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Algorithms of Common Solutions to Modified Generalized System of Variational Inclusion Problem and Hierarchical Fixed Point Problem

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Abstract. This manuscript deals with two problems : the first one is a new problem of the system of variational inclusion that is called modified generalized system of variational inclusion problem(MGSVIP) and the other one is a hierarchical fixed point problem in the framework of real Hilbert space. We establish the important lemma that show the relation between fixed point of nonlinear mapping and solution of MGSVIP for proving the main theorem. To approximate the common solution of these problems, we design an iterative scheme under suitable conditions on parameters. A strong convergence result for the proposed iterative scheme is proved. Applying our main result, we prove strong convergence theorems of the modification system of variational inequalities problem and variational inclusion problem. Moreover, we give the numerical example for supporting our results.

1. Introduction

Let H be a real Hilbert space and C be a nonempty closed convex subset of a real space H with the inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. Let $T : C \to C$ be a mapping. Then, T is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in C$. We denote $F(T)$ by the set of fixed points of T, that is $F(T) = \{x \in C : Tx = x\}$. It is well known that $F(T)$ is closed convex and also nonempty.

The variational inequality problem is to find a point $u \in C$ such that

$$\langle Au, v - u \rangle \geq 0, \qquad \forall v \in C. \tag{1}$$

The set of the solutions of the variational inequality problem is denoted by $VI(C, A)$. It is known that variational inequality, as a greatly important tool, has already been studied for a wide class of unilateral, obstacle, and equilibrium problems arising in several branches of pure and applied sciences in a unified and general framework. Many numerical methods have been developed for solving variational inequalities and some related optimization problems; see [4], [5] and the references therein.

By using the concept of the variational inequality problem and fixed point problem, Moudafi and Mainge [1] introduced and studied the following hierarchical fixed point problem(in short, HFPP) for a nonexpansive mapping T with respect to another nonexpansive mapping S on C: Find $x^* \in F(T)$ such that

$$\langle (I - S)x^*, x - x^* \rangle \geq 0, \qquad \forall x \in F(T), \tag{2}$$

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where $S : C \to C$ is a nonexpansive mapping. The solution set of HFPP (2) is denoted by Φ ,i.e., $\Phi = \{x^* \in F(T) : \langle (I - S)x^*, x - x^* \rangle \geq 0,\ \forall x \in F(T)\}$. We note that HFPP(2) covers monotone variational inequality on fixed point sets, minimization problems over equilibrium constraints, hierarchical minimization problem, etc.

One of the important method to solve the hierarchical fixed point problem (2) for nonexpansive mapping S, T on a subset C of a Hilbert space H is Krasnoselki-Mann algorithm which was introduced by Moudafi [2], as follows:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n(\Sigma_n S x_n + (1 - \Sigma_n)T x_n), \quad \forall n \geq 0, \tag{3}$$

where $\{\alpha_n\}$ and $\{\Sigma_n\}$ are two real sequence in (0,1). It is worth mentioning that some algorithms in signal processing and image reconstruction may be written as the Krasnoselki-Mann algorithm.

Let $B : H \to H$ be a mapping and $M : H \to 2^H$ be a multi-valued mapping. The variational inclusion problem is to find $x \in H$ such that

$$\theta \in Bx + Mx, \tag{4}$$

where θ is zero vector in H. The set of the solution of (4) is denoted by $VI(H, B, M)$. This problem has received much attention due to its applications in large variety of problems arising in convex programming, variational inequalities, split feasibility problem, and minimization problem. To be more precise, some concrete problems in machine learning, image processing, and linear inverse problem can be modeled mathematically as this formulation.

A multi-valued mapping $M : H \to 2^H$ is called monotone, if for all $x, y \in H$, $u \in Mx$ and $v \in My$ implies that $\langle u - v, x - y \rangle \geq 0$. A multi-valued mapping $M : H \to 2^H$ is called maximal monotone, if it is monotone and if for any $(x, u) \in H \times H$, $\langle u - v, x - y \rangle \geq 0$ for every $(y, v) \in Graph(M)$ $(Graph(M) := \{(x, u) \in H \times H : u \in Mx\})$ implies that $u \in Mx$.

Let $M : H \to 2^H$ be a multi-valued maximal monotone mapping, then the single-valued mapping $J_{M,\lambda} : H \to H$ defined by

$$J_{M,\lambda}(u) = (I + \lambda M)^{-1}(u), \ \ \forall u \in H,$$

is called the resolvent operator associated with M where λ is positive number and I is an identity mapping, see [3]. Note that $J_{M,\lambda}$ is nonexpansive mapping.

In 2008, Zhang et al.[3] proved a strong convergence theorem for finding a common element of the set of solutions of variational inclusion problem and the set of fixed points of nonexpansive mappings in Hilbert space. They introduced the iterative scheme as follows:

$$\begin{aligned} y_n &= J_{M,\lambda}(x_n - \lambda A x_n), \\ x_{n+1} &= \alpha x + (1 - \alpha_n) S y_n, \quad \forall n \geq 0, \end{aligned}$$

and proved a strong convergence theorem of the sequence $\{x_n\}$ under suitable conditions of parameter $\{\alpha_n\}$ and λ.

Motivated by problem (4), we introduce a new problem of the system of varitional inclusion in a real Hilbert space as follows:

Let a real Hilbert space H and let $A, B : H \to H$ be mappings and $M_A, M_B : H \to 2^H$ be set value mappings. We consider the problem of finding $x^* \in H$ such that

$$\theta \in Ax^* + M_A x^* \ \ and \ \ \theta \in Bx^* + M_B x^*, \tag{5}$$

where θ is zero vector in H, which is called modified generalized system of variational inclusion problem(in short, MGSVIP) . The set of solution of (5) is denoted by Ω ,i.e., $\Omega = \{x^* \in H : \theta \in Ax^* + M_A x^* \ \ and \ \ \theta \in Bx^* + M_B x^*\}$. In particular, if $A = B$ and $M_A = M_B$, then the problem (5) reduces to the problem (4).

