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An Extension of Hirano Inverses in Banach Algebras

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Abstract. We introduce a new class of generalized inverse which is called $\pi-$Hirano inverse. In this paper some elementary properties of the $\pi-$Hirano inverse are obtained. We prove that $a \in \mathcal{A}$ is $\pi-$Hirano invertible if and only if $a - a^{n+1}$ is nilpotent for some positive integer n. Certain multiplicative and additive results for the $\pi-$Hirano inverse in a Banach algebra are presented. We then apply these new results to block operator matrices over Banach spaces.

1. Introduction

Let $\mathcal{A}$ be a Banach algebra with an identity. An element a in $\mathcal{A}$ has Drazin inverse if there is a common solution to the equations $ax = xa, x = xax$ *and* $a^n = a^{n+1}x$ for some $n \in \mathbb{N}$. As is well known, an element $a \in \mathcal{A}$ has Drazin inverse if there exists $x \in \mathcal{A}$ such that

$$ax = xa, x = xax \text{ and } a - a^2x \in N(\mathcal{A}).$$

Here $N(\mathcal{A})$ is the set of all nilpotent elements in $\mathcal{A}$. The preceding x is unique, if such element exists. As usual, it will be denoted by a^D, and called the Drazin inverse of a (For more information see [11] and [5]) .

Recently, several subclasses of the Drazin inverse have been studied. An element $a \in \mathcal{A}$ has strongly Drazin inverse if there is a common solution to the equations

$$ax = xa, x = xax \textit{ and } a - ax \in N(\mathcal{A}).$$

We know that $a \in \mathcal{A}$ has strongly Drazin inverse if and only if it is the sum of an idempotent and a nilpotent that commute (see [2, Theorem 2.1]). This generalized inverse has been studied in [2] and [12] extensively. In a Banach algebra $\mathcal{A}$, $a \in \mathcal{A}$ has strongly Drazin inverse if and only if $a - a^2 \in \mathcal{A}$ is nilpotent [2, Theorem 2.1].

An element $a \in \mathcal{A}$ has Hirano inverse if the following equations hold,

$$ax = xa, x = xax \textit{ and } a^2 - ax \in N(\mathcal{A}).$$

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It was proved that, $a \in \mathcal{A}$ has Hirano inverse if and only if it is the sum of a tripotent and a nilpotent that commute. Here, $p \in \mathcal{A}$ is a tripotent if $p^3 = p$ (see [3, Theorem 3.3]). It was proved that a has Hirano inverse if and only if $a - a^3 \in \mathcal{A}$ is nilpotent. In [17], Zou and Mosic et al. investigated the element $a \in \mathcal{A}$ satisfying the condition $a - a^{n+1} \in N(\mathcal{A})$ for a fixed n. This inspires us to introduce and study a new class of generalized inverse. In fact, it forms a subclass of Drazin inverses in a Banach algebra. We say that an element $a \in \mathcal{A}$ has $\pi-$Hirano inverse if there exists $x \in \mathcal{A}$ such that

$$ax = xa, x = xax \text{ and } a - a^{n+2}x \in N(\mathcal{A})$$

for some $n \in \mathbb{N}$. The preceding x shall be unique, if such element exists. We observed that these inverses form a subclass of Drazin inverses which is related to periodic elements in a Banach algebra $\mathcal{A}$. We denote the set of all $\pi-$Hirano invertible elements in $\mathcal{A}$ by $\mathcal{A}^{\pi H}$.

In Section 2, we investigate some elementary properties of $\pi-$Hirano invertible elements. It is proved that an element $a \in \mathcal{A}$ has $\pi-$Hirano inverse if and only if $a - a^{n+1} \in N(\mathcal{A})$ for some $n \in \mathbb{N}$. The invertibility of the sum of two π-Hirano invertible elements in a Banach algebra under some conditions will be presented. We prove that for any $a, b \in \mathcal{A}^{\pi H}$, if $aba = 0, bab = 0, a^2b^2 = 0$ and $ab^3 = 0$, then $a + b \in \mathcal{A}^{\pi H}$.

