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On the Left (Right) Condition Pseudospectrum of Linear Operators

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Abstract. The aim of this paper is twofold: On the one hand, we suggest a characterization for the left (right) condition pseudospectrum of bounded linear operators. On the other hand, we give an analogue of the spectral mapping theorem for left (right) condition pseudospectrum.

1. Introduction

We assume throughout the present work that X is complex infinite-dimensional Banach spaces. Denote by $\mathcal{L}(X)$ the collection of all bounded linear operators acting on a Banach space X.

The identity operator on X is denoted by I_X and simply by I if the underlying space is clear from the context. Then $\mathcal{D}(T), \mathcal{N}(T), \mathcal{R}(T)$ and T' are, respectively, used to denote the domain, the kernel, the range and the adjoint (if exists) of T.

Recall that an operator T with domain $\mathcal{D}(T) \subset X$, is said to be invertible if there exists an everywhere defined $B \in \mathcal{L}(X)$ such that

$$TB = I \text{ and } BT \subset I.$$

We say that T is right invertible if there exists an everywhere defined $B \in \mathcal{L}(X)$ such that

$$TB = I,$$

which equivalent to the fact that T is surjective and $\mathcal{N}(T)$ admits a topological supplement.
We say that T is left invertible if there is an everywhere defined $C \in \mathcal{L}(X)$ such that

$$CT \subset I,$$

which is equivalent to the fact that T is bounded from below and $\mathcal{R}(T)$ admits a topological supplement (see for example [8], Sec. 2.4]. Then, the left spectrum and the right spectrum of $T \in \mathcal{L}(X)$ are defined respectively as follow:

$$\sigma^l(T) := \left\{\lambda \in \mathbb{C} : \quad \lambda - T \ \text{ is not left invertible}\right\}$$

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and

$$\sigma^r(T) := \Big\{\lambda \in \mathbb{C} : \ \ \lambda - T \ \text{ is not right invertible}\Big\}.$$

The pseudospectrum of $T \in \mathcal{L}(X)$ is denoted by $\sigma_\varepsilon(T)$ and is defined to be the set

$$\sigma_\varepsilon(T) := \sigma(T) \bigcup \Big\{\lambda \in \mathbb{C} : \ \ \|(\lambda - T)^{-1}\| > \frac{1}{\varepsilon}\Big\}$$

with the convention $\|(\lambda - T)^{-1}\| = \infty$ if $\lambda - T$ is not invertible.

For more information about the pseudospectrum, we refer the reader to [11] and [2, 3]. The condition spectrum of $T \in \mathcal{L}(X)$ is denoted by $\Sigma_\varepsilon(T)$ and is defined by

$$\Sigma_\varepsilon(T) := \sigma(T) \bigcup \Big\{\lambda \in \mathbb{C} : \ \|\lambda - T\|\|(\lambda - T)^{-1}\| > \frac{1}{\varepsilon}\Big\},$$

with the convention $\|(\lambda - T)\|\|(\lambda - T)^{-1}\| = \infty$ if $\lambda - T$ is not invertible. The condition pseudospectrum decomposes into two disjoint subsets as follows

(i) The left condition pseudospectrum:

$$\Sigma^l_\varepsilon(T) := \sigma^l(T) \bigcup \Big\{\lambda \in \mathbb{C} : \ \inf\Big\{\|\lambda - T\|\|S_l\| : \ S_l \text{ a left inverse of } \lambda - T\Big\} > \frac{1}{\varepsilon}\Big\},$$

with the convention $\inf\Big\{\|\lambda - T\|\|S_l\| : \ S_l \text{ a left inverse of } \lambda - T\Big\} = \infty$, if $\lambda - T$ is not left invertible.

(ii)The right condition pseudospectrum:

$$\Sigma^r_\varepsilon(T) := \sigma^r(T) \bigcup \Big\{\lambda \in \mathbb{C} : \ \inf\Big\{\|\lambda - T\|\|S_r\| : \ S_r \text{ a right inverse of } \lambda - T\Big\} > \frac{1}{\varepsilon}\Big\},$$

with the convention $\inf\Big\{\|\lambda - T\|\|S_l\| : \ S_l \text{ a right inverse of } \lambda - T\Big\} = \infty,$, if $\lambda - T$ is not right invertible.

