoint operator $A^* : E_2^* \to E_1^*$. The *split common null point problem* (SCNPP) is formulated as finding $x^* \in E_1$ such that

$$0 \in B_1(x^*) \ \text{ and } \ 0 \in B_2(Ax^*). \tag{1}$$

This formalism is also at the core of the modeling of many inverse problems and other real life problems, for instance, in practice as a model in intensity-modulated radiation therapy treatment planning (see [15, 19]) and in sensor networks in computerized tomography and data compression (see [14]).

To solve the SCNPP in two Hilbert spaces H_1 and H_2, Byrne et al. [11] introduced the following algorithms: for $u, x_1 \in H_1$, compute the sequences $\{x_n\}$ generated iteratively by

$$x_{n+1} = J_\lambda(x_n - \gamma A^*(I - Q_\mu)Ax_n), \ \ \forall n \geq 1 \tag{2}$$

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and

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)J_\lambda(x_n - \gamma A^*(I - Q_\mu)Ax_n), \;\; \forall n \geq 1, \tag{3}$$

where J_λ and Q_μ are the resolvent operators of B_1 and B_2 for $\lambda, \mu > 0$, respectively, and the parameter γ satisfies $0 < \gamma < \frac{2}{\|A\|^2}$. They obtained weak and strong convergence results of (2) and (3), respectively under some control conditions.

Alofi et al. [4] introduced the modified Halpern's iteration for solving the SCNPP (1) in the case that E_1 is a Hilbert space and E_2 is a Banach space as follows:

$$\begin{cases} x_1 \in E_1, \\ x_{n+1} = \beta_n x_n + (1 - \beta_n)(\alpha_n u_n + (1 - \alpha_n)J_{\lambda_n}(x_n - \lambda_n A^* J_E(I - Q_{\mu_n})Ax_n)), \;\; \forall n \geq 1, \end{cases} \tag{4}$$

where J_E is the duality mapping on E_2, $\{u_n\}$ is a sequence in E_1 such that $u_n \to u$, and the stepsize λ_n satisfies $0 < a \leq \lambda_n\|A\|^2 \leq b < 2$ for some $a, b > 0$. Under some suitable assumptions, they proved that the sequence $\{x_n\}$ generated by (4) converges strongly to a solution of the SCNPP.

Suantai et al. [49] also proposed the following algorithm for solving the SCNPP (1) between a Hilbert space and a Banach space:

$$\begin{cases} x_1 \in E_1, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n J_{\lambda_n}(x_n - \lambda_n A^* J_E(I - Q_{\mu_n})Ax_n), \;\; \forall n \geq 1, \end{cases} \tag{5}$$

where $f : E_1 \to E_1$ is a contraction and the stepsize λ_n satisfies $0 < a \leq \lambda_n\|A\|^2 \leq b < 2$ for some $a, b > 0$. They proved a strong convergence result of $\{x_n\}$ generated by (5) under some suitable conditions. Recently, some iterative methods have been proposed and invented independently for solving such a problem in many different contexts (see for instance [24, 51, 53, 54, 56, 57]).

However, it is observed that the choice of the stepsize of the above results and other corresponding results depend on the operator norm or the matrix norm (in the finite-dimensional space). As a result, the implementation of such algorithms are usually difficult to handle (see [23]). To overcome this difficulty, López et al. [30] suggested an algorithm so-called a *self-adaptive method* for solving the split feasibility problem (SFP) in Hilbert spaces. We note that the SFP is an interest special case of SCNPP and it is very important in nonlinear analysis. To be more precise, they proposed the following method, which permits the stepsize λ_n being selected self-adaptively in such a way

$$\lambda_n = \frac{\rho_n f(x_n)}{\|\nabla f(x_n)\|^2}, \tag{6}$$

where $\{\rho_n\} \subset (0, 4)$, $f(x_n) = \frac{1}{2}\|(I - P_Q)Ax_n\|^2$ and $\nabla f(x_n) = A^*(I - P_Q)Ax_n$ for all $n \geq 1$ (P_C and P_Q denote the metric projections on C and Q, respectively). They proposed an iterative method for solving the SFP in two Hilbert spaces as follows:

$$\begin{cases} u, x_1 \in C, \\ x_{n+1} = \alpha_n u + (1 - \alpha_n)P_C(x_n - \lambda_n \nabla f(x_n)), \;\; \forall n \geq 1, \end{cases} \tag{7}$$

where the stepsize λ_n is chosen in (6), and also proved that the sequence $\{x_n\}$ generated by (7) converges strongly to a solution of the SFP provided $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$.

On the other hand, let E be a real Banach space. We consider the fixed point problem which is the problem of finding a point

$$x^* \in E \;\text{ such that }\; x^* = Tx^*, \tag{8}$$

where T is a nonlinear mapping on E. In real life, many mathematical models have been formulated as this problem. Currently, many mathematicians are interested in finding solutions of some optimization problems with fixed point constraints (see for instance [18, 25–28, 40–43, 46, 47]).

In this paper, inspired and motivated by the above-mentioned works, we introduce a self-adaptive algorithm for finding a common solution of the SCNPP and the fixed point problem for multivalued Bregman quasi-nonexpansive mappings in the framework of Banach spaces. We prove a strong convergence theorem of the sequence generated by our proposed method under some suitable conditions as shown in Sec. 3. Furthermore, in Sec. 4, the result for solving the split feasibility problem and the fixed point problem in Banach spaces is a consequence of our main result. In the last, Sec. 5, we give some numerical examples to demonstrate the convergence behavior of our algorithm and support our main theorem. The results presented in this paper improve and extend many recent results in the literature.

2. Preliminaries

Let E be a real Banach spaces with its the dual space E^* of E. We write $\langle x, j\rangle$ for the value of a functional j in E^* at x in E. We shall use the notations $x_n \to x$ means that $\{x_n\}$ converges strongly to x and $x_n \rightharpoonup x$ means that $\{x_n\}$ converges weakly to x. Let E_1 and E_2 be real Banach spaces and let $A : E_1 \to E_2$ be a bounded linear operator with its adjoint operator $A^* : E_2^* \to E_1^*$ which is defined by

$$\langle x, A^*\bar{y}\rangle := \langle Ax, \bar{y}\rangle, \quad \forall x \in E_1, \quad \bar{y} \in E_2^*$$

and the equalities $\|A^*\| = \|A\|$ and $\mathcal{N}(A^*) = \mathcal{R}(A)^\perp$ are valid, where $\mathcal{R}(A)^\perp := \{x^* \in E_2^* : \langle u, x^*\rangle = 0, \ \forall u \in \mathcal{R}(A)\}$. For more details on bounded linear operators and their duals, please see ([21, 50]).

Let $1 < q \le 2 \le p < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. The *modulus of convexity* of E is the function $\delta_E : (0, 2] \to [0, 1]$ defined by

$$\delta_E(\epsilon) := \inf\left\{1 - \frac{\|x+y\|}{2} : \|x\| = \|y\| = 1, \|x - y\| \ge \epsilon\right\}.$$

A space E is called *uniformly convex* if $\delta_E(\epsilon) > 0$ for all $\epsilon \in (0, 2]$ and *p-uniformly convex* if there is a $c_p > 0$ such that $\delta_E(\epsilon) \ge c_p\epsilon^p$ for all $\epsilon \in (0, 2]$.

The *modulus of smoothness* of E is the function $\rho_E : \mathbb{R}^+ := [0, \infty) \to \mathbb{R}^+$ defined by

$$\rho_E(\tau) := \sup\left\{\frac{\|x+\tau y\|+\|x-\tau y\|}{2} - 1 : \|x\| = \|y\| = 1\right\}.$$

A space E is called *uniformly smooth* if $\lim_{\tau\to 0} \frac{\rho_E(\tau)}{\tau} = 0$ and *q-uniformly smooth* if there exists a $c_q > 0$ such that $\rho_E(\tau) \le c_q\tau^q$ for all $\tau > 0$. Note that every p-uniformly convex (q-uniformly smooth) space is uniformly convex (uniformly smooth) space. It is known that E is p-uniformly convex (q-uniformly smooth) if and only if its dual E^* is q-uniformly smooth (p-uniformly convex) (see [5]). Furthermore, L_p (or ℓ_p) and the Sobolev spaces are $\min\{p, 2\}$-uniformly smooth for every $p > 1$ while a Hilbert space is 2-uniformly smooth (see [58]).

Definition 2.1. A continuous strictly increasing function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be a *gauge* if $\varphi(0) = 0$ and $\lim_{t\to\infty} \varphi(t) = \infty$.

Definition 2.2. The mapping $J_\varphi : E \multimap E^*$ associated with a gauge function φ defined by

$$J_\varphi(x) := \{f \in E^* : \langle x, f\rangle = \|x\|\varphi(\|x\|), \ \|f\| = \varphi(\|x\|), \ \forall x \in E\},$$

is called the *duality mapping with gauge* φ, where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between E and E^*.

In the particular case $\varphi(t) = t$, the duality mapping $J_\varphi = J$ is called *normalized duality mapping*. In the case $\varphi(t) = t^{p-1}$, where $p > 1$, the duality mapping $J_\varphi = J_p$ is called the *generalized duality mapping* which is defined by

$$J_p(x) := \{f \in E^* : \langle x, f\rangle = \|x\|^p, \|f\| = \|x\|^{p-1}\}.$$

It follows from the definition that $J_\varphi(x) = \frac{\varphi(\|x\|)}{\|x\|}J(x)$ and $J_p(x) = \|x\|^{p-2}J(x)$, $p > 1$. It is well known that if E is uniformly smooth, the generalized duality mapping J_p is norm to norm uniformly continuous on bounded subsets of E (see [35]). Furthermore, J_p is one-to-one, single-valued and satisfies $J_p = J_q^{-1}$, where J_q is the generalized duality mapping of E^* (see [13, 36] for more details).

The following lemma can be found in [5, Theorem 2.8.17] (see also [29, Lemma 5]).

Lemma 2.3. *Let $p > 1$, $r > 0$ and let E be a uniformly convex Banach space. Then there exists a strictly, increasing and convex function $g_r : \mathbb{R}^+ \to \mathbb{R}^+$ with $g(0) = 0$ such that*

$$\|tx + (1-t)y\|^p \le t\|x\|^p + (1-t)\|y\|^p - t(1-t)g_r(\|x-y\|),$$

for all $x, y \in B_r := \{z \in E : \|z\| \le r\}$ and $t \in [0,1]$.