In Section 3, we consider the $\pi-$Hirano inverse of a 2×2 operator matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where $A \in \mathcal{L}(X), B \in \mathcal{L}(X, Y), C \in \mathcal{L}(Y, X)$ and $D \in \mathcal{L}(Y)$. Here, M is a bounded operator on $X \oplus Y$. In Section 4, we present some $\pi-$Hirano inverses for a 2×2 operator matrix M under a number of different conditions.

If $a \in \mathcal{A}$ has $\pi-$Hirano inverse $a^{\pi H}$, then element $p = 1 - aa^{\pi H}$ is called the spectral idempotent of a. In Section 4, we consider the $\pi-$Hirano inverse of a 2×2 operator matrix M under the perturbations on spectral idempotents.

The double commutant of $a \in \mathcal{A}$ is defined by $comm^2(a) = \{x \in \mathcal{A} \mid xy = yx \text{ if } ay = ya \text{ for } y \in \mathcal{A}\}$. $\mathbb{N}$ stands for the set of all natural numbers and $U(\mathcal{A})$ is the set of all invertible elements in $\mathcal{A}$.

2. Cline's Formula

In this section we are concern with additive property of the $\pi-$Hirano inverse of the sum in a Banach algebra $\mathcal{A}$. We begin with

Theorem 2.1. *Let $\mathcal{A}$ be a Banach algebra, and $a \in \mathcal{A}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}^{\pi H}$;
(2) $a - a^{n+1} \in N(\mathcal{A})$ *for some* $n \in \mathbb{N}$;
(3) *There exists* $b \in comm^2(a)$ *such that* $b = b^2a, a - a^{n+2}b \in N(\mathcal{A})$ *for some* $n \in \mathbb{N}$.

Proof. (1) ⇒ (2) Since a has $\pi-$Hirano inverse, we have $b \in \mathcal{A}$ such that

$$ab = ba, b = bab \text{ and } a - a^{n+2}b \in N(\mathcal{A})$$

for some positive integer n. That is, $a \in \mathcal{A}$ is n-strongly Drazin invertible. Hence, $a - a^{n+1} \in N(\mathcal{A})$ for some $n \in \mathbb{N}$ by [17, Theorem 3.2].

(2) ⇒ (3) Since $a - a^{n+1} \in N(\mathcal{A})$, we deduce that $(a^n)^2 - a^n = a^{n-1}(a^{n+1} - a) \in N(\mathcal{A})$. By [1, Lemma 2.1], there exists an idempotent $e \in \mathbb{Z}[a]$ such that $a^n - e = w \in N(\mathcal{A})$ *and* $ae = ea$. Hence we obtain $1 + a^n - e = 1 + w \in U(\mathcal{A})$. Let $b = (1 + a^n - e)^{-1}a^{n-1}e$. Then $b \in comm^2(a)$. Moreover, we have

$$\begin{aligned} b^2a &= (1 + a^n - e)^{-2}a^{2(n-1)}ea \\ &= (1 + a^n - e)^{-2}a^{2n-1}e \\ &= (1 + a^n - e)^{-2}a^n e a^{n-1} \\ &= (1 + a^n - e)^{-2}(1 + a^n - e)a^{n-1}e \\ &= (1 + a^n - e)^{-1}a^{n-1}e \\ &= b, \end{aligned}$$

$$\begin{aligned} a - a^{n+2}b &= a - (1 + a^n - e)^{-1}a^{2n+1}e \\ &= a - (1 + a^n - e)^{-1}(1 + a^n - e)ea^{n+1} \\ &= a - ea^{n+1} \\ &= (a - a^{n+1}) + (1 - e)a^{n+1} \\ &= (a - a^{n+1}) + (1 - e)wa \\ &\in N(\mathcal{A}), \end{aligned}$$

as desired.