The pseudospectrum decomposes into two disjoint subsets as

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with the convention $\inf\Big\{\|S_l\| : \ S_l \text{ a left inverse of } \lambda - T\Big\} = \infty$, if $\lambda - T$ is not left invertible.

(ii)The right pseudospectrum:

$$\sigma^r_\varepsilon(T) := \sigma^r(T) \bigcup \Big\{\lambda \in \mathbb{C} : \ \inf\Big\{\|S_r\| : \ S_r \text{ a right inverse of } \lambda - T\Big\} > \frac{1}{\varepsilon}\Big\},$$

with the convention $\inf\Big\{\|S_l\| : \ S_l \text{ a right inverse of } \lambda - T\Big\} = \infty,$, if $\lambda - T$ is not right invertible.

They can be ordered as

$$\Sigma_\varepsilon(T) = \Sigma^l_\varepsilon(T) \cup \Sigma^r_\varepsilon(T) \text{ and } \sigma_\varepsilon(T) = \sigma^l_\varepsilon(T) \cup \sigma^r_\varepsilon(T).$$

It is clear that for all $T \in \mathcal{L}(X)$, we have

$$\sigma^l(T) \subseteq \Sigma^l_\varepsilon(T) \subseteq \Sigma_\varepsilon(T), \ \ \sigma^r(T) \subseteq \Sigma^r_\varepsilon(T) \subseteq \Sigma_\varepsilon(T),$$

$$\sigma^l(T) \subseteq \sigma^l_\varepsilon(T) \subseteq \sigma_\varepsilon(T) \text{ and } \sigma^r(T) \subseteq \sigma^r_\varepsilon(T) \subseteq \sigma_\varepsilon(T).$$

In particular $\sigma^l(T)$, $\sigma^r(T)$, $\Sigma^l(T)$ and $\Sigma^r(T)$ are non empty sets.

The condition pseudospectra of linear operators play a crucial role in many branches of mathematics and in numerous applications. Analytic information on the condition pseudospectrum is, in general, hard to obtain and numerical approximations may not be reliable, in particular, if the operator is not self-adjoint or normal. For more material about the condition pseudospectrum and other information on the basic theory of algebraic condition pseudospectrum, we refer the reader to [1, 2, 4–6] and [7]. Some other related topics can be found in [3, 14], the interested reader may consult the remarkable books of Jeribi [9, 10, 15]. Besides the recent works [1, 9, 12, 13] on the spectral analysis, we are motivated by several papers where the main interest is focused on the spectral mapping theorem. The spectral mapping theorem is a fundamental result in functional analysis of great importance. The spectral mapping theorem says that if f is an analytic function on an open set containing $\sigma(T)$, then

$$f(\sigma(T)) = \sigma(f(T)).$$

In [12, 13], the author described the analogue of the spectral mapping Theorem for pseudospectrum. An analogue of the spectral mapping theorem for condition spectrum is done in [11]. Then, it is natural to ask whether similiar results can be proven for the left(right) condition pseudospectrum.

The present work is organized as follow: After this introduction where several basic definitions and facts are recalled, in the second section, we devote ourselves to characterize the left (right) condition pseudospectrum of linear operators on a Banach space. Finally, we prove an analogue of the spectral mapping theorem for the left (right) condition pseudospectrum of linear operators on a Banach space.

2. Left (Right) condition pseudospectrum

The following proposition provides some elementary properties of the left (right) condition pseudospectrum. For more details, see [14].

Proposition 2.1. *Let $T \in \mathcal{L}(X)$ and $0 < \varepsilon < 1$.*

(*i*) $\sigma^l(T) = \bigcap_{0<\varepsilon<1} \Sigma^l_\varepsilon(T)$ *and* $\sigma^r(T) = \bigcap_{0<\varepsilon<1} \Sigma^r_\varepsilon(T)$.

(*ii*) *If* $0 < \varepsilon_1 < \varepsilon_2 < 1$, *then* $\Sigma^l_{\varepsilon_1}(T) \subset \Sigma^l_{\varepsilon_2}(T)$ *and* $\Sigma^r_{\varepsilon_1}(T) \subset \Sigma^r_{\varepsilon_2}(T)$.