Lemma 2.4. [58] *Let E be a q-uniformly smooth Banach space. Then there exists a constant $c_q > 0$ which is called the q-uniform smoothness coefficient of E such that*

$$\|x-y\|^q \le \|x\|^q - q\langle y, J_q(x)\rangle + c_q\|y\|^q,$$

for all $x, y \in E$.

Let C be a nonempty, closed and convex subset of a strictly convex, smooth and reflexive Banach space E. The *metric projection* of $x \in E$ onto C is the unique element $P_C(x) \in C$ such that

$$\|x - P_C(x)\| = \min_{y\in C}\|x-y\|.$$

The metric projection can be also characterized by the following variational inequality:

$$\langle y - P_C(x), J_\varphi(x - P_C(x))\rangle \le 0, \quad \forall y \in C.$$

For a gauge φ, the function $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ defined by $\Phi(t) := \int_0^t \varphi(s)ds$ is a continuous, convex and strictly increasing differentiable function on $\mathbb{R}^+$ with $\Phi'(t) = \varphi(t)$ and $\lim_{t\to\infty}\frac{\Phi(t)}{t} = \infty$. Therefore, Φ has a continuous inverse function Φ^{-1}.

We next recall the Bregman distance, which was introduced and studied in [10].

Definition 2.5. Let E be a real smooth Banach space. The Bregman distance $D_\varphi(x,y)$ between x and y in E is defined by

$$D_\varphi(x,y) := \Phi(\|x\|) - \Phi(\|y\|) - \langle x-y, J_\varphi(y)\rangle.$$

We note that $D_\varphi(x,y) \ge 0$ and $D_\varphi(x,y) = 0$ if and only of $x = y$. Moreover, the Bregman distance has the following important properties:

$$D_\varphi(x,y) + D_\varphi(y,x) = \langle x-y, J_\varphi(x) - J_\varphi(y)\rangle, \quad \forall x, y \in E \tag{9}$$

and

$$D_\varphi(x,y) = D_\varphi(x,z) + D_\varphi(z,y) + \langle z-y, J_\varphi(x) - J_\varphi(z)\rangle, \quad \forall x, y, z \in E. \tag{10}$$

For a smooth and uniformly convex Banach space E, then there exists a strictly, increasing and convex function $g : \mathbb{R}^+ \to \mathbb{R}^+$ with $g(0) = 0$ such that

$$g(\|x-y\|) \le D_\varphi(x,y) \tag{11}$$

for all $x, y \in E$ (see [29]).

In the case $\varphi(t) = t^{p-1}$, $p > 1$, we have $\Phi(t) = \int_0^t \varphi(s)ds = \frac{t^p}{p}$. So we have the distance $D_\varphi = D_p$ is called the *p-Lyapunov function* which was studied in [12] and it is given by

$$D_p(x,y) = \frac{1}{p}\|x\|^p - \frac{1}{p}\|y\|^p - \langle x-y, J_p(y)\rangle. \tag{12}$$

It is easy to show that (12) equivalent to the following:

$$D_p(x,y) = \frac{1}{p}\|x\|^p - \langle x, J_p(y)\rangle + \frac{1}{q}\|y\|^p, \tag{13}$$

where $\frac{1}{p}+\frac{1}{q}=1$. If $p=2$, we have $D_2(x,y)=\frac{1}{2}\phi(x,y)$, where ϕ is called the *Lyapunov function* which was introduced by Alber [1].

Following [32], we make use of the function $V_p : E\times E^* \to \mathbb{R}^+$ which is defined by

$$V_p(x,\bar{x}) := \frac{1}{p}\|x\|^p - \langle x,\bar{x}\rangle + \frac{1}{q}\|\bar{x}\|^q, \ \ \forall x\in E,\ \bar{x}\in E^*. \tag{14}$$

Note that V_p is nonnegative and

$$V_p(x,\bar{x}) = D_p(x, J_q(\bar{x})), \ \ \forall x\in E,\ \bar{x}\in E^*. \tag{15}$$

By the subdifferential inequality, we have

$$V_p(x,\bar{x}) + \langle J_q(\bar{x})-x, \bar{y}\rangle \le V_p(x,\bar{x}+\bar{y}), \ \ \forall x\in E,\ \bar{x},\bar{y}\in E^*. \tag{16}$$

Moreover, V_p is convex in the second variable. Then, for all $z\in E$,

$$D_p\Big(z, J_q\Big(\sum_{i=1}^{M} t_i J_p(x_i)\Big)\Big) \le \sum_{i=1}^{M} t_i D_p(z,x_i), \tag{17}$$

where $\{x_i\}_{i=1}^M \subset E$ and $\{t_i\}_{i=1}^M \subset (0,1)$ with $\sum_{i=1}^M t_i = 1$.

Let C be a nonempty, closed and convex subset of a strictly convex, smooth and reflexive Banach space E. The *Bregman projection*, denoted by Π_C^{φ}, is defined as the unique solution of the following minimization problem:

$$\Pi_C^{\varphi}(x) := \operatorname{argmin}_{y\in C} D_{\varphi}(x,y), \ \ x\in E. \tag{18}$$

When $\varphi(t)=t$, we have Π_C^{φ} coincides with the generalized projection which studied in [1]. When $\varphi(t)=t^{p-1}$, where $p>1$, we have Π_C^{φ} becomes the Bregman projection with respect to p and denoted by Π_C.

Proposition 2.6. ([29]) *Let C be a nonempty, closed and convex subset of a strictly convex, smooth and reflexive Banach space E and let $x\in E$. Then the following assertions are equivalent:*

(i) $z=\Pi_C^{\varphi}(x)$ *if and only if* $\langle y-z, J_{\varphi}(x)-J_{\varphi}(z)\rangle \le 0$, $\forall y\in C$.

(ii) $D_{\varphi}(y,\Pi_C^{\varphi}(x)) + D_{\varphi}(\Pi_C^{\varphi}(x),x) \le D_{\varphi}(y,x)$, $\forall y\in C$.

Lemma 2.7. ([33]) *Let E be a smooth and uniformly convex real Banach space. Suppose that $\{x_n\}$ and $\{y_n\}$ are two sequences in E. Then,* $\lim_{n\to\infty} D_p(x_n,y_n)=0$ *if and only if* $\lim_{n\to\infty}\|x_n-y_n\|=0$.

Lemma 2.8. ([38]) *Let E be a smooth and uniformly convex real Banach space. Suppose that $x\in E$, if $\{D_p(x,x_n)\}$ is bounded, then the sequence $\{x_n\}$ is bounded.*

Let C be a nonempty, closed and convex subset of a Banach space E. Let $N(C)$ and $CB(C)$ denote the family of nonempty subsets and nonempty, closed and bounded subsets of C, respectively. Let $\mathcal{H}$ be the Hausdorff metric on $CB(C)$ defined by

$$\mathcal{H}(A,B) := \max\Big\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\Big\},$$

for all $A,B\in CB(C)$, where $d(a,B)=\inf_{b\in B}\{\|a-b\|\}$ is the distance from the point a to the subset B.

Let C be a nonempty subset of E and $T : C \to CB(C)$ be a multi-valued mapping. We denote the set of fixed point of T by $F(T)$, *i.e.*, $F(T) := \{x \in C : x \in Tx\}$. A point $z \in C$ is called an *asymptotic fixed point* of T, if C contains a sequence $\{x_n\}$ such that $x_n \rightharpoonup z$ and $d(x_n, Tx_n) \to 0$ as $n \to \infty$. We denote $\widehat{F}(T)$ by the set of asymptotic fixed points of T. The concept of an asymptotic fixed point was introduced by Reich [37].

We now give the definitions of some classes of Bregman multi-valued mappings.

Definition 2.9. A multivalued mapping $T : C \to CB(C)$ is said to be

(1) *φ-Bregman nonexpansive* if

$$D_\varphi(u, v) \le D_\varphi(x, y), \ \ \forall u \in Tx, \ v \in Ty \text{ and } x, y \in C,$$

(2) *φ-Bregman relatively nonexpansive* if $\widehat{F}(T) = F(T) \neq \emptyset$ and

$$D_\varphi(z, u) \le D_\varphi(z, x), \ \ \forall u \in Tx, \ x \in C \text{ and } z \in F(T),$$

(3) *φ-Bregman quasi-nonexpansive* if $F(T) \neq \emptyset$ and

$$D_\varphi(z, u) \le D_\varphi(z, x), \ \ \forall u \in Tx, \ x \in C \text{ and } z \in F(T).$$

We remark that the class of φ-Bregman quasi-nonexpansive is more general than class of φ-Bregman relatively nonexpansive mappings and φ-Bregman nonexpansive mappings with nonempty fixed point set.

Remark 2.10. *(i) In the case $\varphi(t) = t^{p-1}$, where $p > 1$, we have φ-Bregman quasi-nonexpansive, φ-Bregman relatively nonexpansive and φ-Bregman nonexpansive mappings become Bregman quasi-nonexpansive, Bregman relatively nonexpansive and Bregman nonexpansive mappings, respectively.*

(ii) In a Hilbert space H and $\varphi(t) = t$, a Bregman quasi-nonexpansive mapping and quasi-nonexpansive mapping are equivalent, for $D_2(x, y) := \|x - y\|^2$ for all $x, y \in H$, i.e.,

$$D_2(z, u) \le D_2(z, x) \Longleftrightarrow \|z - u\| \le \|z - x\|, \ \ \forall u \in Tx, x \in C \ \text{ and } \ z \in F(T).$$

Let E be a Banach space and $B : E \multimap E^*$ be a mapping. The effective domain of B is denoted by $\mathcal{D}(B)$, *i.e.*, $\mathcal{D}(B) := \{x \in E : Bx \neq \emptyset\}$ and the range of B is also denoted by $\mathcal{R}(B) := \bigcup_{x \in \mathcal{D}(B)} Bx$. A multi-valued mapping B is said to be *monotone* if

$$\langle x - y, u - v\rangle \ge 0, \ \ \forall x, y \in \mathcal{D}(B), \ u \in Bx \text{ and } v \in By. \tag{19}$$

A monotone operator B on E is said to be *maximal* if its graph is not properly contained in the graph of any other monotone operator on E.