The paper is organised as follows. In Section 2, we recall some basic concepts and establish lemma 2.8 that show the relation between fixed point of nonlinear mapping and solution of MGSVIP under suitable conditions on parameters. Moreover, we give some examples to support Lemma 2.8 and show that Lemma

2.8 is not true if some condition fails. In Section 3, we prove the strong convergence theorem for finding common element of the solution sets of HFPP(2) and MGSVIP(5) under some proper conditions. In Section 4, we apply our main theorem to prove strong convergence theorem for finding solutions of modification system of variational inequalities problem and variational inclusion problem. In Section 5, we give a numerical example for supporting our result.

2. Preliminaries

In this section, we give some useful lemmas that will be needed to prove our main result.

Let C be a nonempty closed convex subset of a real Hilbert space H. We denote weak convergence and strong convergence by notations $\rightharpoonup$and $\to$, respectively. For every $x \in H$, there exists a unique nearest point $P_C xinC$ such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C.$$

P_C is called metric projection of H onto C.

Lemma 2.1. *[6] Given $x \in H$ and $y \in C$. Then, $y = P_C x$ if and only if there holds the inequality*

$$\langle x - y, y - z\rangle \geq 0, \quad \forall z \in C.$$

Lemma 2.2. *In real Hilbert spaces H, the following well-known results hold:*
(i) For all $x, y \in H$ and $\alpha \in [0, 1]$,

$$\|\alpha x + (1 - \alpha)y\|^2 = \alpha\|x\|^2 + (1 - \alpha)\|y\|^2 - \alpha(1 - \alpha)\|x - y\|^2,$$

(ii) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$ for all $x, y \in H$.

Lemma 2.3. *[11] Let C be a nonempty closed and convex subset of a real Hilbert space H. If $T : C \to C$ is a nonexpansive mapping with $Fix(T) \neq \emptyset$, then the mapping $I - T$ is demiclosed at 0, i.e., if $\{x_n\}$ is a sequence in C weakly converging to $x \in C$ and if $\{x_n - Tx_n\}$ converges strongly to 0, then $x \in Fix(T)$.*

Lemma 2.4. *[9] Let $\{a_n\}, \{c_n\} \subset \mathbb{R}^+, \{\alpha_n\} \subset (0, 1)$ and $\{b_n\} \subset \mathbb{R}$ be sequences such that*

$$a_{n+1} = (1 - \alpha_n)a_n + b_n + c_n, \quad \forall n \geq 0$$

Assume $\sum_{n=0}^{\infty} c_n < \infty$. Then the following results hold:
(a) if $b_n \leq \alpha_n C$ where $C \geq 0$, then $\{a_n\}$ is a bounded sequence.
(b) if $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\limsup_{n\to 0} \frac{b_n}{\alpha_n} \leq 0$, then $\lim_{n\to 0} a_n = 0$.

Lemma 2.5. *[7] Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \leq (1 - \alpha_n)s_n + \delta_n, \quad \forall \geq 0,$$

where $\{\alpha_n\}$ is a sequence in (0,1) and $\{\delta_n\}$ is a sequence such that
(1) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(2) $\limsup_{n\to 0} \frac{\delta_n}{\alpha_n} \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n| = \infty$;
Then, $\lim_{n\to 0} s_n = 0$.

Lemma 2.6. *Each Hilbert space H satisfies Opial's condition, i.e., for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality*

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for every $y \in H$ with $x \neq y$.

Lemma 2.7. *[3] $u \in H$ is a solution of variational inclusion (4) if and only if $u = J_{M,\lambda}(u - \lambda Bu)$, $\forall \lambda > 0$, i.e.,*

$$VI(H, B, M) = Fix(J_{M,\lambda}(I - \lambda B)), \forall \lambda > 0,$$

if $\lambda \in (0, 2\alpha]$, then $VI(H, B, M)$ is closed convex subset in H.

The next lemma present associate between fixed point of nonlinear mapping and solution of MGSVIP under suitable conditions on parameters.

Lemma 2.8. *Let H be a real Hilbert space and let $A, B : H \to H$ be α and β-inverse strongly monotone mappings with $\eta = min\{\alpha, \beta\}$. Let $M_A, M_B : H \to 2^H$ be multivalue maximal monotone mappings with $\Omega \neq \emptyset$.*
If $x^* \in \Omega$ if and only if $x^* = Qx^*$, where $Q : H \to H$ be a mapping defined by

$$Q(x) = J_{M_A,\lambda_A}(I - \lambda_A A)(ax + (1 - a)J_{M_B,\lambda_B}(I - \lambda_B B)x)$$

for all $x \in H$, $a \in (0, 1)$ and $\lambda_A, \lambda_B \in (0, 2\eta)$. Moreover, we have Q is a nonexpansive mapping.

Proof. Let conditions hold.
($\to$) Let $x^* \in \Omega$, we have $x \in H$ such that $\theta \in Ax^* + M_A x^*$ and $\theta \in Bx^* + M_B x^*$,
that is $x^* \in VI(H, A, M_A)$ and $x^* \in VI(H, B, M_B)$.
From lemma 2.7, we have
$x^* \in Fix(J_{M_A,\lambda_A}(I - \lambda_A A))$ and $x^* \in Fix(J_{M_B,\lambda_B}(I - \lambda_B B))$. It implies that

$$x^* = J_{M_A,\lambda_A}(I - \lambda_A A)x^* \tag{6}$$

and

$$x^* = J_{M_B,\lambda_B}(I - \lambda_B B)x^*. \tag{7}$$

By definition of Q ,(6) and (7) we have

$$\begin{aligned} Q(x^*) &= J_{M_A,\lambda_A}(I - \lambda_A A)(ax^* + (1 - a)J_{M_B,\lambda_B}(I - \lambda_B B)x^*) \\ &= x^*. \end{aligned}$$