(*iii*) $\Sigma^l_\varepsilon(T)$ *and* $\Sigma^r_\varepsilon(T)$ *are non-empty compact subsets of* $\mathbb{C}$.

(*iv*) *If* $\alpha \in \mathbb{C}$, *then* $\Sigma^l_\varepsilon(T + \alpha I) = \alpha + \Sigma^l_\varepsilon(T)$ *and* $\Sigma^r_\varepsilon(T + \alpha I) = \alpha + \Sigma^r_\varepsilon(T)$.

(*v*) *If* $\alpha \in \mathbb{C}\backslash\{0\}$, *then* $\Sigma^l_\varepsilon(\alpha T) = \alpha\Sigma^l_\varepsilon(T)$ *and* $\Sigma^r_\varepsilon(\alpha T) = \alpha\Sigma^r_\varepsilon(T)$.

The following theorem establishes a the relationship between left (right) condition pseudospectrum and left (right) pseudospectrum of an bounded linear operator $T \in \mathcal{L}(X)$.

Theorem 2.2. *Let $T \in \mathcal{L}(X)$ such that $T \neq \lambda I$ for all $0 < \varepsilon < 1$. Then,*

$$\Sigma^l_\varepsilon(T) \subseteq \sigma^l_{\gamma_\varepsilon}(T) \subseteq \Sigma^l_{\upsilon_\varepsilon}(T)$$

where,

$$\gamma_\varepsilon = \frac{2\varepsilon\|T\|}{1-\varepsilon}$$

and

$$0 < \upsilon_\varepsilon = \frac{2\varepsilon\|T\|}{(1-\varepsilon)\delta_T} < 1. \qquad \diamond$$

Proof. Let $\lambda \in \Sigma^l_\varepsilon(T)$, then for all S_l a left inverse of $\lambda - T$ we have that

$$\|\lambda - T\|\|S_l\| > \frac{1}{\varepsilon}.$$

Thus

$$\|S_l\| > \frac{1}{\varepsilon\|\lambda - T\|} > \frac{1}{\varepsilon(|\lambda| + \|T\|)}.$$

Since $\lambda \in \Sigma_\varepsilon(T)$, using Lemma 2.1 in [2], we obtain that

$$\|S_l\| \geq \frac{1-\varepsilon}{2\varepsilon\|T\|}.$$

Hence

$$\lambda \in \sigma^l_{\gamma_\varepsilon}(T).$$

For the second inclusion, let $\lambda \in \sigma^l_{\gamma_\varepsilon}(T)$. Then, for all S_l a left inverse of $\lambda - T$ we have that

$$\|S_l\| \geq \frac{1-\varepsilon}{2\varepsilon\|T\|}.$$

Also, we have $\|\lambda - T\| \geq \inf\left\{\|\lambda - T\| : \lambda \in \mathbb{C}\right\} := \delta_T > 0$, hence for all S_l a left inverse of $\lambda - T$

$$\|\lambda - T\|\|S_l\| > \delta_T \frac{1-\varepsilon}{2\varepsilon\|T\|}.$$

Therefore, $\lambda \in \Sigma^l_{\upsilon_\varepsilon}(T)$. □

Remark 2.3. *The Theorem 2.2 remain true if we replace $\Sigma^l_\varepsilon(T)$ with $\Sigma^r_\varepsilon(T)$, $\sigma^l_{\gamma_\varepsilon}(T)$ with $\sigma^r_{\gamma_\varepsilon}(T)$ and $\Sigma^l_{\upsilon_\varepsilon}(T)$ with $\Sigma^r_{\upsilon_\varepsilon}(T)$.*

Lemma 2.4. *Let $T \in \mathcal{L}(X)$, $0 < \varepsilon < 1$ and $\lambda \notin \sigma^l(T)$ Then, $\lambda \in \Sigma^l_\varepsilon(T)$ if, and only if, there exists $x \in X$, such that*

$$\|(\lambda - T)x\| < \varepsilon\|\lambda - T\|\|x\|. \qquad \diamond$$

Proof. Let $\lambda \in \Sigma^l_\varepsilon(T) \setminus \sigma^l(T)$, then for all S_l a left inverse of $\lambda - T$ we have

$$\|S_l\| > \frac{1}{\varepsilon\|\lambda - T\|}.$$

In other words,

$$\sup_{y \in X\setminus\{0\}} \frac{\|S_l y\|}{\|y\|} > \frac{1}{\varepsilon\|\lambda - T\|}.$$