Definition 2.11. Let E be a strictly convex, smooth and reflexive Banach space and let $B : E \multimap E^*$ be a maximal monotone operator. For $\lambda > 0$, the *φ-metric resolvent* of B is operator $Q_\lambda^\varphi : E \to \mathcal{D}(B)$ defined by

$$Q_\lambda^\varphi(x) := (I + \lambda J_\varphi^{-1} B)^{-1}(x) \ \text{ for all } \ x \in E. \tag{20}$$

The set of null points of B is defined by $B^{-1}0 := \{z \in E : 0 \in Bz\}$ and it is known that $B^{-1}0$ is closed and convex (see [50]). We see that

$$0 \in J_\varphi(Q_\lambda^\varphi(x) - x) + \lambda B Q_\lambda^\varphi(x) \tag{21}$$

and $F(Q_\lambda^\varphi) = B^{-1}0$ for $\lambda > 0$. By (21), we see that

$$\frac{J_\varphi(x - Q_\lambda^\varphi(x))}{\lambda} \in B Q_\lambda^\varphi(x) \tag{22}$$

and

$$\frac{J_\varphi(y - Q_\lambda^\varphi(y))}{\lambda} \in BQ_\lambda^\varphi(y) \tag{23}$$

for all $x, y \in E$. Adding up (22) with (23) and using the monotonicity of B, we obtain

$$\langle Q_\lambda^\varphi x - Q_\lambda^\varphi y, J_\varphi(x - Q_\lambda^\varphi x) - J_\varphi(y - Q_\lambda^\varphi y)\rangle \geq 0, \tag{24}$$

for all $x, y \in E$. It is also known that, if $B^{-1}0 \neq \emptyset$, then

$$\langle Q_\lambda^\varphi x - z, J_\varphi(x - Q_\lambda^\varphi x)\rangle \geq 0, \tag{25}$$

for all $x \in E$ and $z \in B^{-1}0$ (see [6]).

In fact, let $\{x_n\}$ be a bounded sequence in E. From (25), we have

$$\begin{aligned} \|x_n - z\|\varphi(\|x_n - Q_\lambda^\varphi x_n\|) &\geq \langle x_n - z, J_\varphi(x_n - Q_\lambda^\varphi x_n)\rangle \\ &\geq \langle x_n - Q_\lambda^\varphi x_n, J_\varphi(x_n - Q_\lambda^\varphi x_n)\rangle \\ &= \|x_n - Q_\lambda^\varphi x_n\|\varphi(\|x_n - Q_\lambda^\varphi x_n\|), \end{aligned}$$

which implies that

$$\|x_n - Q_\lambda^\varphi x_n\| \leq \|x_n - z\|,$$

for $z \in B^{-1}0$. Hence, $\{x_n - Q_\lambda^\varphi x_n\}$ is bounded. Moreover, let $x_n \to x$ as $n \to \infty$, then from (11) and (24), we have

$$\begin{aligned} &\langle x_n - x, J_\varphi(x_n - Q_\lambda^\varphi x_n) - J_\varphi(x - Q_\lambda^\varphi x)\rangle \\ \geq\ &\langle x_n - Q_\lambda^\varphi x_n - (x - Q_\lambda^\varphi x), J_\varphi(x_n - Q_\lambda^\varphi x_n) - J_\varphi(x - Q_\lambda^\varphi x)\rangle \\ =\ &D_\varphi(x_n - Q_\lambda^\varphi x_n, x - Q_\lambda^\varphi x) + D_\varphi(x - Q_\lambda^\varphi x, x_n - Q_\lambda^\varphi x_n) \\ \geq\ &g(\|x_n - Q_\lambda^\varphi x_n - (x - Q_\lambda^\varphi x)\|) + g(\|x - Q_\lambda^\varphi x - (x_n - Q_\lambda^\varphi x_n)\|) \\ =\ &2g(\|x_n - Q_\lambda^\varphi x_n - (x - Q_\lambda^\varphi x)\|). \end{aligned}$$

Since $x_n \to x$ and by the property of g, then $Q_\lambda^\varphi x_n \to Q_\lambda^\varphi x$. Hence, Q_λ^φ is continuous.

In the case $\varphi(t) = t^{p-1}$, where $p > 1$, we shall denote Q_λ^φ by $Q_\lambda := (I + \lambda J_p^{-1}B)^{-1}$.

Definition 2.12. ([29]) Let C be a nonempty, closed and convex subset of a smooth Banach space E and let $J_\varphi : E \to E^*$ be the duality mapping with gauge φ. Suppose that $B : E \multimap E^*$ is an operator satisfying the range condition

$$\mathcal{D}(B) \subset C \subset J_\varphi^{-1}\mathcal{R}(J_\varphi + rB), \tag{26}$$

where $r > 0$. For each $r > 0$, the φ-resolvent associated with operator B is the operator $R_r^\varphi : C \multimap E$ defined by

$$R_r^\varphi(x) := \{z \in E : J_\varphi(x) \in (J_\varphi + rB)z\}, \quad x \in C.$$

In addition, it is easy to show that $F(R_r^\varphi) = B^{-1}0$.

Proposition 2.13. ([29]) *Let C be a nonempty, closed and convex subset of a reflexive, strictly convex and smooth Banach space E and let $J_\varphi : E \to E^*$ be the duality mapping with gauge φ. Let $B : E \multimap E^*$ be a monotone operator satisfying* (26). *Let R_r^φ be a resolvent operator of B for $r > 0$, then $\widehat{F}(R_r^\varphi) = F(R_r^\varphi)$.*

Lemma 2.14. ([7]) *Let E be a uniformly convex and smooth Banach space. Let $B : E \multimap E^*$ be a monotone operator. Then, B is maximal if and only if for each $r > 0$,*

$$\mathcal{R}(J_\varphi + rB) = E^*,$$

where $\mathcal{R}(J_\varphi + rB)$ is the range of $J_\varphi + rB$.

Remark 2.15. *(i) If B is maximal monotone, then we see that the range condition holds for $C = \overline{\mathcal{D}(A)}$.*

(ii) By the smoothness and strict convexity of E, we obtain that $R_r^{\varphi,B}$ is single-valued. The range condition ensures that R_λ^φ is single-valued operator from C into $\overline{\mathcal{D}(A)}$. In other words,

$$R_r^\varphi(x) := (J_\varphi + rB)^{-1} J_\varphi(x), \ \ \forall x \in C.$$

For a smooth Banach space E, when $\varphi(t) = t^{p-1}$, where $p > 1$, we denote R_r^φ by $R_r := (J_p + rB)^{-1} J_p$.

Lemma 2.16. ([29]) *Let $B : E \multimap E^*$ be a maximal monotone operator with $B^{-1}0 \neq \emptyset$. Let R_r^φ be a resolvent operator of B for $r > 0$, then*

$$D_\varphi(z, R_r^\varphi x) + D_\varphi(R_r^\varphi x, x) \leq D_\varphi(z, x),$$

for all $x \in E$ and $z \in B^{-1}0$.

Lemma 2.17. ([59]) *Assume that $\{a_n\}$ is a nonnegative real sequence such that*

$$a_{n+1} \leq (1 - \gamma_n) a_n + \gamma_n \delta_n, \ \ \forall n \geq 1,$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a real sequence such that $\sum_{n=1}^\infty \gamma_n = \infty$ and $\limsup_{n\to\infty} \delta_n \leq 0$. Then, $\lim_{n\to\infty} a_n = 0$.

Lemma 2.18. ([31]) *Let $\{\Gamma_n\}$ be a real sequence that does not decrease at infinity in the sense that there exists a subsequence $\{\Gamma_{n_i}\}$ of $\{\Gamma_n\}$ which satisfies $\Gamma_{n_i} < \Gamma_{n_i+1}$ for all $i \in \mathbb{N}$. Define the sequence $\{\tau(n)\}_{n \geq n_0}$ of integers as follows:*

$$\tau(n) := \max\{k \leq n : \Gamma_k < \Gamma_{k+1}\},$$

where $n_0 \in \mathbb{N}$ such that $\{k \leq n_0 : \Gamma_k < \Gamma_{k+1}\} \neq \emptyset$. Then the following hold:

(i) $\tau(n_0) \leq \tau(n_0 + 1) \leq ...$ and $\tau(n) \to \infty$;

(ii) $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ and $\Gamma_n \leq \Gamma_{\tau(n)+1}$, $\forall n \geq n_0$.

3. Main Result

Throughout this paper, we denote by J_p^E and $J_q^{E^*}$ the duality mappings of a Banach space E and its dual space, respectively, where $1 < q \leq 2 \leq p < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. We assume that E_1 is a p-uniformly convex and uniformly smooth Banach space, E_2 is a uniformly convex and smooth Banach space, $B_1 : E_1 \multimap E_1^*$, $B_2 : E_2 \multimap E_2^*$ are two maximal monotone operators, R_r is a resolvent operator of B_1 for $r > 0$, Q_λ is a metric resolvent operator of B_2 for $\lambda > 0$, $A : E_1 \to E_2$ is a bounded linear operator with its adjoint $A^* : E_2^* \to E_1^*$, and $T : E_1 \to CB(E_1)$ is a multivalued Bregman quasi-nonexpansive mapping such that $I - T$ is demiclosed at zero. We introduce an iterative method (Algorithm 3.1) for solving the following problem:

$$\text{Find an element } x^* \in B_1^{-1}0 \cap F(T) \text{ such that } Ax^* \in B_2^{-1}0. \tag{27}$$

The solution set of the problem (27) is denoted by Ω.