(3) ⇒ (1) It is obvious as $comm^2(a) \subseteq comm(a)$. □

Corollary 2.2. *Every $\pi-$Hirano invertible element in a Banach algebra has Drazin inverse.*

Proof. Let $a \in \mathcal{A}$ has $\pi-$Hirano inverse. By Theorem 2.1, $a - a^{n+1} \in N(\mathcal{A})$. Then there exists some $m \in \mathbb{N}$ such that $(a - a^{n+1})^m = 0$. Hence we can find some polynomial $f(x)$ such that $a^n = a^{n+1}f(a)$ and so a is strongly $\pi-$ regular which is Drazin invertible. □

Theorem 2.3. *Let $a, b, c, d \in \mathcal{A}$ satisfying*

$$(ac)^2a = (db)^2a;$$
$$(ac)^2d = (db)^2d.$$

Then $ac \in \mathcal{A}^{\pi H}$ if and only if $bd \in \mathcal{A}^{\pi H}$. In this case, $(bd)^{\pi H} = b[(ac)^{\pi H}]^2d$.

Proof. $\Longrightarrow$ Let $aca = a^{'}, c = c^{'}, dbd = d^{'}$ and $b = b^{'}$. We easily verify that

$$a'c'a' = d'b'a';$$
$$a'c'd' = d'b'd'.$$

Since $ac \in \mathcal{A}^{\pi H}$, then ac is n-strongly Drazin invertible for some $n \in \mathbb{N}$. So $a'c'$ is n-strongly Drazin invertible. In view of [17, Theorem 3.7], $b'd' \in \mathcal{A}$ is n-strongly Drazin invertible. Hence, $b'd' - (b'd')^{n+1} \in N(\mathcal{A})$. We check that

$$bd[bd - (bd)^{2n+1}] = (bd)^2 - (bd)^{2n+2} \in N(\mathcal{A}).$$

Therefore

$$[bd - (bd)^{2n+1}]^2 = bd[bd - (bd)^{2n+1}][1 - (bd)^{2n}] \in N(\mathcal{A}),$$

and then $bd - (bd)^{2n+1} \in N(\mathcal{A})$. By using Theorem 2.1, $bd \in \mathcal{A}^{\pi H}$. Also we know that, if $a \in \mathcal{A}^{\pi H}$, then $a^2 \in \mathcal{A}^{\pi H}$ and we have $(a^2)^{\pi H} = (a^{\pi H})^2$. Then,

$$\begin{aligned} (bd)^{\pi H} &= [(bd)^2]^{\pi H}bd \\ &= (b'd')^{\pi H}bd = b'[(a'c')^{\pi H}]^2d'bd \\ &= b[(ac)^{\pi H}]^4(db)^2d \\ &= b[(ac)^{\pi H}]^2d. \end{aligned}$$

.

$\Longleftarrow$ This is symmetric. □

Corollary 2.4. *Let $a, b \in \mathcal{A}$. If ab has $\pi-$Hirano inverse, then so does ba.*

Proof. It follows directly from Theorem 2.3. □

Corollary 2.5. *Let $a \in \mathcal{A}$ and $m \in \mathbb{N}$. Then $a \in \mathcal{A}^{\pi H}$ if and only if $a^m \in \mathcal{A}^{\pi H}$.*

Proof. By Theorem 2.1 and induction, we get the result easily. □

Lemma 2.6. *Let $a, b \in \mathcal{A}^{\pi H}$ and $ab = 0$. Then $a + b \in \mathcal{A}^{\pi H}$.*

Proof. Since $a, b \in \mathcal{A}^{\pi H}$, there exist $k, l \in \mathbb{N}$ such that

$$a - a^{k+1}, b - b^{l+1} \in N(\mathcal{A}).$$

Then

$$\begin{array}{rcl} a - a^{kl+1} & = & (a - a^{k+1}) + (a^{k+1} - a^{2k+1}) + \cdots + (a^{(l-1)k+1} - a^{lk+1}) \\ & = & [1 + a^k + \cdots + a^{(l-1)k}](a - a^{k+1}) \\ & \in & N(\mathcal{A}). \end{array}$$

Likewise, we have

$$b - b^{lk+1} \in N(\mathcal{A}).$$

Let $n = lk$. Then $a - a^{n+1}, b - b^{n+1} \in N(\mathcal{A})$. Thus a and b are n-strongly Drazin invertible. By virtue of [17, Theorem 4.2], $a + b$ is n-strongly Drazin invertible. Therefore $a + b$ is $\pi-$Hirano invertible. □