($\leftarrow$) Let $x^* \in Q(x^*)$.
We will show that $J_{M_A,\lambda_A}(I - \lambda_A A)$ and $J_{M_B,\lambda_B}(I - \lambda_B B)$ are nonexpansive mapping.
Since A, B are α and β-inverse strongly monotone mappings with $\eta = min\{\alpha, \beta\}$,
we have

$$\begin{aligned} \|J_{M_A,\lambda_A}(I - \lambda_A A)x - J_{M_A,\lambda_A}(I - \lambda_A A)y\|^2 &\leq \|(I - \lambda_A A)x - (I - \lambda_A A)y\|^2 \\ &= \|(x - y) - \lambda_A(Ax - Ay)\|^2 \\ &= \|x - y\|^2 - 2\lambda_A\langle x - y, Ax - Ay\rangle + \lambda_A^2\|Ax - Ay\|^2 \\ &\leq \|x - y\|^2 - 2\lambda_A\alpha\|Ax - Ay\|^2 + \lambda_A^2\|Ax - Ay\|^2 \\ &\leq \|x - y\|^2 - \lambda_A(\lambda_A - 2\eta)\|Ax - Ay\|^2 \\ &\leq \|x - y\|^2. \end{aligned}$$

Hence, we obtain $J_{M_A,\lambda_A}(I - \lambda_A A)$ is nonexpansive mapping.
Similarly, we can show that $J_{M_B,\lambda_B}(I - \lambda_B B)$ is also nonexpansive mapping.
Since $x^* \in Q(x^*)$, we have

$$x^* = Q(x^*) = J_{M_A,\lambda_A}(I - \lambda_A A)(ax^* + (1 - a)J_{M_B,\lambda_B}(I - \lambda_B B)x^*).$$

Let $y \in \Omega$, we have $\theta \in Ay + M_A y$ and $\theta \in By + M_B y$.
From Lemma 2.7, it implies that

$y \in Fix(J_{M_A,\lambda_A}(I-\lambda_A A)) \cap Fix(J_{M_B,\lambda_B}(I-\lambda_B B))$. Then

$$\begin{aligned}
\|x^*-y\|^2 &= \|J_{M_A,\lambda_A}(I-\lambda_A A)(ax^*+(1-a)J_{M_B,\lambda_B}(I-\lambda_B B)x^*)-y\|^2 \\
&= \|J_{M_A,\lambda_A}(I-\lambda_A A)(ax^*+(1-a)J_{M_B,\lambda_B}(I-\lambda_B B)x^*)-J_{M_A,\lambda_A}(I-\lambda_A A)y\|^2 \\
&\leq \|(ax^*+(1-a)J_{M_B,\lambda_B}(I-\lambda_B B)x^*)-y\|^2 \\
&= \|a(x^*-y)+(1-a)(J_{M_B,\lambda_B}(I-\lambda_B B)x^*-y)\|^2 \\
&= a\|x^*-y\|^2+(1-a)\|J_{M_B,\lambda_B}(I-\lambda_B B)x^*-y\|^2 \\
&\quad -a(1-a)\|x^*-J_{M_B,\lambda_B}(I-\lambda_B B)x^*\|^2 \\
&\leq a\|x^*-y\|^2+(1-a)\|x^*-y\|^2-a(1-a)\|x^*-J_{M_B,\lambda_B}(I-\lambda_B B)x^*\|^2 \\
&= \|x^*-y\|^2-a(1-a)\|x^*-J_{M_B,\lambda_B}(I-\lambda_B B)x^*\|^2.
\end{aligned}$$

It impiles that $\|x^*-J_{M_B,\lambda_B}(I-\lambda_B B)x^*\|=0$.
That is $x^* \in Fix(J_{M_B,\lambda_B}(I-\lambda_B B))$.
Since $x^*=Q(x^*)$ and $x^* \in Fix(J_{M_B,\lambda_B}(I-\lambda_B B))$.
We have

$$\begin{aligned}
x^* &= Q(x^*) \\
&= J_{M_A,\lambda_A}(I-\lambda_A A)(ax^*+(1-a)x^*) \\
&= J_{M_A,\lambda_A}(I-\lambda_A A)x^*.
\end{aligned}$$

Therefore $x^* \in Fix(J_{M_A,\lambda_A}(I-\lambda_A A))$.
From Lemma 2.7, $x^* \in Fix(J_{M_A,\lambda_A}(I-\lambda_A A))$ and $x^* \in Fix(J_{M_B,\lambda_B}(I-\lambda_B B))$, we have
$\theta \in Ax^*+M_A x^*$ and $\theta \in Bx^*+M_B x^*$.
Then $x^* \in \Omega$.
Next, we claim that Q is nonexpansive mapping. From the definition of Q and $J_{M_A,\lambda_A}(I-\lambda_A A)$ and $J_{M_B,\lambda_B}(I-\lambda_B B)$ are nonexpansive mapping, we have

$$\begin{aligned}
\|Q(x)-Q(y)\| &= \|J_{M_A,\lambda_A}(I-\lambda_A A)(ax+(1-a)J_{M_B,\lambda_B}(I-\lambda_B B)x) \\
&\quad -J_{M_A,\lambda_A}(I-\lambda_A A)(ay+(1-a)J_{M_B,\lambda_B}(I-\lambda_B B)y)\| \\
&\leq \|(ax+(1-a)J_{M_B,\lambda_B}(I-\lambda_B B)x)-(ay+(1-a)J_{M_B,\lambda_B}(I-\lambda_B B)y)\| \\
&= \|a(x-y)+(1-a)[J_{M_B,\lambda_B}(I-\lambda_B B)x)-J_{M_B,\lambda_B}(I-\lambda_B B)y)]\| \\
&\leq a\|x-y\|+(1-a)\|J_{M_B,\lambda_B}(I-\lambda_B B)x)-J_{M_B,\lambda_B}(I-\lambda_B B)y)\| \\
&\leq a\|x-y\|+(1-a)\|x-y\| \\
&= \|x-y\|.
\end{aligned}$$

Hence Q is nonexpansive mapping.

We give some examples to support Lemma 2.8 and show that Lemma 2.8 is not true if some condition fails.