Then, there exists a nonzero $y \in X$, such that

$$\|S_l y\| > \frac{\|y\|}{\varepsilon\|\lambda - T\|}.$$

Putting $y = (\lambda - T)x$, we have the result. Conversely, we assume there exists $x \in X$ such that

$$\|(\lambda - T)x\| < \varepsilon\|\lambda - T\|\|x\|.$$

Let $\lambda \notin \sigma^l(T)$, then $\lambda - T$ is left invertible. Let S_l be any left inverse, then $x = S_l(\lambda - T)x$, Therefore,

$$\|x\| \leq \|S_l\|\|(\lambda - T)x\|.$$

Moreover,

$$1 < \varepsilon\|\lambda - T\|\|S_l\|.$$

So, $\lambda \in \Sigma^l_\varepsilon(T) \setminus \sigma^l(T)$. □

In the following theorem, we investigate the relation between the left condition pseudospectrum and the usual left spectrum in a complex Banach space.

Theorem 2.5. *Let $T \in \mathcal{L}(X)$, $\lambda \in \mathbb{C}$, and $0 < \varepsilon < 1$. If there is $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon\|\lambda - T\|$ and $\lambda \in \sigma^l(T+D)$. Then, $\lambda \in \Sigma_\varepsilon^l(T)$.*

Proof. We assume that there exists D such that $\|D\| < \varepsilon\|\lambda - T\|$ and $\lambda \in \sigma^l(T+D)$. Let $\lambda \notin \Sigma_\varepsilon^l(T)$, then for all S_l a left inverse of $\lambda - T$ we have

$$\|\lambda - T\|\|S_l\| \leq \frac{1}{\varepsilon}.$$

Now, we define the operator $S : X \longrightarrow X$ by

$$S := \sum_{n=0}^{\infty} S_l\Big(DS_l\Big)^n.$$

Since,

$$\|DS_l\| < 1,$$

we can write

$$S = S_l\Big(I - DS_l\Big)^{-1}.$$

Then, for all $y \in X$ we have.

$$S\Big(I - DS_l\Big)y = S_l y.$$

Let $y = (\lambda - T)x$. Then,

$$S(\lambda - T - D)x = x$$

for every $x \in X$. Hence, $\lambda - T - D$ is left invertible, so

$$\lambda \in \Sigma_\varepsilon^l(T). \quad \square$$

Theorem 2.6. *Suppose X is a complex Banach space with the following property:*

For all left invertible operator $T \in \mathcal{L}(X)$ there exists a non-left-invertible $B \in \mathcal{L}(X)$ and a left inverse S_l such that

$$\|T - D\| = \frac{1}{\|S_l\|}.$$

Then, if $\lambda \in \Sigma_\varepsilon^l(T)$ there exists $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon\|\lambda - T\|$ and $\lambda \in \sigma^l(T+D)$.

Proof. Suppose $\lambda \in \Sigma_\varepsilon^l(T)$. We will discuss these two cases:
1^{st} case : If $\lambda \in \sigma^l(T)$, then it is sufficient to take $D = 0$.
2^{nd} case : If $\lambda \in \Sigma_\varepsilon^l(T)\backslash\sigma^l(T)$. Then,

$$\inf\Big\{\|\lambda - T\|\|S_l\| : \ S_l \text{ a left inverse of } \lambda - T\Big\} > \frac{1}{\varepsilon}.$$

Hence there exists S_l a left inverse of $\lambda - T$ such that

$$\|\lambda - T\|\|S_l\| > \frac{1}{\varepsilon}.$$

By assumption, there exists $B \in \mathcal{L}(X)$ such that

$$\|\lambda - T - B\| = \frac{1}{\|S_l\|}.$$

Let $D = \lambda - T - B$. Then

$$\|D\| = \frac{1}{\|S_l\|} < \varepsilon\|\lambda - T\|.$$

Also $B = \lambda - (T + D)$, is not left invertible. So, $\lambda \in \sigma^l(T + D)$. $\square$

Remark 2.7. *The Lemma 2.4 and Theorems 2.5 and 2.6 remain true if we replace* $\Sigma^l_\varepsilon(T)$ *with* $\Sigma^r_\varepsilon(T)$.