Lemma 2.7. *Let $a, b \in \mathcal{A}^{\pi H}$. If $aba = 0$ and $ab^2 = 0$, then $a + b \in \mathcal{A}^{\pi H}$.*

Proof. Let $p = a^2 + ab$ and $q = ba + b^2$. Since $(ab)^2 = 0$, we have $ab - (ab)^2 \in \mathcal{A}$ is nilpotent. Hence $ab \in \mathcal{A}^{\pi H}$. By using Corollary 2.4, $ba \in \mathcal{A}^{\pi H}$. Clearly $(ab)a^2 = (ba)b^2 = 0$. It follows by Lemma 2.6, that $p, q \in \mathcal{A}^{\pi H}$. Furthermore, we check that

$$pq = (a^2 + ab)(ba + b^2) = a^2ba + a^2b^2 + abba + abb^2 = 0$$

and then $p + q = (a + b)^2 \in \mathcal{A}^{\pi H}$ by using Lemma 2.6 again. According to Corollary 2.5, $a + b \in \mathcal{A}^{\pi H}$, as required. □

Theorem 2.8. *Let $a, b \in \mathcal{A}^{\pi H}$. If $aba = 0, bab = 0, a^2b^2 = 0$ and $ab^3 = 0$, then $a + b \in \mathcal{A}^{\pi H}$.*

Proof. Let

$$M = \begin{pmatrix} a^3 + a^2b + ab^2 & a^3b \\ a^2 + ab + ba + b^2 & a^2b + ab^2 + b^3 \end{pmatrix}.$$

Then

$$\begin{array}{rcl} M & = & \begin{pmatrix} a^2b + ab^2 & a^3b \\ 0 & a^2b + ab^2 \end{pmatrix} + \begin{pmatrix} a^3 & 0 \\ a^2 + ab + ba + b^2 & b^3 \end{pmatrix} \\ & := & G + F. \end{array}$$

Obviously, we have $G^3 = 0$. Further, we see that

$$\begin{array}{rcl} F & = & \begin{pmatrix} a^3 & 0 \\ a^2 + ab + ba + b^2 & b^3 \end{pmatrix} \\ & = & \begin{pmatrix} a^3 & 0 \\ a^2 + ba & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ b^2 + ab & b^3 \end{pmatrix} \\ & := & H + K. \end{array}$$

By hypothesis, we compute that

$$\begin{array}{rcl} GH & = & \begin{pmatrix} a^2b + ab^2 & a^3b \\ 0 & a^2b + ab^2 \end{pmatrix} \begin{pmatrix} a^3 & 0 \\ a^2 + ba & 0 \end{pmatrix} \\ & = & \begin{pmatrix} ab^2a^3 & 0 \\ a^2b^2a & 0 \end{pmatrix}, \\ GK & = & \begin{pmatrix} a^2b + ab^2 & a^3b \\ 0 & a^2b + ab^2 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ b^2 + ab & b^3 \end{pmatrix} \\ & = & 0. \end{array}$$

Hence $FGF = FGH + FGK = 0$ and $FG^2 = 0$. It is easy to verify that

$$H = \begin{pmatrix} a^3 & 0 \\ a^2 + ba & 0 \end{pmatrix} = \begin{pmatrix} a^2 \\ a + b \end{pmatrix} (a, 0).$$

Since $(a, 0)\begin{pmatrix} a^2 \\ a + b \end{pmatrix} = a^3 \in \mathcal{A}^{\pi H}$, by Theorem 2.3, H has $\pi-$Hirano inverse. Similarly, K has $\pi-$Hirano inverse. Obviously, $HK = 0$. In light of Lemma 2.6, F has $\pi-$Hirano inverse. According to Lemma 2.7, M has $\pi-$Hirano inverse. Also we compute that $M = \left(\begin{pmatrix} a \\ 1 \end{pmatrix}(1, b)\right)^3$. By using Corollary 2.4, $\left(1, b\right)\begin{pmatrix} a \\ 1 \end{pmatrix}$ has π-Hirano inverse, which implies that $a + b$ has π-Hirano inverse, as required. □