Example 2.9. *Let $\mathbb{R}$ be the set of real numbers and A, B: $\mathbb{R} \to \mathbb{R}$ be defined by $Ax = x-5$ and $Bx = \frac{x}{2}$, for all $x \in \mathbb{R}$. Let $M_A : \mathbb{R} \to 2^{\mathbb{R}}$ be defined by $M_A x = \{2x-1\}$ for all $x \in \mathbb{R}$ and $M_B : \mathbb{R} \to 2^{\mathbb{R}}$ be defined by $M_B x = \{\frac{3x}{2}-4\}$ for all $x \in \mathbb{R}$.*

Solution *It's obvious that $\Omega = 2$. Choose $\lambda_A = \frac{1}{2}$. From $M_A(x) = \{2x-1\}$ and the resolvent of M_A , $J_{M_A,\frac{1}{2}}x = (I+\frac{1}{2}M_A)^{-1}x$ for all $x \in \mathbb{R}$, we have*

$$J_{M_A,\frac{1}{2}}(x) = \frac{x}{2}+\frac{1}{4}, \tag{8}$$

for all $x \in \mathbb{R}$. Choose $\lambda_B = 1$. From $M_B(x) = \{2x-1\}$ and the resolvent of M_B , $J_{M_B,1}x = (I+1M_B)^{-1}x$ for all $x \in \mathbb{R}$, we have

$$J_{M_B,1}(x) = \frac{2x}{5}+\frac{8}{5}, \tag{9}$$

for all $x \in \mathbb{R}$*. From definitions of A and B, we have*

$$\langle (x-5)-(y-5), x-y\rangle \quad \geq \quad (1)\|(x-5)-(y-5)\|^2, \tag{10}$$

and

$$\langle \frac{x}{2}-\frac{y}{2}, x-y\rangle \quad \geq \quad (2)\|\frac{x}{2}-\frac{y}{2}\|^2, \tag{11}$$

for all $x \in \mathbb{R}$*. From* (10) *and* (11)*, then A and B are 1-inverse strongly monotone mapping and 2-inverse strongly monotone mapping, respectively.*
Choose $a = 0.5$*. From* (8) *and* (9)*, we have*

$$\begin{aligned} Q(x) &= J_{M_A,\frac{1}{2}}(I-\frac{1}{2}A)(0.5x+0.5J_{M_B,1}(I-1B)x) \\ &= \frac{3x}{20}+\frac{34}{20}. \end{aligned}$$

Then, we have $2 \in F(Q)$*.*

Example 2.10. *Let* $\mathbb{R}$ *be the set of real numbers and* $A, B\colon \mathbb{R} \to \mathbb{R}$ *defined by* $Ax = x-5$ *and* $Bx = \frac{x}{2}$*, for all* $x \in \mathbb{R}$*. Let* $M_A : \mathbb{R} \to 2^{\mathbb{R}}$ *defined by* $M_A x = \{2x-1\}$ *for all* $x \in \mathbb{R}$ *and* $M_B : \mathbb{R} \to 2^{\mathbb{R}}$ *defined by* $M_B x = \{\frac{3x}{2}-4\}$ *for all* $x \in \mathbb{R}$*.*
Solution *It's obvious that* $\Omega = 2$*. Choose* $\lambda_A = 2$*. From* $M_A(x) = \{2x-1\}$ *and the resolvent of* M_A *,* $J_{M_A,2}x = (I+2M_A)^{-1}x$ *for all* $x \in \mathbb{R}$*, we have*

$$J_{M_A,2}(x) = \frac{x}{5}+\frac{2}{5}, \tag{12}$$

for all $x \in \mathbb{R}$*. Choose* $\lambda_B = 4$*. From* $M_A(x) = \{2x-1\}$ *and the resolvent of* M_B *,* $J_{M_B,4}x = (I+4M_B)^{-1}x$ *for all* $x \in \mathbb{R}$*, we have*

$$J_{M_B,4}(x) = \frac{x}{7}+\frac{16}{7}, \tag{13}$$

for all $x \in \mathbb{R}$*. From Example* 2.9*, we have A and B are 1-inverse strongly monotone mapping and 2-inverse strongly monotone mapping, respectively. Choose* $a = 0.5$*. From* (12) *and* (13)*, we have*

$$\begin{aligned} Q(x) &= J_{M_A,2}(I-1A)(0.5x+0.5J_{M_B,4}(I-4B)x) \\ &= \frac{-3}{5}. \end{aligned}$$

Then, we have $2 \notin F(Q)$*.*

3. Main Result

In this section, we prove strong convergence of the sequence acquired from the proposed iterative methods for finding a common element of the set of hierarchical fixed point problem and the set of solution of the proposed problem.

Theorem 3.1. *Let H be a real Hilbert space. Let* $S,T : H \to H$ *be two nonexpansive mappings. Let* $A,B : H \to H$ *be* α *and* β *- inverse strongly monotone mappings with* $\eta = min\{\alpha, \beta\}$*. Define the mapping* $Q : H \to H$ *by* $Q(x) = J_{M_A,\lambda_A}(I-\lambda_A A)(ax+(1-a)J_{M_B,\lambda_B}(I-\lambda_B B)x)$ *for all* $x \in H$ *,* $a \in (0,1)$ *and* $\lambda_A, \lambda_B \in (0, 2\eta)$*. Assume that* $\Gamma = \Phi \cap \Omega \neq \emptyset$*. Let* $\{u_n\}$ *and* $\{x_n\}$ *be generated by iterative algorithm :*

$$\begin{cases} x_0 \in H; \\ u_n = (1-\beta_n)x_n + \beta_n(\sigma_n S x_n + (1-\sigma_n)T x_n) \\ x_{n+1} = \alpha_n z + (1-\alpha_n)Q u_n \end{cases} \tag{14}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\sigma_n\} \in (0,\ 1)$ *and* $0 < a < \beta_n < b < 1$, *for some* $a, b > 0$.
Suppose the following conditions hold:
(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$
(ii) $\sum_{n=0}^{\infty} \sigma_n < \infty$
(iii) $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$,$\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$
(iv) $\lim_{n\to\infty} \frac{\|x_n - u_n\|}{\beta_n \sigma_n} = 0$.
Then $\{x_n\}$ *converges strongly to* $x^* \in \Gamma$ *where* $x^* = P_\Gamma z$.