Algorithm 3.1. *For $u \in E_1$, let $\{x_n\}_{n=1}^{\infty}$ be a sequence generated by $x_1 \in E_1$ and*

$$\begin{cases} y_n = R_r(J_q^{E_1^*}(J_p^{E_1}(x_n) - \lambda_n \nabla f(x_n))) \\ x_{n+1} = J_q^{E_1^*}\Big(\alpha_n J_p^{E_1}(u) + (1-\alpha_n)(\beta_n J_p^{E_1}(y_n) + (1-\beta_n)J_p^{E_1}(u_n))\Big), \ \ \forall n \geq 1, \end{cases}$$

where $u_n \in Ty_n$, $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $(0,1)$ and the stepsize λ_n is chosen in such a way that

$$\lambda_n = \begin{cases} \frac{\rho_n f^{p-1}(x_n)}{\|\nabla f(x_n)\|^p}, & \text{if } f(x_n) \neq 0; \\ 0, & \text{otherwise,} \end{cases} \tag{28}$$

where $f(x_n) = \frac{1}{p}\|(I - Q_\lambda)Ax_n\|^p$ and $\{\rho_n\} \subset \Big(0, (\frac{pq}{c_q})^{\frac{1}{q-1}}\Big)$.

Remark 3.2. *Note that the choice in* (28) *of the stepsize λ_n is independent of the norm $\|A\|$.*

Lemma 3.3. *The stepsize λ_n defined by* (28) *is well-defined.*

Proof. Since $I - Q_\lambda$ is continuous, we have $\nabla f(x) = A^* J_p^{E_2}(I - Q_\lambda)Ax$ for all $x \in E_1$ (see [22, Proposition 5.7]). Let $z \in \Omega$, *i.e.*, $z \in B_1^{-1}0$ and $Az \in B_2^{-1}0$. Then, from (25), we have

$$\begin{aligned} \|x_n - z\|\|\nabla f(x_n)\| &\geq \langle x_n - z, \nabla f(x_n)\rangle \\ &= \langle x_n - z, A^* J_p^{E_2}(I - Q_\lambda)Ax_n\rangle \\ &= \langle Ax_n - Az, J_p^{E_2}(I - Q_\lambda)Ax_n\rangle \\ &\geq \langle Ax_n - Az, J_p^{E_2}(I - Q_\lambda)Ax_n\rangle + \langle Az - Q_\lambda(Ax_n), J_p^{E_2}(I - Q_\lambda)Ax_n\rangle \\ &= \langle Ax_n - Q_\lambda(Ax_n), J_p^{E_2}(I - Q_\lambda)Ax_n\rangle \\ &= \|(I - Q_\lambda)Ax_n\|^p = pf(x_n). \end{aligned} \tag{29}$$

We see that $\|\nabla f(x_n)\| > 0$, when $f(x_n) \neq 0$. This implies that $\|\nabla f(x_n)\| \neq 0$. That is λ_n is well-defined. □

The following proposition is needed before proving our main result.

Proposition 3.4. *Let E be a uniformly convex and uniformly smooth Banach space. Let $T : E \to CB(E)$ be a multivalued Bregman quasi-nonexpansive mapping with $F(T) \neq \emptyset$. Then, $F(T)$ is closed and convex.*

Proof. First, we show that $F(T)$ is closed. Let $\{x_n\}$ be a sequence in $F(T)$, such that $x_n \to x$. Since T is a multivalued Bregman quasi-nonexpansive mapping, then for all $v \in Tx$ and by (9), we have

$$\begin{aligned} D_p(v, x_n) &\leq D_p(x, x_n) \\ &\leq \langle x - x_n, J_p^E(x) - J_p^E(x_n)\rangle \\ &\leq \|x - x_n\|\|J_p^E(x) - J_p^E(x_n)\| \to 0. \end{aligned}$$

This implies that $\lim_{n\to\infty} D_p(v, x_n) = 0$ and by Lemma 2.7, we have $\lim_{n\to\infty} \|x_n - v\| = 0$. We see that $x = v$. Hence, $x \in F(T)$, *i.e.*, $F(T)$ is closed.

Next, we show that $F(T)$ is convex. Let $x, y \in F(T)$ and $w = tx + (1-t)y$ for $t \in (0,1)$. Let $z \in Tw$, then we have

$$\begin{aligned}
D_p(w,z) &= \frac{1}{p}\|w\|^p - \frac{1}{p}\|z\|^p - \langle w - z, J_p^E(z)\rangle \\
&= \frac{1}{p}\|w\|^p - \frac{1}{p}\|z\|^p - \langle t(x-z) + (1-t)(y-z), J_p^E(z)\rangle \\
&= \frac{1}{p}\|w\|^p + tD_p(x,z) + (1-t)D_p(y,z) - t\frac{\|x\|^p}{p} - (1-t)\frac{\|y\|^p}{p} \\
&\leq \frac{1}{p}\|w\|^p + tD_p(x,w) + (1-t)D_p(y,w) - t\frac{\|x\|^p}{p} - (1-t)\frac{\|y\|^p}{p} \\
&= \frac{1}{p}\|w\|^p + t\Big(\frac{1}{p}\|x\|^p - \frac{1}{p}\|w\|^p - \langle x - w, J_p^E(w)\rangle\Big) + (1-t)\Big(\frac{1}{p}\|y\|^p - \frac{1}{p}\|w\|^p - \langle y - w, J_p^E(w)\rangle\Big) \\
&\quad -t\frac{\|x\|^p}{p} - (1-t)\frac{\|y\|^p}{p} \\
&= -\langle tx + (1-t)y - w, J_p^E(w)\rangle = 0,
\end{aligned}$$

which implies that $z = w$. Hence, $w \in F(T)$, *i.e.*, $F(T)$ is convex. Therefore, $F(T)$ is closed and convex. □

We now prove a strong convergence theorem of Algorithm 3.1, which is the main result of this paper.

Theorem 3.5. *Let $\{x_n\}$ be the sequence generated by Algorithm 3.1. Suppose that $\Omega \neq \emptyset$ and the following conditions hold:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C2) $0 < a \leq \beta_n \leq b < 1$ *for some* $a, b \in (0,1)$;

(C3) $\liminf_{n\to\infty} \rho_n\Big(p - \frac{\rho_n^{q-1} c_q}{q}\Big) > 0$.

Then, $\{x_n\}$ converges strongly to a common element $x^* = \Pi_\Omega u$, where Π_Ω is the Bregman projection from E_1 onto Ω.

Proof. Put $v_n := J_q^{E_1^*}(J_p^{E_1}(x_n) - \lambda_n \nabla f(x_n))$ for all $n \geq 1$. Since $(p-1)q = p$. Then, by (29) and Lemma 2.4, we have

$$\begin{aligned}
D_p(z, y_n) &\leq D_p(z, v_n) \\
&= D_p\Big(z, J_q^{E_1^*}(J_p^{E_1}(x_n) - \lambda_n \nabla f(x_n))\Big) \\
&= \frac{1}{p}\|z\|^p - \langle z, J_p^{E_1}(x_n)\rangle + \lambda_n\langle z, \nabla f(x_n)\rangle + \frac{1}{q}\|J_p^{E_1}(x_n) - \lambda_n \nabla f(x_n))\|^q \\
&\leq \frac{1}{p}\|z\|^p - \langle z, J_p^{E_1}(x_n)\rangle + \lambda_n\langle z, \nabla f(x_n)\rangle + \frac{1}{q}\|J_p^{E_1}(x_n)\|^q - \lambda_n\langle x_n, \nabla f(x_n)\rangle + \frac{c_q\lambda_n^q}{q}\|\nabla f(x_n)\|^q \\
&= \frac{1}{p}\|z\|^p - \langle z, J_p^{E_1}(x_n)\rangle + \frac{1}{q}\|x_n\|^p - \lambda_n\langle x_n - z, \nabla f(x_n)\rangle + \frac{c_q\lambda_n^q}{q}\|\nabla f(x_n)\|^q \\
&\leq D_p(z, x_n) - \lambda_n p f(x_n) + \frac{c_q\lambda_n^q}{q}\|\nabla f(x_n)\|^q \\
&= D_p(z, x_n) - \frac{\rho_n p f^p(x_n)}{\|\nabla f(x_n)\|^p} + \frac{\rho_n^q c_q}{q}\frac{f^p(x_n)}{\|\nabla f(x_n)\|^p} \\
&= D_p(z, x_n) - \rho_n\Big(p - \frac{\rho_n^{q-1} c_q}{q}\Big)\frac{f^p(x_n)}{\|\nabla f(x_n)\|^p}.
\end{aligned}$$

Put $z_n := J_q^{E_1^*}(\beta_n J_p^{E_1}(y_n) + (1-\beta_n)J_p^{E_1}(u_n))$ for all $n \geq 1$. From Lemmas 2.3 and 2.16, we have

$$\begin{aligned}
D_p(z, z_n) &= D_p(z, J_q^{E_1^*}(\beta_n J_p^{E_1}(y_n) + (1-\beta_n)J_p^{E_1}(u_n)) \\
&= \frac{1}{p}\|z\|^q - \beta_n\langle z, J_p^{E_1}(y_n)\rangle - (1-\beta_n)\langle z, J_p^{E_1}(u_n)\rangle + \frac{1}{q}\|\beta_n J_p^{E_1}(y_n) + (1-\beta_n)J_p^{E_1}(u_n)\|^q \\
&\leq \frac{1}{p}\|z\|^q - \beta_n\langle z, J_p^{E_1}(y_n)\rangle - (1-\beta_n)\langle z, J_p^{E_1}(u_n)\rangle \\
&\quad + \frac{1}{q}\Big[\beta_n\|J_p^{E_1}(y_n)\|^q + (1-\beta_n)\|J_p^{E_1}(u_n)\|^q - \beta_n(1-\beta_n)g_r(\|J_p^{E_1}(y_n) - J_p^{E_1}(u_n)\|)\Big] \\
&= \beta_n\Big(\frac{1}{p}\|z\|^p - \langle z, J_p^{E_1}(y_n)\rangle + \frac{1}{q}\|y_n\|^p\Big) + (1-\beta_n)\Big(\frac{1}{p}\|z\|^p - \langle z, J_p^{E_1}(u_n)\rangle + \frac{1}{q}\|u_n\|^p\Big) \\
&\quad - \frac{\beta_n(1-\beta_n)}{q}g_r(\|J_p^{E_1}(y_n) - J_p^{E_1}(u_n)\|) \\
&= \beta_n D_p(z, y_n) + (1-\beta_n)D_p(z, u_n) - \frac{\beta_n(1-\beta_n)}{q}g_r(\|J_p^{E_1}(y_n) - J_p^{E_1}(u_n)\|) \\
&\leq \beta_n D_p(z, y_n) + (1-\beta_n)D_p(z, y_n) - \frac{\beta_n(1-\beta_n)}{q}g_r(\|J_p^{E_1}(y_n) - J_p^{E_1}(u_n)\|) \\
&= D_p(z, R_r v_n) - \frac{\beta_n(1-\beta_n)}{q}g_r(\|J_p^{E_1}(y_n) - J_p^{E_1}(u_n)\|) \\
&\leq D_p(z, v_n) - D_p(R_r v_n, v_n) - \frac{\beta_n(1-\beta_n)}{q}g_r(\|J_p^{E_1}(y_n) - J_p^{E_1}(u_n)\|) \\
&\leq D_p(z, x_n) - \rho_n\Big(p - \frac{\rho_n^{q-1}c_q}{q}\Big)\frac{f^p(x_n)}{\|\nabla f(x_n)\|^p} - D_p(R_r v_n, v_n) \\
&\quad - \frac{\beta_n(1-\beta_n)}{q}g_r(\|J_p^{E_1}(y_n) - J_p^{E_1}(u_n)\|), \qquad (30)
\end{aligned}$$