Proposition 2.9. *Let $a, b, ab \in \mathcal{A}^{\pi H}$. If $a^2b = 0$ and $ab^2 = 0$, then $a + b \in \mathcal{A}^{\pi H}$.*

Proof. Since $ab \in \mathcal{A}^{\pi H}$, we see that $ba \in \mathcal{A}^{\pi H}$ by applying Corollary 2.4. As $a^2(ab) = 0$, it follows by Lemma 2.6, that $p = a^2 + ab \in \mathcal{A}^{\pi H}$. Likewise, $q = ba + b^2 \in \mathcal{A}^{\pi H}$. Indeed $pq = 0$. In light of Lemma 2.6, $(a + b)^2 = p + q \in \mathcal{A}^{\pi H}$. According to Corollary 2.5, $a + b \in \mathcal{A}^{\pi H}$. □

We are now ready to prove:

Theorem 2.10. *Let $a, b \in \mathcal{A}^{\pi H}$. If $ab^2 = 0, a^2ba = 0$ and $(ba)^2 = 0$, then $a + b \in \mathcal{A}^{\pi H}$.*

Proof. Let $p = a^2 + ba$ and $q = ab + b^2$. Since $(ba)^2 = 0$, we see that $ba \in \mathcal{A}^{\pi H}$. By Corollary 2.4, $ab \in \mathcal{A}^{\pi H}$. In view of Corollary 2.5, a^2, $b^2 \in \mathcal{A}^{\pi H}$. Since $a^2(ba) = 0$, it follows by Lemma 2.6, that $p \in \mathcal{A}^{\pi H}$. As $ab(b^2) = 0$, we see that $q \in \mathcal{A}^{\pi H}$.

One easily checks that $pqp = 0$, $pq^2 = 0$. According to Lemma 2.7, $(a + b)^2 = p + q \in \mathcal{A}^{\pi H}$. Therefore $a + b \in \mathcal{A}^{\pi H}$, by Corollary 2.5. □

Let $a, b \in \mathcal{A}^{\pi H}$. If $a^2b = 0, bab^2 = 0$ and $(ba)^2 = 0$, then $a + b \in \mathcal{A}^{\pi H}$. This can be proved in a symmetric way in Theorem 2.10. Contrasting to preceding results, we now record the following.

Proposition 2.11. *Let $a, b \in \mathcal{A}^{\pi H}$. If $ab^2 = 0, aba^2 = 0$ and $(ba)^2 = 0$, then $a + b \in \mathcal{A}^{\pi H}$*

Proof. Let $p = a^2 + ab$ and $q = ba + b^2$. Clearly, $ba \in \mathcal{A}^{\pi H}$, and $ab \in \mathcal{A}^{\pi H}$. In view of Corollary 2.5, $a^2, b^2 \in \mathcal{A}^{\pi H}$. Since $(ba)b^2 = 0$, it follows by Lemma 2.6, that $q \in \mathcal{A}^{\pi H}$. As $aba^2 = 0$, we see that $p \in \mathcal{A}^{\pi H}$.

One easily checks that

$$pq^2 = (a^2 + ba)(ba + b^2)^2 = 0, pqp = (a^2 + ab)(ba + b^2)(a^2 + ab) = 0.$$

According to Lemma 2.7, $(a + b)^2 = p + q \in \mathcal{A}^{\pi H}$. Therefore $a + b \in \mathcal{A}^{\pi H}$, by Corollary 2.5. □

It is obvious by Theorem 2.1, that every Hirano invertible element is $\pi-$Hirano invertible. In the next example we show that the converse is not true.

Example 2.12. *Let $\mathcal{A} = \mathbb{C}^{2\times 2}$ and $a = \begin{pmatrix} -i & 1 \\ 0 & 0 \end{pmatrix} \in \mathcal{A}$. Then a has $\pi-$ Hirano inverse but it is not Hirano invertible.*

Proof. It is obvious that $a = a^5$ and so $a - a^5 \in N(\mathcal{A})$. Then by Theorem 2.1, a has $\pi-$Hirano inverse. If a has Hirano inverse, it follows by [3, Theorem 2.1], $a - a^3 = \begin{pmatrix} -2i & 2 \\ 0 & 0 \end{pmatrix}$ is nilpotent. This gives a contradiction. □

3. Operator matrices

To illustrate the preceding results, we are concerned with the $\pi-$Hirano inverse for an operator matrix. Throughout this section, the operator matrix M is given by (1.1), i.e.,

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where $A \in \mathcal{L}(X)^{\pi H}, B \in \mathcal{L}(X,Y), C \in \mathcal{L}(Y,X)$ *and* $D \in \mathcal{L}(Y)^{\pi H}$. Using different splitting approach, we will obtain various conditions for the $\pi-$Hirano inverse of M.