Proof. We divide the proof into five steps:
Step 1. We show that $\{x_n\}$ and $\{u_n\}$ are bounded.
Let $x^* \in \Gamma$ From the definition of $\{u_n\}$, we have

$$\begin{aligned}
\|u_n - x^*\| &= \|(1-\beta_n)x_n + \beta_n(\sigma_n S x_n + (1-\sigma_n)T x_n) - x^*\| \\
&= \|(1-\beta_n)(x_n - x^*) + \beta_n(\sigma_n(S x_n - x^*) + (1-\sigma_n)(T x_n - x^*))\| \\
&\leq (1-\beta_n)\|x_n - x^*\| + \beta_n\sigma_n\|S x_n - x^*\| + \beta_n(1-\sigma_n)\|T x_n - x^*\| \\
&\leq (1-\beta_n)\|x_n - x^*\| + \beta_n\sigma_n\|S x_n - x^*\| + \beta_n(1-\sigma_n)\|x_n - x^*\| \\
&= (1-\beta_n\sigma_n)\|x_n - x^*\| + \beta_n\sigma_n\|S x_n - x^*\| \\
&= (1-\beta_n\sigma_n)\|x_n - x^*\| + \beta_n\sigma_n\|S x_n - S x^* + S x^* - x^*\| \\
&\leq (1-\beta_n\sigma_n)\|x_n - x^*\| + \beta_n\sigma_n\|S x_n - S x^*\| + \beta_n\sigma_n\|S x^* - x^*\| \\
&\leq (1-\beta_n\sigma_n)\|x_n - x^*\| + \beta_n\sigma_n\|x_n - x^*\| + \beta_n\sigma_n\|S x^* - x^*\| \\
&= \|x_n - x^*\| + \beta_n\sigma_n\|S x^* - x^*\| \qquad (15)
\end{aligned}$$

From the definition of $\{x_n\}$ and (15), we have

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \|\alpha_n z + (1-\alpha_n)Q u_n + \alpha_n x^* - \alpha_n x^* - x^*\| \\
&= \|\alpha_n(z - x^*) + (1-\alpha_n)(Q u_n - x^*)\| \\
&\leq \alpha_n\|z - x^*\| + (1-\alpha_n)\|Q u_n - x^*\| \\
&\leq \alpha_n\|z - x^*\| + (1-\alpha_n)\|u_n - x^*\| \\
&\leq \alpha_n\|z - x^*\| + (1-\alpha_n)[\|x_n - x^*\| + \beta_n\sigma_n\|S x^* - x^*\|] \\
&= (1-\alpha_n)\|x_n - x^*\| + \alpha_n\|z - x^*\| + (1-\alpha_n)\beta_n\sigma_n\|S x^* - x^*\| \\
&\leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n\|z - x^*\| + \beta_n\sigma_n\|S x^* - x^*\|
\end{aligned}$$

From the condition (ii) and Lemma 2.4 (a), we conclude that the sequence $\{x_n\}$ is bounded and so are $\{u_n\}$, $\{Q x_n\}$ and $\{T x_n\}$.

Step 2. We will show that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$. By the definition of $\{u_n\}$,we obtain

$$\begin{aligned}
\|u_n-u_{n-1}\| &= \|(1-\beta_n)x_n+\beta_n(\sigma_nSx_n+(1-\sigma_n)Tx_n)\\
&\quad -[(1-\beta_{n-1})x_{n-1}+\beta_{n-1}(\sigma_{n-1}Sx_{n-1}+(1-\sigma_{n-1})Tx_{n-1})]\|\\
&= \|(1-\beta_n)(x_n-x_{n-1})+(\beta_n-\beta_{n-1})(x_{n-1}-Tx_{n-1})+\beta_n\sigma_n(Sx_n-Sx_{n-1})\\
&\quad +(\beta_n-\beta_n\sigma_n)(Tx_n-Tx_{n-1})+(\beta_n\sigma_n-\beta_{n-1}\sigma_{n-1})(Sx_{n-1}-Tx_{n-1})\|\\
&\le (1-\beta_n)\|x_n-x_{n-1}\|+|\beta_n-\beta_{n-1}|\|x_{n-1}-Tx_{n-1}\|+\beta_n\sigma_n\|Sx_n-Sx_{n-1}\|\\
&\quad +(\beta_n-\beta_n\sigma_n)\|Tx_n-Tx_{n-1}\|+|\beta_n\sigma_n-\beta_{n-1}\sigma_{n-1}|\|Sx_{n-1}-Tx_{n-1}\|\\
&\le (1-\beta_n)\|x_n-x_{n-1}\|+|\beta_n-\beta_{n-1}|\|x_{n-1}-Tx_{n-1}\|+\beta_n\sigma_n\|x_n-x_{n-1}\|\\
&\quad +\beta_n(1-\sigma_n)\|x_n-x_{n-1}\|+|\beta_n\sigma_n-\beta_{n-1}\sigma_{n-1}|\|Sx_{n-1}-Tx_{n-1}\|\\
&= \|x_n-x_{n-1}\|+|\beta_n-\beta_{n-1}|\|x_{n-1}-Tx_{n-1}\|\\
&\quad +|\beta_n\sigma_n-\beta_{n-1}\sigma_{n-1}|\|Sx_{n-1}-Tx_{n-1}\|\\
&= \|x_n-x_{n-1}\|+|\beta_n-\beta_{n-1}|\|x_{n-1}-Tx_{n-1}\|\\
&\quad +|\beta_n(\sigma_n-\sigma_{n-1})-\sigma_{n-1}(\beta_n-\beta_{n-1})|\|Sx_{n-1}-Tx_{n-1}\|\\
&\le \|x_n-x_{n-1}\|+|\beta_n-\beta_{n-1}|\|x_{n-1}-Tx_{n-1}\|\\
&\quad +\beta_n|\sigma_n-\sigma_{n-1}|\|Sx_{n-1}-Tx_{n-1}\|+\sigma_{n-1}|\beta_n-\beta_{n-1}|\|Sx_{n-1}-Tx_{n-1}\|\\
&\le \|x_n-x_{n-1}\|+|\beta_n-\beta_{n-1}|\|x_{n-1}-Tx_{n-1}\|\\
&\quad +|\sigma_n-\sigma_{n-1}|\|Sx_{n-1}-Tx_{n-1}\|+|\beta_n-\beta_{n-1}|\|Sx_{n-1}-Tx_{n-1}\| \qquad (16)
\end{aligned}$$