which implies that

$$D_p(z, z_n) \leq D_p(z, x_n).$$

Then, it follows that

$$\begin{aligned}
D_p(z, x_{n+1}) &= D_p\Big(z, J_q^{E_1^*}(\alpha_n J_p^{E_1}(u) + (1-\alpha_n)J_p^{E_1}(z_n))\Big) \\
&\leq \alpha_n D_p(z, u) + (1-\alpha_n)D_p(z, z_n) \\
&\leq \alpha_n D_p(z, u) + (1-\alpha_n)D_p(z, x_n) \\
&\leq \max\{D_p(z, u), D_p(z, x_n)\} \\
&\vdots \\
&\leq \max\{D_p(z, u), D_p(z, x_1)\}. \qquad (31)
\end{aligned}$$

Hence, $\{D_p(z, x_n)\}$ is bounded and so $\{x_n\}$ is bounded by Lemma 2.8.

Let $x^* = \Pi_\Omega u$. Using (16) and (30), we have the following estimation:

$$\begin{aligned}
D_p(x^*, x_{n+1}) &= D_p\Big(x^*, J_q^{E_1^*}(\alpha_n J_p^{E_1}(u) + (1-\alpha_n)J_p^{E_1}(z_n))\Big)\\
&= V_p(x^*, \alpha_n J_p^{E_1}(u) + (1-\alpha_n)J_p^{E_1}(z_n))\\
&\leq V_p(x^*, \alpha_n J_p^{E_1}(u) + (1-\alpha_n)J_p^{E_1}(z_n) - \alpha_n(J_p^{E_1}(u) - J_p^{E_1}(x^*)))\\
&\quad +\alpha_n\langle x_{n+1} - x^*, J_p^{E_1}(u) - J_p^{E_1}(x^*)\rangle\\
&= V_p(x^*, \alpha_n J_p^{E_1}(x^*) + (1-\alpha_n)J_p^{E_1}(z_n)) + \alpha_n\langle x_{n+1} - x^*, J_p^{E_1}(u) - J_p^{E_1}(x^*)\rangle\\
&= D_p(x^*, J_q^{E_1^*}(\alpha_n J_p^{E_1}(x^*) + (1-\alpha_n)J_p^{E_1}(z_n))) + \alpha_n\langle x_{n+1} - x^*, J_p^{E_1}(u) - J_p^{E_1}(x^*)\rangle\\
&\leq \alpha_n D_p(x^*, x^*) + (1-\alpha_n)D_p(x^*, z_n) + \alpha_n\langle x_{n+1} - x^*, J_p^{E_1}(u) - J_p^{E_1}(x^*)\rangle\\
&\leq (1-\alpha_n)\Big[D_p(x^*, x_n) - \rho_n\Big(p - \frac{\rho_n^{q-1}c_q}{q}\Big)\frac{f^p(x_n)}{\|\nabla f(x_n)\|^p} - D_p(R_r v_n, v_n)\\
&\quad -\frac{\beta_n(1-\beta_n)}{q}g_r(\|J_p^{E_1}(y_n) - J_p^{E_1}(u_n)\|)\Big] + \alpha_n\langle x_{n+1} - x^*, J_p^{E_1}(u) - J_p^{E_1}(x^*)\rangle\\
&= (1-\alpha_n)D_p(x^*, x_n) - (1-\alpha_n)\rho_n\Big(p - \frac{\rho_n^{q-1}c_q}{q}\Big)\frac{f^p(x_n)}{\|\nabla f(x_n)\|^p} - (1-\alpha_n)D_p(R_r v_n, v_n)\\
&\quad -\frac{(1-\alpha_n)\beta_n(1-\beta_n)}{q}g_r(\|J_p^{E_1}(y_n) - J_p^{E_1}(u_n)\|) + \alpha_n\langle x_{n+1} - x^*, J_p^{E_1}(u) - J_p^{E_1}(x^*)\rangle. \qquad (32)
\end{aligned}$$

For each $n \geq 1$, we set

$$\begin{aligned}
\Gamma_n &:= D_p(x^*, x_n),\\
\eta_n &:= (1-\alpha_n)\rho_n\Big(p - \frac{\rho_n^{q-1}c_q}{q}\Big)\frac{f^p(x_n)}{\|\nabla f(x_n)\|^p} + (1-\alpha_n)D_p(R_r v_n, v_n)\\
&\quad +\frac{(1-\alpha_n)\beta_n(1-\beta_n)}{q}g_r(\|J_p^{E_1}(y_n) - J_p^{E_1}(u_n)\|),\\
\delta_n &:= \alpha_n\langle x_{n+1} - x^*, J_p^{E_1}(u) - J_p^{E_1}(x^*)\rangle.
\end{aligned}$$

Then (32) reduces to the following formulae:

$$\Gamma_{n+1} \leq (1-\alpha_n)\Gamma_n - \eta_n + \delta_n, \ \ \forall n \geq 1 \qquad (33)$$

and

$$\Gamma_{n+1} \leq (1-\alpha_n)\Gamma_n + \delta_n, \ \ \forall n \geq 1. \qquad (34)$$

We now show that $\Gamma_n \to 0$ as $n \to \infty$ by considering two possible cases:

Case 1. Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{\Gamma_n\}_{n=n_0}^\infty$ is non-increasing. This implies that $\{\Gamma_n\}_{n=1}^\infty$ is convergent. From (33), we have

$$\eta_n \leq \Gamma_n - \Gamma_{n+1} + \delta_n - \alpha_n\Gamma_n. \qquad (35)$$

Since $\alpha_n \to 0$, $\liminf_{n\to\infty} \rho_n\Big(p - \frac{\rho_n^{q-1}c_q}{q}\Big) > 0$ and $\beta_n \in (a, b)$. This implies that $\lim_{n\to\infty} \eta_n = 0$. Then, we have

$$\frac{f^p(x_n)}{\|\nabla f(x_n)\|^p} \to 0, \ \ D_p(R_r v_n, v_n) \to 0 \ \text{ and } \ g_r(\|J_p^{E_1}(y_n) - J_p^{E_1}(u_n)\|) \to 0. \qquad (36)$$

Since $\{\|\nabla f(x_n)\|\}$ is bounded, there exists $M > 0$ such that $\|\nabla f(x_n)\| \leq M$. Thus we have

$$\frac{f^p(x_n)}{M^p} \leq \frac{f^p(x_n)}{\|\nabla f(x_n)\|^p} \to 0 \ \text{ as } \ n \to \infty.$$

This implies that

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} \|(I - Q_\lambda)Ax_n\| = 0. \tag{37}$$

Moreover, we have

$$\lim_{n\to\infty} \|R_r v_n - v_n\| = 0 \tag{38}$$

and

$$\lim_{n\to\infty} \|J_p^{E_1}(y_n) - J_p^{E_1}(u_n)\| = 0. \tag{39}$$

It follows from (39) that

$$\|J_p^{E_1}(z_n) - J_p^{E_1}(y_n)\| = (1-\beta_n)\|J_p^{E_1}(u_n) - J_p^{E_1}(y_n)\| \to 0 \ \text{ as } \ n \to \infty.$$

Since $J_q^{E_1^*}$ is norm-to-norm uniformly continuous on bounded subsets of E_1^*, we have

$$\lim_{n\to\infty} \|y_n - u_n\| = 0 \tag{40}$$

and

$$\lim_{n\to\infty} \|z_n - y_n\| = 0. \tag{41}$$

From (37), we see that

$$\|J_p^{E_1}(v_n) - J_p^{E_1}(x_n)\| = \frac{\rho_n f^{p-1}(x_n)}{\|\nabla f(x_n)\|^p}\|\nabla f(x_n)\| = \frac{\rho_n f^{p-1}(x_n)}{\|\nabla f(x_n)\|^{p-1}} \to 0 \ \text{ as } \ n \to \infty.$$

So we have

$$\lim_{n\to\infty} \|v_n - x_n\| = 0. \tag{42}$$

From (38) and (42), we have

$$\|y_n - x_n\| \le \|y_n - v_n\| + \|v_n - x_n\| \to 0 \ \text{ as } \ n \to \infty. \tag{43}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \hat{x} \in E_1$ as $k \to \infty$ and

$$\limsup_{n\to\infty} \langle x_n - x^*, J_p^{E_1}(u) - J_p^{E_1}(x^*)\rangle = \lim_{k\to\infty} \langle x_{n_k} - x^*, J_p^{E_1}(u) - J_p^{E_1}(x^*)\rangle, \tag{44}$$

where $x^* = \Pi_\Omega u$. Since $\|y_n - x_n\| \to 0$ as $n \to \infty$, we also have $y_{n_k} \rightharpoonup \hat{x}$. So, by (40), we see that

$$d(y_n, Ty_n) \le \|y_n - u_n\| \to 0 \ \text{ as } \ n \to \infty \tag{45}$$

and by the demiclosedness of $I - T$ at zero, we get $\hat{x} \in F(T)$. Since $x_{n_k} \rightharpoonup \hat{x}$ and by (42), we also get $v_{n_k} \rightharpoonup \hat{x}$. Then from (38), we get $\hat{x} \in F(R_r)$. From (24) and (37), we see that

$$\begin{aligned}
& \langle Q_\lambda Ax_n - Q_\lambda A\hat{x}, J_p^{E_2}(I - Q_\lambda)A\hat{x}\rangle \\
\le\ & \langle Q_\lambda Ax_n - Q_\lambda A\hat{x}, J_p^{E_2}(I - Q_\lambda)Ax_n\rangle \\
\le\ & \|Q_\lambda Ax_n - Q_\lambda A\hat{x}\|\|(I - Q_\lambda)Ax_n\|^{p-1} \to 0 \ \text{ as } \ n \to \infty.
\end{aligned} \tag{46}$$