Lemma 3.1. *Let $A, BC \in \mathcal{L}(X)^{\pi H}$. If $ABC = 0$, then*

$$M = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$$

has $\pi-$Hirano inverse.

Proof. Consider the splitting of M,

$$M = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} = P + Q$$

Claim 1. P has $\pi-$Hirano inverse.
Claim 2. Q has $\pi-$Hirano inverse. According to the assumptions, we have,

$$PQ^2 = \begin{pmatrix} 0 & AB \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} = \begin{pmatrix} ABC & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$P^2QP = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & AB \\ 0 & 0 \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$(QP)^2 = \begin{pmatrix} 0 & 0 \\ CA & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ CA & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore

$$PQ^2 = 0, P^2QP = 0, (QP)^2 = 0.$$

Applying Theorem 2.10, $M = P + Q \in \mathcal{A}^{\pi H}$, as asserted. □

Theorem 3.2. *Let $A, BC \in \mathcal{L}(X)^{\pi H}$ and $D \in \mathcal{L}(Y)^{\pi H}$. If $ABC = 0, DCA = 0$ and $DCB = 0$, then*

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

has $\pi-$Hirano inverse.

Proof. Write

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} + \begin{pmatrix} A & B \\ C & 0 \end{pmatrix} = P + Q.$$

Clearly, Q has $\pi-$Hirano inverse. By Theorem 3.1, P has $\pi-$Hirano inverse. We check that

$$PQ^2 = \begin{pmatrix} 0 & 0 \\ DC & 0 \end{pmatrix}\begin{pmatrix} A & B \\ C & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ DCA & DCB \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$P^2QP = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} 0 & 0 \\ DC & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$(QP)^2 = \begin{pmatrix} 0 & BD \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & BD \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Hence we have

$$PQ^2 = 0, P^2QP = 0, (QP)^2 = 0.$$

Applying Theorem 2.10, $M = P + Q \in \mathcal{A}^{\pi H}$. □

Corollary 3.3. *If $ABC = 0$ and $DC = 0$, then $M \in \mathcal{A}^{\pi H}$.*

Proof. If $DC = 0$ then $DCA = 0$ and $DCB = 0$. So we get the result by Theorem 3.2. □

Corollary 3.4. *Let $A, BC \in \mathcal{L}(X)^{\pi H}$ and $D \in \mathcal{L}(Y)^{\pi H}$. If $DCB = 0$ and $AB = 0$, then $M \in \mathcal{A}^{\pi H}$.*

Proof. Applying Corollary 3.3, $\begin{pmatrix} D & C \\ B & A \end{pmatrix} \in \mathcal{A}^{\pi H}$. Observing that

$$M = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} D & C \\ B & A \end{pmatrix} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix},$$

we obtain the result. □

Theorem 3.5. *Let $A, BC \in \mathcal{L}(X)^{\pi H}$ and $D \in \mathcal{L}(Y)^{\pi H}$. If $ABC = 0, BD^2 = 0, ABD = 0$ and $CBD = 0$, then*

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

has $\pi-$ Hirano inverse.