From the definition of $\{x_n\}$ and (16), we have

$$\begin{aligned}
\|x_n-x_{n-1}\| &= \|\alpha_nz+(1-\alpha_n)Qu_n-[\alpha_{n-1}z+(1-\alpha_{n-1})Qu_{n-1}]\|\\
&= \|(\alpha_n-\alpha_{n-1})(z-Qu_{n-1})+(1-\alpha_n)(Qu_n-Qu_{n-1}\|\\
&\le |\alpha_n-\alpha_{n-1}|\|z-Qu_{n-1}\|+(1-\alpha_n)\|Qu_n-Qu_{n-1}\|\\
&\le |\alpha_n-\alpha_{n-1}|\|z-Qu_{n-1}\|+(1-\alpha_n)\|u_n-u_{n-1}\|\\
&\le |\alpha_n-\alpha_{n-1}|\|z-Qu_{n-1}\|+(1-\alpha_n)[\|x_n-x_{n-1}\|+|\beta_n-\beta_{n-1}|\|x_{n-1}-Tx_{n-1}\|\\
&\quad +|\sigma_n-\sigma_{n-1}|\|Sx_{n-1}-Tx_{n-1}\|+|\beta_n-\beta_{n-1}|\|Sx_{n-1}-Tx_{n-1}\|]\\
&\quad +|\beta_n-\beta_{n-1}|\|Sx_{n-1}-Tx_{n-1}\|]\\
&\le (1-\alpha_n)\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|\|z-Qu_{n-1}\|\\
&\quad +|\beta_n-\beta_{n-1}|\|x_{n-1}-Tx_{n-1}\|+|\sigma_n-\sigma_{n-1}|\|Sx_{n-1}-Tx_{n-1}\|\\
&\quad +|\beta_n-\beta_{n-1}|\|Sx_{n-1}-Tx_{n-1}\|\\
&\le (1-\alpha_n)\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|M_1+|\beta_n-\beta_{n-1}|M_1+|\sigma_n-\sigma_{n-1}|M_1\\
&\quad +|\beta_n-\beta_{n-1}|M_1, \qquad (17)
\end{aligned}$$

where $M_1:=\mathbf{Max}_{n\in\mathbb{N}}\{\|z-Qu_n\|,\|x_n-Tx_n\|,\|Sx_n-Tx_n\|\}$.
Applying lemma 2.5, (17) and the conditions (i),(ii), we have

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \qquad (18)$$

Step 3. We will show that $\lim_{n\to\infty}\|u_n-Qu_n\|=0, \lim_{n\to\infty}\|u_n-x_n\|=0, \lim_{n\to\infty}\|x_n-Tx_n\|=0$.
From the definition of x_n, we have

$$\begin{aligned}
x_{n+1}-x_n &= \alpha_nz+(1-\alpha_n)Qu_n-x_n\\
&= \alpha_n(z-x_n)+(1-\alpha_n)(Qu_n-x_n). \qquad (19)
\end{aligned}$$

From the condition (i) , (18) and (19), we have

$$\lim_{n\to\infty}\|Qu_n-x_n\|=0. \qquad (20)$$

From the definition of u_n, we obtain

$$\begin{aligned}
\|u_n - x^*\|^2 &= \|(1-\beta_n)x_n + \beta_n(\sigma_n Sx_n + (1-\sigma_n)Tx_n) - x^*\|^2\\
&= \|(1-\beta_n)(x_n - x^*) + \beta_n(\sigma_n(Sx_n - x^*) + (1-\sigma_n)(Tx_n - x^*))\|^2\\
&= (1-\beta_n)\|x_n - x^*\|^2 + \beta_n\|\sigma_n(Sx_n - x^*) + (1-\sigma_n)(Tx_n - x^*)\|^2\\
&\quad -\beta_n(1-\beta_n)\|x_n - x^* - (\sigma_n(Sx_n - x^*) + (1-\sigma_n)(Tx_n - x^*))\|^2\\
&\leq (1-\beta_n)\|x_n - x^*\|^2 + \beta_n[\sigma_n\|Sx_n - x^*\|^2 + (1-\sigma_n)\|Tx_n - x^*\|^2\\
&\quad -\sigma_n(1-\sigma_n)\|Sx_n - Tx_n\|^2] - \beta_n(1-\beta_n)\|\sigma_n(Sx_n - x_n) + (1-\sigma_n)(Tx_n - x_n)\|^2\\
&\leq (1-\beta_n)\|x_n - x^*\|^2 + \beta_n\sigma_n\|Sx_n - x^*\|^2 + \beta_n\|x_n - x^*\|^2\\
&\quad -\beta_n\sigma_n(1-\sigma_n)\|Sx_n - Tx_n\|^2\\
&\quad -\beta_n(1-\beta_n)\|\sigma_n(Sx_n - x_n) + (1-\sigma_n)(Tx_n - x_n)\|^2\\
&\leq \|x_n - x^*\|^2 + \beta_n\sigma_n\|Sx_n - x^*\|^2\\
&\quad -\beta_n(1-\beta_n)\|\sigma_n(Sx_n - x_n) + (1-\sigma_n)(Tx_n - x_n)\|^2. \qquad (21)
\end{aligned}$$

From the definition of x_n and (21), we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|\alpha_n z + (1-\alpha_n)Qu_n - x^*\|^2\\
&= \|\alpha_n(z - x^*) + (1-\alpha_n)(Qu_n - x^*)\|^2\\
&\leq \alpha_n\|z - x^*\|^2 + (1-\alpha_n)\|Qu_n - x^*\|^2\\
&\leq \alpha_n\|z - x^*\|^2 + \|Qu_n - x^*\|^2\\
&\leq \alpha_n\|z - x^*\|^2 + \|u_n - x^*\|^2\\
&\leq \alpha_n\|z - x^*\|^2 + \|x_n - x^*\|^2 + \beta_n\sigma_n\|Sx_n - x^*\|^2\\
&\quad -\beta_n(1-\beta_n)\|\sigma_n(Sx_n - x_n) + (1-\sigma_n)(Tx_n - x_n)\|^2.
\end{aligned}$$