Theorem 2.8. *Let* $T \in \mathcal{L}(X)$, $\lambda \in \mathbb{C}$, *and* $0 < \varepsilon < 1$. *Then,*

(i) *Let* $\lambda_n \notin \Sigma^l_\varepsilon(T)$*and let* $\lambda \in \Sigma^l_\varepsilon(T)$ *be such that* $\lambda_n \to \lambda$. *Then*

$$\inf\Big\{\|\lambda - T\|\|S_l\| : \ S_l \textit{ a left inverse of } \lambda - T\Big\} = \infty.$$

(ii) *Let* $\lambda_n \notin \Sigma^r_\varepsilon(T)$*and let* $\lambda \in \Sigma^r_\varepsilon(T)$ *be such that* $\lambda_n \to \lambda$. *Then*

$$\inf\Big\{\|\lambda - T\|\|S_r\| : \ S_r \textit{ a right inverse of } \lambda - T\Big\} = \infty.$$ ◊

Proof. (i) Suppose $\inf\Big\{\|\lambda - T\|\|S_l\| : \ S_l \text{ a left inverse of } \lambda - T\Big\} \leq c$ for some $c \in \mathbb{R}$ and $\lambda \in \Sigma^l_\varepsilon(T)$ be such that $\lambda_n \to \lambda$, then there exists $n_0 \in \mathbb{N}$ such that

$$|\lambda_n - \lambda| < \frac{1}{c+1} < \frac{1}{c} \leq \frac{1}{\inf\Big\{\|\lambda - T\|\|S_l\| : \ S_l \text{ a left inverse of } \lambda - T\Big\}} \text{ for all } n \geq n_0.$$

Hence, $\lambda \notin \Sigma^l_\varepsilon(T)$. This is a contradiction.

(ii) The proof of (ii) may be achieved in the same way as the proof of (i). □

Theorem 2.9. *Let* $T \in \mathcal{L}(X)$ *such that* $T \neq \lambda I$ *for every* $\lambda \in \mathbb{C}$ *and* $0 < \varepsilon < 1$. *Then,*

(i) $\Sigma^l_\varepsilon(T)$ *has no isolated points.*

(ii) $\Sigma^r_\varepsilon(T)$ *has no isolated points.*

Proof. (i) Suppose $\Sigma^l_\varepsilon(T)$ has an isolated point μ. Then there exists an $\delta > 0$ such that for all $\lambda \in \mathbb{C}$ with $0 < |\lambda - \mu| < \delta$ and $\inf\Big\{\|\lambda - T\|\|S_l\| : \ S_l \text{ a left inverse of } \lambda - T\Big\} \leq \dfrac{1}{\varepsilon}$. Let $\mu \in \Sigma^l_\varepsilon(T)\backslash\sigma^l(T)$. Then, using the Hahn-Banach theorem, there exist $x' \in X'$ and $y' \in X'$ such that

$$x'(\mu - T) = \|\mu - T\| \text{ with } \|x'\| = 1$$

and

$$y'(S_l) = \|S_l\| \text{ with } \|y'\| = 1.$$

Now, we define

$$\begin{cases} f : \mathbb{C}\backslash\Sigma^l(T) \longrightarrow \mathbb{C}, \\ \quad \lambda \longrightarrow f(\lambda) = \inf\Big\{x'(\lambda - T)y'(S_l) : \ S_l \text{ a left inverse of } \lambda - T\Big\}. \end{cases}$$

It is clear that f is analytic in $B(\mu, \delta)$ and for all $\lambda \in \mathbb{C}$ with $0 < |\lambda - \mu| < \delta$, we have

$$\begin{aligned} |f(\lambda)| &= \Big|\inf\Big\{x'(\lambda - T)y'(S_l) : \ S_l \text{ a left inverse of } \lambda - T\Big\}\Big| \\ &\leq \inf\Big\{\|\lambda - T\|\|S_l\| : \ S_l \text{ a left inverse of } \lambda - T\Big\} \leq \frac{1}{\varepsilon}. \end{aligned}$$

But, $f(\mu) = \inf\Big\{\|\mu - T\|\|S_l\| : \ S_l \text{ a left inverse of } \mu - T\Big\} > \dfrac{1}{\varepsilon}$. This contradicts the maximum modulus principle.