Moreover, we have

$$\begin{aligned}
\|(I - Q_\lambda)A\hat{x}\|^p &= \langle (I - Q_\lambda)A\hat{x}, J_p^{E_2}(I - Q_\lambda)A\hat{x}\rangle \\
&= \langle A\hat{x} - Ax_{n_k}, J_p^{E_2}(I - Q_\lambda)A\hat{x}\rangle + \langle Ax_{n_k} - Q_\lambda Ax_{n_k}, J_p^{E_2}(I - Q_\lambda)A\hat{x}\rangle \\
&\quad + \langle Q_\lambda Ax_{n_k} - Q_\lambda A\hat{x}, J_p^{E_2}(I - Q_\lambda)A\hat{x}\rangle.
\end{aligned} \tag{47}$$

Since A is continuous, we have $Ax_{n_k} \rightharpoonup A\hat{x}$ as $k \to \infty$, by (37) and (46), we have

$$\|A\hat{x} - Q_\lambda A\hat{x}\| = 0.$$

This shows that $A\hat{x} \in F(Q_\lambda)$. Hence $\hat{x} \in F(T) \cap B_1^{-1}0 \cap A^{-1}(B_2^{-1}0) = \Omega$. Note that

$$\begin{aligned} D_p(z_n, x_{n+1}) &= D_p(z_n, J_q^{E_1^*}\big(\alpha_n J_p^{E_1}(u) + (1-\alpha_n) J_p^{E_1}(z_n)\big)) \\ &\leq \alpha_n D_p(z_n, u) + (1-\alpha_n) D_p(z_n, z_n) \to 0 \text{ as } n \to \infty \end{aligned}$$

and so

$$\|x_{n+1} - z_n\| \to 0 \text{ as } n \to \infty. \tag{48}$$

It follows from (41), (43) and (48) that

$$\|x_{n+1} - x_n\| \leq \|x_{n+1} - z_n\| + \|z_n - y_n\| + \|y_n - x_n\| \to 0 \text{ as } n \to \infty. \tag{49}$$

Then by (44) and Proposition 2.6, we obtain

$$\limsup_{n\to\infty} \langle x_{n+1} - x^*, J_p^{E_1}(u) - J_p^{E_1}(x^*)\rangle \leq 0. \tag{50}$$

This together with (34) and (50), we conclude by Lemma 2.17 that $\Gamma_n \to 0$ as $n \to \infty$. Hence, $x_n \to x^*$ as $n \to \infty$.

Case 2. Suppose that there exists a subsequence $\{\Gamma_{n_i}\}$ of $\{\Gamma_n\}$ such that $\Gamma_{n_i} < \Gamma_{n_i+1}$ for all $i \in \mathbb{N}$. Let us define a mapping $\tau : \mathbb{N} \to \mathbb{N}$ by

$$\tau(n) := \max\{k \leq n : \Gamma_k < \Gamma_{k+1}\}.$$

Then by Lemma 2.18, we have

$$\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1} \text{ and } \Gamma_n \leq \Gamma_{\tau(n)+1}.$$

Put $\Gamma_n := D_p(x_n, x^*)$ for all $n \in \mathbb{N}$. From (31), we have

$$\begin{aligned} 0 &\leq \lim_{n\to\infty} (D_p(x^*, x_{\tau(n)+1}) - D_p(x^*, x_{\tau(n)})) \\ &\leq \lim_{n\to\infty} (D_p(x^*, u) + (1-\alpha_{\tau(n)}) D_p(x^*, x_{\tau(n)}) - D_p(x^*, x_{\tau(n)})) \\ &= \lim_{n\to\infty} \alpha_{\tau(n)}\big(D_p(x^*, u) - D_p(x^*, x_{\tau(n)})\big) = 0, \end{aligned}$$

which implies that

$$\lim_{n\to\infty} (D_p(x^*, x_{\tau(n)+1}) - D_p(x^*, x_{\tau(n)})) = 0. \tag{51}$$

Following the proof line in **Case 1**, we can show that

$$\limsup_{n\to\infty} \langle x_{\tau(n)+1} - x^*, J_p^{E_1}(u) - J_p^{E_1}(x^*)\rangle \leq 0.$$

Since $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ and $\alpha_{\tau(n)} > 0$, by (34), we have

$$D_p(x_n, x^*) \leq \langle x_{\tau(n)+1} - x^*, J_p^{E_1}(u) - J_p^{E_1}(x^*)\rangle.$$

Thus we have

$$\limsup_{n\to\infty} D_p(x^*, x_{\tau(n)}) \leq 0$$

and so

$$\lim_{n\to\infty} D_p(x^*, x_{\tau(n)}) = 0.$$

Since $\Gamma_n \leq \Gamma_{\tau(n)+1}$. Then from (51), we have

$$D_p(x^*, x_n) \leq D_p(x^*, x_{\tau(n)+1}) = D_p(x^*, x_{\tau(n)}) + (D_p(x^*, x_{\tau(n)+1}) - D_p(x^*, x_{\tau(n)})) \to 0 \text{ as } n \to \infty.$$

Therefore, $x_n \to x^*$ as $n \to \infty$. We thus complete the proof. $\square$

Corollary 3.6. *Let E_1 be a p-uniformly convex and uniformly smooth Banach space and E_2 a uniformly convex and smooth Banach space. Let $B_1 : E_1 \multimap E_1^*$ and $B_2 : E_2 \multimap E_2^*$ be two maximal monotone operators such that R_r is a resolvent operator of B_1 for $r > 0$ and Q_λ is a metric resolvent operator of B_2 for $\lambda > 0$. Let $A : E_1 \to E_2$ be a bounded linear operator with its adjoint $A^* : E_2^* \to E_1^*$ and let $T : E_1 \to CB(E_1)$ be a multivalued Bregman relatively nonexpansive mapping. Suppose that $\Omega \neq \emptyset$. For $u \in E_1$, let $\{x_n\}$ be the sequence generated by $x_1 \in E_1$ and*

$$\begin{cases} y_n = R_r(J_q^{E_1^*}(J_p^{E_1}(x_n) - \lambda_n \nabla f(x_n))) \\ x_{n+1} = J_q^{E_1^*}\Big(\alpha_n J_p^{E_1}(u) + (1-\alpha_n)(\beta_n J_p^{E_1}(y_n) + (1-\beta_n)J_p^{E_1}(u_n))\Big), \quad \forall n \geq 1, \end{cases}$$

where $u_n \in Ty_n$, $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $(0,1)$ and the stepsize λ_n is chosen in such a way that

$$\lambda_n = \begin{cases} \frac{\rho_n f^{p-1}(x_n)}{\|\nabla f(x_n)\|^p}, & \text{if } f(x_n) \neq 0; \\ 0, & \text{otherwise}, \end{cases} \tag{52}$$

where $f(x_n) = \frac{1}{p}\|(I - Q_\lambda)Ax_n\|^p$ and $\{\rho_n\} \subset \Big(0, (\frac{pq}{c_q})^{\frac{1}{q-1}}\Big)$. Suppose that the following conditions hold:

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*

(C2) $0 < a \leq \beta_n \leq b < 1$ *for some* $a, b \in (0,1)$*;*

(C3) $\liminf_{n\to\infty} \rho_n\Big(p - \frac{\rho_n^{q-1} c_q}{q}\Big) > 0.$

Then, $\{x_n\}$ converges strongly to a common element $x^* = \Pi_\Omega u$, where Π_Ω is the Bregman projection from E_1 onto Ω.

If we take $T = I$ is a single-valued mapping in Theorem 3.5, then we obtain the following result.

Corollary 3.7. *Let E_1 be a p-uniformly convex and uniformly smooth Banach space and E_2 a uniformly convex and smooth Banach space. Let $B_1 : E_1 \multimap E_1^*$ and $B_2 : E_2 \multimap E_2^*$ be two maximal monotone operators such that R_r is a resolvent operator of B_1 for $r > 0$ and Q_λ is a metric resolvent operator of B_2 for $\lambda > 0$. Let $A : E_1 \to E_2$ be a bounded linear operator with its adjoint $A^* : E_2^* \to E_1^*$. Suppose that $\Lambda := \{x \in B_1^{-1}0 : Ax \in B_2^{-1}0\} \neq \emptyset$. For $u \in E_1$, let $\{x_n\}$ be the sequence generated by $x_1 \in E_1$ and*

$$\begin{cases} y_n = J_q^{E_1^*}(J_p^{E_1}(x_n) - \lambda_n \nabla f(x_n)) \\ x_{n+1} = J_q^{E_1^*}\Big(\alpha_n J_p^{E_1}(u) + (1-\alpha_n)J_p^{E_1}(R_r y_n)\Big), \quad \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\}$ is a sequences in $(0,1)$ and the stepsize λ_n is chosen in such a way that

$$\lambda_n = \begin{cases} \frac{\rho_n f^{p-1}(x_n)}{\|\nabla f(x_n)\|^p}, & \text{if } f(x_n) \neq 0; \\ 0, & \text{otherwise}, \end{cases} \tag{53}$$

where $f(x_n) = \frac{1}{p}\|(I - Q_\lambda)Ax_n\|^p$ and $\{\rho_n\} \subset \Big(0, (\frac{pq}{c_q})^{\frac{1}{q-1}}\Big)$. Suppose that the following conditions hold:

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*

(C2) $\liminf_{n\to\infty} \rho_n\Big(p - \frac{\rho_n^{q-1} c_q}{q}\Big) > 0.$

Then, $\{x_n\}$ converges strongly to an element $x^* = \Pi_\Lambda u$, where Π_Λ is the Bregman projection from E_1 onto Λ.