It implies that

$$\begin{aligned}
\beta_n(1-\beta_n)\|\sigma_n(Sx_n - x_n) + (1-\sigma_n)(Tx_n - x_n)\|^2 &\leq \alpha_n\|z - x^*\|^2 + \beta_n\sigma_n\|Sx_n - x^*\|^2\\
&\quad +\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2\\
&\leq \alpha_n\|z - x^*\|^2 + \beta_n\sigma_n\|Sx_n - x^*\|^2\\
&\quad +(\|x_n - x^*\| + \|x_{n+1} - x^*\|)\|x_{n+1} - x_n\|. \qquad (22)
\end{aligned}$$

From condition (i), (ii) and (22) , we have

$$\lim_{n\to\infty}\|\sigma_n(Sx_n - x_n) + (1-\sigma_n)(Tx_n - x_n)\|^2 = 0. \qquad (23)$$

Since

$$\begin{aligned}
u_n - x_n &= (1-\beta_n)x_n + \beta_n(\sigma_n Sx_n + (1-\sigma_n)Tx_n) - x_n\\
&= \beta_n(\sigma_n(Sx_n - x_n) + (1-\sigma_n)(Tx_n - x_n)))
\end{aligned}$$

From above and (23), we have

$$\lim_{n\to\infty}\|u_n - x_n\| = 0. \qquad (24)$$

Observe that

$$\|u_n - Qu_n\| \leq \|u_n - x_n\| + \|x_n - Qu_n\|. \qquad (25)$$

From (20) , (23) and (25), we have

$$\lim_{n\to\infty}\|u_n - Qu_n\| = 0. \qquad (26)$$

Observe that

$$\|x_n - Tx_n\| \leq \|x_n - u_n\| + \|u_n - Tx_n\|. \tag{27}$$

Since $\{x_n\}$ is bounded and S, T are nonexpansive, then there exists $C > 0$ such that $\|Sx_n - Tx_n\| \leq C, \ \ \forall n \geq 0$. Now, we estimate

$$\begin{aligned}\|u_n - Tx_n\| &= \|(1-\beta_n)x_n + \beta_n(\sigma_n Sx_n + (1-\sigma_n)Tx_n) - Tx_n\| \\ &= \|(1-\beta_n)(x_n - Tx_n) + \beta_n\sigma_n(Sx_n - Tx_n)\| \\ &\leq (1-\beta_n)\|x_n - Tx_n\| + \beta_n\sigma_n\|Sx_n - Tx_n\| \\ &\leq (1-\beta_n)\|x_n - u_n\| + (1-\beta_n)\|u_n - Tx_n\| + \beta_n\sigma_n\|Sx_n - Tx_n\|\end{aligned}$$

which implie

$$\begin{aligned}\beta_n\|u_n - Tx_n\| &\leq (1-\beta_n)\|x_n - u_n\| + \beta_n\sigma_n\|Sx_n - Tx_n\| \\ &\leq \|x_n - u_n\| + \beta_n\sigma_n C.\end{aligned}$$

Hence, we have

$$\|u_n - Tx_n\| \leq \frac{\|x_n - u_n\|}{\beta_n} + \beta_n\sigma_n C. \tag{28}$$

It follows from condition (ii),(iii) that

$$\lim_{n\to\infty}\frac{\|x_n - u_n\|}{\beta_n} = \lim_{n\to\infty}\sigma_n\frac{\|x_n - u_n\|}{\beta_n\sigma_n} = 0.$$

Hence, condition (iv) and (28) implies that

$$\lim_{n\to\infty}\|u_n - Tx_n\| = 0. \tag{29}$$

Thus, it follows from (25), (27) and (29) that

$$\lim_{n\to\infty}\|x_n - Tx_n\| = 0. \tag{30}$$

Step 4. We show that $\limsup_{n\to\infty}\langle z - x^*, x_n - x^*\rangle \ \leq \ 0$ where $x^* = P_\Gamma z$.
To show this, choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle z - x^*, x_n - x^*\rangle = \lim_{i\to\infty}\langle z - x^*, x_{n_i} - x^*\rangle \tag{31}$$

Since $\{x_n\}$ is bounded, without loss of generality, we can assume that $x_{n_i} \rightharpoonup q$ as $i \to \infty$ where $q \in H$. We may assume that

$$\begin{aligned}\liminf_{n\to\infty}\langle -x_n, x - \frac{u_n - x_n}{\beta_n} - x_n\rangle &= \lim_{i\to\infty}\langle -x_{n_i}, x - \frac{u_{n_i} - x_{n_i}}{\beta_{n_i}} - x_{n_i}\rangle \\ \liminf_{n\to\infty}\langle Sx_n, x - \frac{u_n - x_n}{\beta_n} - x_n\rangle &= \lim_{i\to\infty}\langle Sx_{n_i}, x - \frac{u_{n_i} - x_{n_i}}{\beta_{n_i}} - x_{n_i}\rangle.\end{aligned}$$

From (24) and $x_{n_i} \rightharpoonup q$ as $i \to \infty$, we get that $u_{n_i} \rightharpoonup q$ as $i \to \infty$. From (24) and $x_{n_i} \rightharpoonup q$ as $i \to \infty$, we have

$$\limsup_{n\to\infty}\langle z - x^*, x_n - x^*\rangle = \lim_{i\to\infty}\langle z - x^*, q - x^*\rangle \tag{32}$$