(ii) The proof of (ii) may be checked in the same way as in the proof of (i). □

3. Left (Right) pseudospectral mapping Theorem

The following is a left (right) pseudospectral mapping theorem for complex analytic functions. It is sharp in the sense that the functions φ and ϕ measure the sizes of the left (right) pseudospectra are optimal. Actually, the theorem is an easy consequence of the definitions of these functions. Let $T \in \mathcal{L}(X)$ and let f be an analytic function defined on D, an open set containing $\Sigma^l_\varepsilon(T)$ (resp. $\Sigma^r_\varepsilon(T)$). For every $0 < \varepsilon < 1$, we define

$$\varphi(\varepsilon) = \sup_{\lambda \in \Sigma^l_\varepsilon(T)} \left\{ \frac{1}{\inf\left\{ \|f(\lambda) - f(T)\| \|S_{1,l}\| : \ S_{1,l} \text{ a left inverse of } f(\lambda) - f(T) \right\}} \right\}.$$

Assuming the existence of a $0 < \varepsilon_0 < 1$ such that $\Sigma^l_{\varepsilon_0}(f(T)) \subseteq f(D)$. it is also possible to define

$$\phi(\varepsilon) = \sup_{\mu \in f^{-1}(\Sigma^l_\varepsilon(T)) \cap D} \left\{ \frac{1}{\inf\left\{ \|\mu - T\| \|S_l\| : \ S_l \text{ a left inverse of } \mu - T \right\}} \right\}.$$

Lemma 3.1. *Let $T \in \mathcal{L}(X)$ and $0 < \varepsilon < 1$. Then*

(*i*) *If $f(\lambda)I \neq f(T)$, then $\varphi(\varepsilon)$ is well defined , $\lim_{\varepsilon \to 0} \varphi(\varepsilon) = 0$ and $0 \leq \varphi(\varepsilon) \leq 1$.*

(*ii*) *If $\mu I \neq T$, then $\phi(\varepsilon)$ is well defined , $\lim_{\varepsilon \to 0} \phi(\varepsilon) = 0$ and $0 \leq \phi(\varepsilon) \leq 1$.*

Proof. The proof is a straightforward adaption of the proof of Theorem 2.1 in [11]. □

Theorem 3.2. *Let $T \in \mathcal{L}(X)$ such that $f(\lambda) \neq f(T)$, for every $\lambda \in \mathbb{C}$, and for $0 < \varepsilon < 1$ satisfying $\varphi(\varepsilon) < 1$. Then, we have*

$$f(\Sigma^l_\varepsilon(T)) \subseteq \Sigma^l_{\varphi(\varepsilon)}(f(T)).$$

Proof. In the order to prove that $\varphi(\varepsilon)$ is well defined, we define $h : \mathbb{C} \to \mathbb{R}$

$$h(\lambda) = \frac{1}{\inf\left\{ \|f(\lambda) - f(T)\| \|S_{1,l}\| : \ S_{1,l} \text{ a left inverse of } f(\lambda) - f(T) \right\}}.$$

Since $h(\lambda)$ is continuous and $\Sigma^l_\varepsilon(T)$ is a compact subset of $\mathbb{C}$, then it is clear that

$$\varphi(\varepsilon) = \sup\left\{ h(\lambda) : \lambda \in \Sigma^l_\varepsilon(T) \right\}.$$

We obtain that, $\varphi(\varepsilon)$ is well defined. Using the fact that $\varphi(\varepsilon)$ is a monotonically non-decreasing function and $\varphi(\varepsilon)$ goes to zero as ε goes to zero. Now, let ε be sufficiently small so that $0 < \varphi(\varepsilon) < 1$ and let $\lambda \in \Sigma^l_\varepsilon(T)$. Then $h(\lambda) \leq \varphi(\varepsilon)$. Hence

$$\inf\left\{ \|f(\lambda) - f(T)\| \|S_{1,l}\| : \ S_{1,l} \text{ a left inverse of } f(\lambda) - f(T) \right\} = \frac{1}{h(\lambda)} \geq \frac{1}{\varphi(\varepsilon)}.$$