In addition, we consequently obtain the following result in Hilbert spaces.

Corollary 3.8. *Let H_1 and H_2 be two Hilbert spaces. Let $B_1 : H_1 \multimap H_1$ and $B_2 : H_2 \multimap H_2$ be two maximal monotone operators such that R_r and Q_λ are resolvent operators of B_1 for $r > 0$ and B_2 for $\lambda > 0$, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator with its adjoint $A^* : H_2 \to H_1$ and let $T : H_1 \to CB(H_1)$ be a multivalued quasi-nonexpansive mapping such that $I - T$ is demiclosed at zero. Suppose that $\Omega \neq \emptyset$. For $u \in H_1$, let $\{x_n\}$ be the sequence generated by $x_1 \in H_1$ and*

$$\begin{cases} y_n = R_r(x_n - \lambda_n \nabla f(x_n)) \\ x_{n+1} = \alpha_n u + (1-\alpha_n)(\beta_n y_n + (1-\beta_n)u_n), \quad \forall n \geq 1, \end{cases}$$

where $u_n \in Ty_n$, $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $(0,1)$ and the stepsize λ_n is chosen in such a way that

$$\lambda_n = \begin{cases} \frac{\rho_n f(x_n)}{\|\nabla f(x_n)\|^2}, & \text{if } f(x_n) \neq 0; \\ 0, & \text{otherwise}, \end{cases} \tag{54}$$

where $f(x_n) = \frac{1}{2}\|(I - Q_\lambda)Ax_n\|^2$ and $\{\rho_n\} \subset (0,4)$. Suppose that the following conditions hold:

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C2) $0 < a \leq \beta_n \leq b < 1$ *for some* $a, b \in (0,1)$;

(C3) $\liminf_{n\to\infty} \rho_n(4 - \rho_n) > 0$.

Then, $\{x_n\}$ converges strongly to a common element $x^* = P_\Omega u$, where P_Ω is the metric projection from H_1 onto Ω.

4. Application to Split Feasibility Problems

Let E_1 and E_2 be p-uniformly convex and uniformly smooth Banach spaces. Let C and Q be nonempty, closed and convex subsets of E_1 and E_2, respectively. Let $A : E_1 \to E_2$ be a bounded linear operator with its adjoint A^*. The *split feasibility problem* (SFP) is formulated as finding an element

$$x^* \in C \text{ such that } Ax^* \in Q. \tag{55}$$

We denote by $\Gamma := \{x \in C : Ax \in Q\} = C \cap A^{-1}(Q)$ the set of solutions of the SFP. This problem was first introduced, in a finite dimensional Hilbert space, by Censor-Elfving [15] for modeling inverse problems which arise from phase retrieval and in medical image reconstruction. Moreover, the SFP has applications in signal processing, in image recovery, in radiation therapy, in data denoising and in data compression (see for instance [8, 9, 19, 20]).

In order to solve the SFP in Banach spaces, Schöpfer et al. [48] first introduced the following algorithm: for $x_1 \in E_1$ and

$$x_{n+1} = \Pi_C J_{E_1}^*(J_{E_1}(x_n) - \lambda_n A^* J_{E_2}(Ax_n - P_Q(Ax_n))), \quad \forall n \geq 1, \tag{56}$$

where $\{\lambda_n\}$ is a positive sequence, Π_C denotes the generalized projection on E_1, P_Q is the metric projection on E_2, J_{E_1} is the duality mapping on E_1 and $J_{E_1}^*$ is the duality mapping on E_1^*. It was proved that the sequence $\{x_n\}$ converges weakly to a solution of the SFP under some mild conditions.

To obtain a strong convergence theorem, Shehu [39] introduced the following iterative algorithm for solving the SFP in p-uniformly convex and uniformly smooth Banach spaces: for $u, x_1 \in E$ and

$$\begin{cases} y_n = J_q^{E_1^*}(J_p^{E_1}(x_n) - \lambda_n A^* J_p^{E_2}(I - P_Q)Ax_n), \\ x_{n+1} = \Pi_C J_q^{E_1^*}(\alpha_n J_p^{E_1}(u) + (1-\alpha_n)J_p^{E_1}(y_n)), \quad \forall n \geq 1, \end{cases} \tag{57}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$ and the stepsize λ_n satisfies $0 < a \le \lambda_n \le b < \left(\frac{q}{\kappa_q \|A\|^q}\right)^{\frac{1}{q-1}}$ for some $a, b > 0$. Under suitable assumptions, he proved that the sequence $\{x_n\}$ generated by (57) converges strongly to a solution of the SFP.

Let C be a closed and convex subset of a strictly convex, smooth and reflexive Banach space E. Recall that the *indicator function* of C given by

$$i_C(x) := \begin{cases} 0, & \text{if } x \in C; \\ \infty, & \text{if } x \notin C. \end{cases} \tag{58}$$

It is known that i_C is proper convex, lower semicontinuous and convex function with its subdifferential ∂i_C is maximal monotone (see [34]). From [5], we know that

$$\partial i_C(z) = N_C(z) := \{u \in E^* : \langle y - z, u\rangle \le 0, \ \forall y \in C\}, \tag{59}$$

where N_C is the normal cone for C at a point $z \in C$. Thus, we can define the resolvent R_r of ∂i_C for $r > 0$ by

$$R_r(x) := (J_p + r\partial i_C)^{-1} J_p(x), \ \ \forall x \in E.$$

So we have for any $x \in E$ and $z \in C$,

$$\begin{aligned} z = R_r(x) \ &\Leftrightarrow \ J_p(x) \in J_p(z) + rN_C(z) \\ &\Leftrightarrow \ J_p(x) - J_p(z) \in rN_C(z) \\ &\Leftrightarrow \ \langle y - z, J_p(x) - J_p(z)\rangle \le 0, \ \ \forall y \in C \\ &\Leftrightarrow \ z = \Pi_C(x), \end{aligned}$$

where Π_C is the Bregman projection from E onto C. Moreover, we can define the metric resolvent Q_λ of ∂i_C for $\lambda > 0$ by

$$Q_\lambda(x) := (I + \lambda J_p^{-1} \partial i_C)^{-1}(x), \ \ \forall x \in E.$$

So we have for any $x \in E$ and $z \in C$,

$$\begin{aligned} z = Q_\lambda(x) \ &\Leftrightarrow \ x \in z + \lambda J_p^{-1} N_C(z) \\ &\Leftrightarrow \ x - z \in \lambda J_p^{-1} N_C(z) \\ &\Leftrightarrow \ J_\varphi(x - z) \in N_C(z) \\ &\Leftrightarrow \ \langle y - z, J_p(x - z)\rangle, \ \ \forall y \in C \\ &\Leftrightarrow \ z = P_C(x), \end{aligned}$$

where P_C is the metric projection from E onto C.

In fact, we set $B_1 := \partial i_C$ and $B_2 := \partial i_Q$, then $R_r = \Pi_C$ and $Q_\lambda = P_Q$ for $\lambda_1, \lambda_2 > 0$. We also have $F(R_r) = B_1^{-1}0 = C$ and $F(Q_\lambda) = B_2^{-1}0 = Q$. So we obtain the following result.

Theorem 4.1. *Let E_1 and E_2 be p-uniformly convex and uniformly smooth Banach spaces. Let C and Q be nonempty, closed and convex subsets of E_1 and E_2, respectively. Let $A : E_1 \to E_2$ be a bounded linear operator with its adjoint $A^* : E_2^* \to E_1^*$ and let $T : C \to CB(C)$ be a multivalued Bregman quasi-nonexpansive mapping such that $I - T$ is demiclosed at zero. Suppose that $\Theta := F(T) \cap \Gamma \neq \emptyset$. For $u \in C$, let $\{x_n\}$ be the sequence generated by $x_1 \in C$ and*

$$\begin{cases} y_n = \Pi_C(J_q^{E_1^*}(J_p^{E_1}(x_n) - \lambda_n \nabla f(x_n))), \\ x_{n+1} = J_q^{E_1^*}\Big(\alpha_n J_p^{E_1}(u) + (1-\alpha_n)(\beta_n J_p^{E_1}(y_n) + (1-\beta_n) J_p^{E_1}(u_n))\Big), \ \ \forall n \ge 1, \end{cases}$$

where $u_n \in Ty_n$, $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $(0,1)$ and the stepsize λ_n is chosen in such a way that

$$\lambda_n = \begin{cases} \frac{\rho_n f^{p-1}(x_n)}{\|\nabla f(x_n)\|^p}, & \text{if } f(x_n) \neq 0; \\ 0, & \text{otherwise}, \end{cases} \tag{60}$$

where $f(x_n) = \frac{1}{p}\|(I - P_Q)Ax_n\|^p$ and $\{\rho_n\} \subset \left(0, (\frac{pq}{c_q})^{\frac{1}{q-1}}\right)$. Suppose that the following conditions hold:

(C1) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$;

(C2) $0 < a \le \beta_n \le b < 1$ *for some* $a, b \in (0,1)$;

(C3) $\liminf_{n\to\infty}\rho_n\Big(p - \frac{\rho_n^{q-1}c_q}{q}\Big) > 0$.

Then, $\{x_n\}$ *converges strongly to a common element* $x^* = \Pi_\Theta u$, *where* Π_Θ *is the Bregman projection from* E_1 *onto* Θ.

5. Numerical Results

In this section, we first give a numerical example to demonstrate the performance of Algorithm 3.1.