Proof. Clearly, we have

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} = P + Q.$$

Then by Theorem 3.2, P and Q have $\pi-$ Hirano inverse. We compute that

$$PQ^2 = \begin{pmatrix} 0 & BD \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} = \begin{pmatrix} 0 & BD^2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$P^2QP = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix} \begin{pmatrix} 0 & BD \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & 0 \end{pmatrix} = \begin{pmatrix} ABDC & 0 \\ CBDC & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$(QP)^2 = \begin{pmatrix} 0 & 0 \\ DC & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ DC & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

That is,

$$PQ^2 = 0, P^2QP = 0, (QP)^2 = 0.$$

Then by Theorem 2.10, we complete the proof and $P + Q = M \in \mathcal{A}^{\pi H}$. □

Corollary 3.6. *Let $A, BC \in \mathcal{L}(X)^{\pi H}$ and $D \in \mathcal{L}(Y)^{\pi H}$. If $ABC = 0$ and $BD = 0$, then $M \in \mathcal{A}^{\pi H}$.*

Proof. If $BD = 0$ then $ABD = 0$ and $CBD = 0$. So we get the result by Theorem 3.5. □

Corollary 3.7. *Let $A, BC \in \mathcal{L}(X)^{\pi H}$ and $D \in \mathcal{L}(Y)^{\pi H}$. If $DCB = 0$ and $CA = 0$, then $M \in \mathcal{A}^{\pi H}$.*

Proof. Similarly to Corollary 3.4, we complete the proof by Corollary 3.6. □

4. perturbations

Let M be an operator matrix M given by (1.1). It is of interest to consider the $\pi-$Hirano inverse of M under generalized Schur condition $D = CA^{\pi H}B$ (see [9, Theorem 2.1]). Let $W = AA^{\pi H} + A^{\pi H}BCA^{\pi H}$. We now derive

Theorem 4.1. *Let $A \in \mathcal{L}(X)^{\pi H}, D \in \mathcal{L}(Y)^{\pi H}$ and M be given by* (1.1). *If $CAA^{\pi}B = 0, A^2A^{\pi}BC = 0, A^{\pi}BCA^2 = 0, A^{\pi}BCB = 0$ and $D = CA^{\pi H}B$. If AW has $\pi-$Hirano inverse, then $M \in \mathcal{L}(X \oplus Y)^{\pi H}$.*

Proof. Clearly, we have

$$M = \begin{pmatrix} A & B \\ C & CA^{\pi H}B \end{pmatrix} = P + Q$$

where

$$P = \begin{pmatrix} AA^{\pi} & 0 \\ 0 & 0 \end{pmatrix}, \ Q = \begin{pmatrix} A^2A^{\pi H} & B \\ C & CA^{\pi H}B \end{pmatrix}$$

By assumption, we verify that $PQP = 0,\ QPQ = 0,\ P^2Q^2 = 0$ and $PQ^3 = 0$. Obviously, P is nilpotent, and then it has π-Hirano inverse. Moreover, we see that

$$Q = Q_1 + Q_2,\ Q_1 = \begin{pmatrix} A^2A^{\pi H} & AA^{\pi H}B \\ CAA^{\pi H} & CA^{\pi H}B \end{pmatrix},\ Q_2 = \begin{pmatrix} 0 & A^{\pi}B \\ CA^{\pi} & 0 \end{pmatrix}$$

and $Q_2Q_1 = 0$. Since $A^{\pi}BCA^2 = 0$ and $A^{\pi}BCB = 0$, we have

$$\begin{aligned} (A^{\pi}BCA^{\pi})^2 &= A^{\pi}BCBCA^{\pi} - A^{\pi}BCA^2(A^{\pi H})^2BCA^{\pi} \\ &= 0, \\ (CA^{\pi}B)^2 &= CA^{\pi}BC(I - AA^{\pi H})B \\ &= CA^{\pi}BCB - CA^{\pi}BCA^2(A^{\pi H})^2B \\ &= 0. \end{aligned}$$

Therefore $Q_2^4 = 0$. Moreover, we have

$$Q_1 = \begin{pmatrix} AA^{\pi H} \\ CA^{\pi H} \end{pmatrix} \begin{pmatrix} A & AA^{\pi H}B \end{pmatrix}$$

by hypothesis, we see that

$$\begin{pmatrix} A & AA^{\pi H}B \end{pmatrix} \begin{pmatrix} AA^{\pi H} \\ CA^{\pi H} \end{pmatrix} = A^2A^{\pi H} + AA^{\pi H}BCA^{\pi H} = AW$$

has $\pi-$Hirano inverse. Obviously, Q_1 has $\pi-$Hirano inverse. Therefore Q has $\pi-$ Hirano inverse. By virtue of Theorem 2.8, $M \in \mathcal{L}(X \oplus Y)^{\pi H}$, as required. □