Thus, $f(\lambda) \in \Sigma^l_{\varphi(\varepsilon)}(f(T))$. This means that

$$f(\Sigma^l_\varepsilon(T)) \subseteq \Sigma^l_{\varphi(\varepsilon)}(f(T)).$$

□

Theorem 3.3. *Let $T \in \mathcal{L}(X)$ such that $T \neq I$ for every $\lambda \in \mathbb{C}$, and for $0 < \varepsilon < 1$ satisfying $\phi(\varepsilon) < 1$. Then, we have*

$$\Sigma^l_\varepsilon(f(T)) \subseteq f(\Sigma^l_{\phi(\varepsilon)}(T)).$$

Proof. We assume that there exists ε_0 with $0 < \varepsilon_0 < 1$ such that

$$\Sigma^l_{\varepsilon_0}(f(T)) \subseteq f(D).$$

We show that for $0 < \varepsilon < \varepsilon_0$, $\phi(\varepsilon)$ is well defined. Define $g : \mathbb{C} \to \mathbb{R}$,

$$g(\mu) = \frac{1}{\inf\left\{\|\mu - T\|\|S_l\| : \ S_l \text{ a left inverse of } \mu - T\right\}}.$$

Since g is continuous and $g(\mu) \leq 1$ for all $\mu \in \mathbb{C}$, $\phi(\varepsilon)$ is well defined and $0 \leq \phi(\varepsilon) \leq 1$. It is also clear that $\phi(\varepsilon)$ is a monotonically non-decreasing function and $\phi(\varepsilon)$ goes to zero as ε goes to zero. Now, if we take ε sufficiently small so we obtain that $0 < \phi(\varepsilon) < 1$. Let $\lambda \in \Sigma^l_{\varepsilon}(f(T)) \subseteq \Sigma^l_{\varepsilon_0}(f(T)) \subseteq f(D)$. Consider $\mu \in D$ such that $\lambda = f(\mu)$. Then $\mu \in f^{-1}(\Sigma^l_{\varepsilon}(f(T)))$, hence $g(\mu) \leq \phi(\varepsilon)$. Therefore,

$$\inf\left\{\|\mu - T\|\|S_l\| : \ S_l \text{ a left inverse of } \mu - T\right\} = \frac{1}{g(\mu)} \geq \frac{1}{\phi(\varepsilon)}.$$

Thus, $\mu \in \Sigma^l_{\phi(\varepsilon)}(T)$. Then, $\lambda = f(\mu) \in f(\Sigma^l_{\phi(\varepsilon)}(T))$. This means that

$$\Sigma^l_{\varepsilon}(f(T)) \subseteq f(\Sigma^l_{\phi(\varepsilon)}(T)).$$

□

Corollary 3.4. *Combining the two inclusions in Theorems 3.2 and 3.3, we get*

$$f(\Sigma^l_{\varepsilon}(T)) \subseteq \Sigma^l_{\varphi(\varepsilon)}(f(T)) \subseteq f(\Sigma^l_{\phi(\varphi(\varepsilon))}(T)$$

and

$$\Sigma^l_{\varepsilon}(f(T)) \subseteq f(\Sigma^l_{\phi(\varepsilon)}(T)) \subseteq \Sigma^l_{\varphi(\phi(\varepsilon))}(f(T)).$$

Theorem 3.5.

$$f(\Sigma^r_{\varepsilon}(T)) \subseteq \Sigma^r_{\varphi(\varepsilon)}(f(T)) \subseteq f(\Sigma^r_{\phi(\varphi(\varepsilon))}(T)$$

and

$$\Sigma^r_{\varepsilon}(f(T)) \subseteq f(\Sigma^r_{\phi(\varepsilon)}(T)) \subseteq \Sigma^r_{\varphi(\phi(\varepsilon))}(f(T)).$$

Proof. Proof of the following Theorem goes similar to the proof of the Theorem 3.2 and 3.3. □

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