Example 5.1. Let $E_1 = \mathbb{R}$ and $E_2 = \mathbb{R}^3$ with the usual norms. Define a multi-valued mapping $T : \mathbb{R} \to CB(\mathbb{R})$ by

$$Tx := \begin{cases} \Big[0, \Big|\frac{5}{6}x\sin\big(\frac{1}{x}\big)\Big|\Big], & \text{if } x \neq 0, \\ \{0\}, & \text{if } x = 0. \end{cases}$$

One can show that T is (Bregman) quasi-nonexpansive and it also satisfies the demiclosedness principle. Define a multi-valued mapping $B_1 : \mathbb{R} \multimap \mathbb{R}$ by

$$B_1(x) := \begin{cases} \Big\{y \in \mathbb{R} : z^2 + xz - 2x^2 \ge (z - x)y,\ \forall z \in [-9,3]\Big\}, & x \in [-9,3], \\ \emptyset, & \text{otherwise.} \end{cases}$$

By [55, Theorem 4.2], B_1 is a maximal monotone operator. Let $g : \mathbb{R}^3 \to \mathbb{R}$ be a function defined by $g(z_1, z_2, z_3) = \frac{1}{2}|5z_1 - 3z_2 + 2z_3|^2$. Let $B_2 : \mathbb{R}^3 \multimap \mathbb{R}^3$ be a subdifferential of g, that is,

$$B_2(x) = \partial g(x) := \Big\{y \in \mathbb{R}^3 : \langle y, z - x\rangle \le g(z) - g(x),\ \forall z \in \mathbb{R}^3\Big\}.$$

Since g is a proper, lower semicontinuous and convex function, then B_2 is a maximal monotone operator (see [34]). The explicit forms of the resolvent operators of B_1 and B_2 can be written by $R_r(x) = \frac{x}{4}$ and $Q_\lambda = M^{-1}$, where

$$M = \begin{pmatrix} 26 & -15 & 10 \\ -15 & 10 & -6 \\ 10 & -6 & 5 \end{pmatrix}$$

(see [17, 44, 55] for more details). Next, define a bounded linear operator $A : \mathbb{R} \to \mathbb{R}^3$ by $Ax := (-8x, -3x, x)$ and let $\Omega := F(T) \cap B_1^{-1}0 \cap A^{-1}(B_2^{-1}0)$.

Take $\alpha_n = \frac{1}{8500n}$, $\beta_n = \frac{n}{2n+1}$, $\rho_n = \frac{2n}{n+1}$, $r = \lambda = 1$ and $u = \frac{1}{2}$. If $y_n \neq 0$, then we choose $u_n = \Big|\frac{5}{12}y_n\sin\big(\frac{1}{y_n}\big)\Big|$; otherwise, $u_n = 0$. Now, Algorithm 3.1 becomes

$$\begin{cases} y_n = \frac{1}{4}\Big(x_n - \lambda_n A^\top\big(I - M^{-1}\big)Ax_n\Big) \\ x_{n+1} = \frac{1}{2(8500n)} + \Big(1 - \frac{1}{8500n}\Big)\Big(\frac{n}{2n+1}y_n + \frac{n+1}{2n+1}u_n\Big), \ \forall n \ge 1, \end{cases} \tag{61}$$

where

$$\lambda_n = \begin{cases} \frac{n}{n+1}\frac{\|(I-M^{-1})Ax_n\|^2}{\|A^\top(I-M^{-1})Ax_n\|^2}, & \text{if } Ax_n \neq M^{-1}(Ax_n), \\ 0, & \text{otherwise.} \end{cases}$$

Let us start with the initial point $x_1 = 10$ and the stopping criterion for our testing method is set as: $E_n := |x_{n+1} - x_n| < 10^{-7}$. Now, we show the numerical experiment of the method (61) and plot the number of iterations n against E_n as seen in Table 1 and Figure 1. It is observed that our algorithm converges to a solution, i.e., $x_n \to 0 \in \Omega$.

n	y_n	x_{n+1}	E_n
1	1.2820513	0.6777851	9.3222149
2	0.0593786	0.0372240	0.6405611
3	0.0025055	0.0011798	0.0360442
4	0.0000650	0.0000475	0.0011323
5	0.0000022	0.0000131	0.0000344
6	0.0000005	0.0000101	0.0000029
7	0.0000004	0.0000087	0.0000015
8	0.0000003	0.0000075	0.0000012
9	0.0000002	0.0000067	0.0000008
$\vdots$	$\vdots$	$\vdots$	$\vdots$
25	3.909E-08	0.0000024	1.028E-07
26	3.665E-08	0.0000023	8.833E-08

Table 1: Numerical experiment of the iterative method (61)

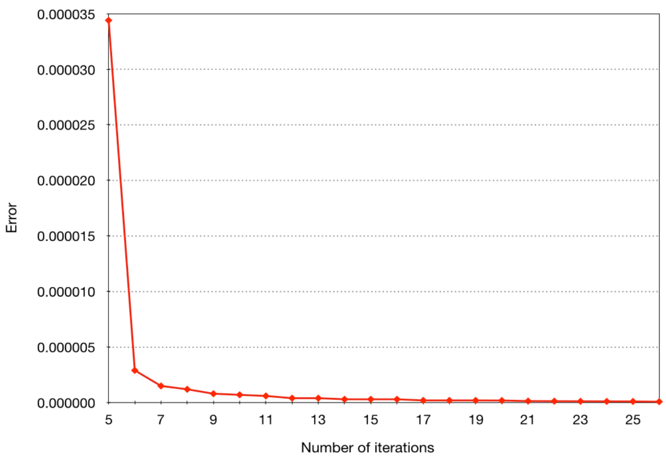

Figure 1: A gragh of error of the iterative method (61)

Finally, we give an example established in the infinite-dimensional space L_p but not a Hilbert space for supporting Theorem 3.5.

Example 5.2. For $p > 2$, let $E_1 = E_2 = L_p([\alpha, \beta])$. From [3], we have the duality mapping of E_1 is the function $J_p^{E_1} : L_p([\alpha, \beta]) \to L_q([\alpha, \beta])$ given by $J_p^{E_1}(x) = |x|^{p-2} \cdot x$ and the Bregman function $D(\cdot, \cdot)$ given by

$$D_p(x, y) = \frac{||x||^p}{p} + \frac{||y||^p}{q} - \langle x, |y|^{p-2} \cdot y \rangle.$$

Consider a hyperplane C of $L_p([\alpha, \beta])$

$$C := \{x \in L_p([\alpha, \beta]) : \langle a, x \rangle = b\},$$

where $a(t) \in L_q([\alpha, \beta])$, $b \in \mathbb{R}$ and $t \in [\alpha, \beta]$. Let $B_1 = \partial i_C$, where ∂i_C is the subdifferential of the indicator function of C. Then the resolvent operator R_r of B_1 becomes the Bregman projection operator Π_C given by [2]

$$\Pi_C(x) = \begin{cases} u_k, & \text{if } x \notin C; \\ x, & \text{if } x \in C, \end{cases}$$

where $u_k \in L_p([\alpha,\beta])$ is a solution of the problem: find $k \in \mathbb{R}$ such that $\langle a, u_k\rangle = b$ and

$$u_k := |k\cdot a + |x|^{p-2}\cdot x|^{q-2}\cdot (k\cdot a + |x|^{p-2}\cdot x).$$

Let a closed ball centered at $v \in L_p([\alpha,\beta])$ and radius $d > 0$ be defined by

$$Q := \{x \in L_p([\alpha,\beta]) : \|x - v\| \leq d\}.$$

Let $B_2 = \partial i_Q$, where ∂i_Q is the subdifferential of the indicator function of Q. Then the metric resolvent operator Q_λ of B_2 becomes the metric projection operator P_Q given by

$$P_Q(x) = \begin{cases} v + d\frac{x-v}{\|x-v\|}, & \text{if } x \notin Q; \\ x, & \text{if } x \in Q. \end{cases}$$

Let $\{\rho_n\}$ be a sequence in $\left(0, (\frac{pq}{c_q})^{\frac{1}{q-1}}\right)$ such that $\liminf_{n\to\infty} \rho_n\left(p - \frac{\rho_n^{q-1}c_q}{q}\right) > 0$, where $c_q = (1+t_q^{q-1})(1+t_q)^{1-q}$ and t_q is the unique solution of the equation $(q-2)t^{q-1} + (q-1)t^{q-2} - 1 = 0,\ 0 < t < 1$ (see [58]). In particular, we consider the following SFP and the fixed point problem:

$$\text{Find } x^* \in C \text{ such that } Ax^* \in Q \text{ and } x^* \in Tx^*$$

with its solution set $\Theta := \Gamma \cap F(T)$. Let

$$C = \{x \in L_3([0,1]) : \langle 1, x\rangle = 0\}$$

and

$$Q = \{x \in L_3([0,1]) : \|x\| \leq 1\}.$$

Let $A : L_3([0,1]) \to L_3([0,1])$ be defined by $(Ax)(t) = \frac{x(t)}{2}, \forall x \in L_3([0,1])$. We see that A is bounded and linear with $A^* = A$. Let $T : C \to CB(C)$ be defined by

$$Tx := \begin{cases} \{y \in C : x - \frac{1}{2} \leq y \leq x - \frac{1}{4}\}, & \text{if } x > 1; \\ \{0\}, & \text{otherwise.} \end{cases}$$

It is shown in [45] that T is a multivalued Bregman quasi-nonexpansive mapping with $F(T) = \{0\}$ and T is demiclosed at zero. We see that $x^* = 0$ is solution in Γ and it is a fixed point of T. Hence, $x^* = 0 \in \Theta$. Suppose that $\alpha_n = \frac{n}{n^2+1}, \beta_n = \frac{n}{2n+1}$. So our Algorithm 3.1 has the following form:

$$\begin{cases} y_n = \Pi_C(J_q^{E_1^*}(J_p^{E_1}(x_n) - \lambda_n A^* J_p^{E_2}(I - P_Q)Ax_n)) \\ z_n \in J_q^{E_1^*}\left(\frac{n}{2n+1}J_p^{E_1}(y_n) + \frac{n+1}{2n+1}J_p^{E_1}(Ty_n)\right) \\ x_{n+1} = J_q^{E_1^*}\left(\frac{n}{n^2+1}J_p^{E_1}(u) + \frac{n^2-n+1}{n^2+1}J_p^{E_1}(z_n)\right), \ \forall n \geq 1, \end{cases} \tag{62}$$

where the stepsize λ_n is chosen in such a way that

$$\lambda_n = \begin{cases} \frac{\rho_n f^{p-1}(x_n)}{\|\nabla f(x_n)\|^p}, & \text{if } f(x_n) \neq 0; \\ 0, & \text{otherwise,} \end{cases} \tag{63}$$

where $f(x_n) = \frac{1}{p}\|(I - P_Q)Ax_n\|^p$. By Theorem 3.5, the sequence $\{x_n\}$ generated by (62) converges strongly to $x^* = 0 \in \Theta$.

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