Corollary 4.2. *Let $A \in \mathcal{L}(X)^{\pi H}, D \in \mathcal{L}(Y)^{\pi H}$ and M be given by* (1.1). *If $CAA^{\pi}B = 0, A^{\pi}BC = 0$ and $D = CA^{\pi H}B$. If AW has $\pi-$ Hirano inverse, then $M \in \mathcal{L}(X \oplus Y)^{\pi H}$.*

Proof. This is obvious by Theorem 4.1. □

Regarding a complex matrix as the operator matrix on $\mathbb{C} \times \cdots \times \mathbb{C}$, we now present a numerical example to demonstrate Theorem 4.1.

Example 4.3. *Let*

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 \\ 1 & -1 \\ -1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{pmatrix}, D = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

be complex matrices and set

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

then

$$A^{\pi H} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, A^{\pi} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We easily check that

$$CAA^{\pi}B = 0, A^2A^{\pi}BC = 0, A^{\pi}BCA^2 = 0, A^{\pi}BCB = 0$$

and $D = CA^{\pi H}B$. *In this case, A and D have* $\pi-$*Hirano inverses.*

Theorem 4.4. *Let* $A \in \mathcal{L}(X)^{\pi H}, D \in \mathcal{L}(Y)^{\pi H}$ *and M be given by* (1.1). *If* $A^2A^{\pi}BC = 0, BCA^{\pi}BC = 0, CAA^{\pi}BC = 0$ *and* $D = CA^{\pi H}B$. *If AW has* $\pi-$*Hirano inverse, then* $M \in \mathcal{L}(X \oplus Y)^{\pi H}$.

Proof. We easily see that

$$M = \begin{pmatrix} A & B \\ C & CA^{\pi H}B \end{pmatrix} = P + Q$$

where

$$P = \begin{pmatrix} A & AA^{\pi H}B \\ C & CA^{\pi H}B \end{pmatrix}, Q = \begin{pmatrix} 0 & A^{\pi}B \\ 0 & 0 \end{pmatrix}$$

then we check that $P^2QP = 0, (QP)^2 = 0, Q^2 = 0$. Clearly, Q has $\pi-$Hirano inverse. Moreover, we have

$$P = P_1 + P_2,\ P_1 = \begin{pmatrix} A^2A^{\pi H} & AA^{\pi H}B \\ CAA^{\pi H} & CA^{\pi H}B \end{pmatrix}, P_2 = \begin{pmatrix} AA^{\pi} & 0 \\ CA^{\pi} & 0 \end{pmatrix}$$

$P_2P_1 = 0$ and P_2 is nilpotent. We easily check that

$$P_1 = \begin{pmatrix} AA^{\pi H} \\ CA^{\pi H} \end{pmatrix} \begin{pmatrix} A & AA^{\pi H}B \end{pmatrix}.$$

By hypothesis, we see that

$$\begin{pmatrix} A & AA^{\pi H}B \end{pmatrix} \begin{pmatrix} AA^{\pi H} \\ CA^{\pi H} \end{pmatrix} = A^2A^{\pi H} + AA^{\pi H}BCA^{\pi H}.$$

By hypothesis, $AW = A^2A^{\pi H} + AA^{\pi H}BCA^{\pi H}$ has $\pi-$Hirano inverse. Therefore P_1 has $\pi-$Hirano inverse. According to Theorem 2.10, $M \in \mathcal{L}(X \oplus Y)^{\pi H}$. □

Corollary 4.5. *Let* $A \in \mathcal{L}(X)^{\pi H}, D \in \mathcal{L}(Y)^{\pi H}$ *and M be given by* (1.1). *If* $A^{\pi}BC = 0$ *and* $D = CA^{\pi H}B$. *If AW has* $\pi-$*Hirano inverse, then* $M \in \mathcal{L}(X \oplus Y)^{\pi H}$.

Proof. This is obvious by Theorem 4.4. □

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