In order to show $\langle z - x^*, q - x^* \rangle \leq 0$, we need to show that $q \in \Gamma = \Omega \cap \Phi$.
First , we show that $q \in \Omega = F(Q)$.
Assume that $q \notin F(Q)$. Then , we have $q \neq Qq$. From (26) and Opial's property, we have

$$\begin{aligned}
\liminf_{i\to\infty} \|u_{n_i} - q\| &< \liminf_{i\to\infty} \|u_{n_i} - Qq\| \\
&\leq \liminf_{i\to\infty} (\|u_{n_i} - Qu_{n_i}\| + \|Qu_{n_i} - Qq\|) \\
&\leq \liminf_{i\to\infty} (\|u_{n_i} - Qu_{n_i}\| + \|u_{n_i} - q\|) \\
&\leq \liminf_{i\to\infty} \|u_{n_i} - q\|.
\end{aligned}$$

This is a contradiction,

$$q \in F(Q). \tag{33}$$

We show that $q \in F(T)$.
Assume that $q \notin F(T)$. Then, we have $q \neq Tq$. From (30) and Opial's property, we have

$$\begin{aligned}
\liminf_{i\to\infty} \|x_{n_i} - q\| &< \liminf_{i\to\infty} \|x_{n_i} - Tq\| \\
&\leq \liminf_{i\to\infty} (\|x_{n_i} - Tx_{n_i}\| + \|Tx_{n_i} - Tq\|) \\
&\leq \liminf_{i\to\infty} (\|x_{n_i} - Tx_{n_i}\| + \|x_{n_i} - q\|) \\
&\leq \liminf_{i\to\infty} \|x_{n_i} - q\|.
\end{aligned}$$

This is a contradiction,

$$q \in F(T). \tag{34}$$

Next, we show that $q \in \Phi$. Consider

$$\begin{aligned}
u_n - x_n &= (1-\beta_n)x_n + \beta_n(\sigma_n S x_n + (1-\sigma_n)T x_n) - x_n \\
&= \beta_n\sigma_n(Sx_n - x_n) + \beta_n(1-\sigma_n)(Tx_n - x_n),
\end{aligned}$$

implies that

$$\begin{aligned}
Sx_n - x_n &= \frac{u_n - x_n}{\beta_n\sigma_n} - \frac{\beta_n(1-\sigma_n)(Tx_n - x_n)}{\beta_n\sigma_n} \\
&= \frac{u_n - x_n}{\beta_n\sigma_n} + \frac{(1-\sigma_n)(I-T)x_n}{\sigma_n}.
\end{aligned}$$

It follows that

$$Sx_n - x_n - \frac{u_n - x_n}{\beta_n\sigma_n} = \frac{(1-\sigma_n)(I-T)x_n}{\sigma_n}.$$

Since T is nonexpansive, we have $I-T$ is monotone. Let $x \in Fix(T)$, we have

$$\begin{aligned}
&\langle Sx_n - x_n - \frac{u_n - x_n}{\beta_n\sigma_n}, x - \frac{u_n - x_n}{\beta_n} - x_n\rangle \\
&= \frac{(1-\sigma_n)}{\sigma_n}\langle (I-T)x_n, x - \frac{u_n - x_n}{\beta_n} - x_n\rangle \\
&= \frac{(1-\sigma_n)}{\sigma_n}\langle (I-T)x_n - (I-T)(x - \frac{u_n - x_n}{\beta_n}) + (I-T)(x - \frac{u_n - x_n}{\beta_n}), \\
&\quad x - \frac{u_n - x_n}{\beta_n} - x_n\rangle \\
&= \frac{(1-\sigma_n)}{\sigma_n}\Big[\langle (I-T)x_n - (I-T)(x - \frac{u_n - x_n}{\beta_n}), x - \frac{u_n - x_n}{\beta_n} - x_n\rangle \\
&\quad + \langle (I-T)(x - \frac{u_n - x_n}{\beta_n}), x - \frac{u_n - x_n}{\beta_n} - x_n\rangle\Big] \\
&\leq \frac{(1-\sigma_n)}{\sigma_n}\langle (I-T)(x - \frac{u_n - x_n}{\beta_n}), x - \frac{u_n - x_n}{\beta_n} - x_n\rangle \\
&\leq \frac{(1-\sigma_n)}{\sigma_n}\Big\|(I-T)(x - \frac{u_n - x_n}{\beta_n})\Big\|\Big\|x - \frac{u_n - x_n}{\beta_n} - x_n\Big\| \\
&= \frac{(1-\sigma_n)}{\sigma_n}\Big\|(I-T)(x - \frac{u_n - x_n}{\beta_n}) - (I-T)x\Big\|\Big\|x - \frac{u_n - x_n}{\beta_n} - x_n\Big\| \\
&\leq 2(1-\sigma_n)\Big\|\frac{u_n - x_n}{\beta_n\sigma_n}\Big\|\Big\|x - \frac{u_n - x_n}{\beta_n} - x_n\Big\|,
\end{aligned}$$

which implies that

$$\begin{aligned}
\langle Sx_n - x_n, x - \frac{u_n - x_n}{\beta_n} - x_n\rangle &\leq 2(1-\sigma_n)\frac{\|u_n - x_n\|}{\beta_n\sigma_n}\Big\|x - \frac{u_n - x_n}{\beta_n} - x_n\Big\| + \langle \frac{u_n - x_n}{\beta_n\sigma_n}, x - \frac{u_n - x_n}{\beta_n} - x_n\rangle \\
&\leq 3\frac{\|u_n - x_n\|}{\beta_n\sigma_n}\Big\|x - \frac{u_n - x_n}{\beta_n} - x_n\Big\|. \qquad (35)
\end{aligned}$$

Since $\lim_{n\to\infty}\frac{\|x_n - u_n\|}{\beta_n} = 0$, we get

$$\lim_{i\to\infty}\langle \frac{u_{n_i} - x_{n_i}}{\beta_{n_i}}, x - \frac{u_{n_i} - x_{n_i}}{\beta_{n_i}} - x_{n_i}\rangle = 0. \qquad (36)$$

Since the norm H is weakly lower semicontinuous, (36) and $\frac{u_{n_i} - x_{n_i}}{\beta_{n_i}} + x_{n_i} \rightharpoonup q$, we have

$$\begin{aligned}
&\liminf_{n\to\infty}\langle -x_n, x - \frac{u_n - x_n}{\beta_n} - x_n\rangle \\
&= \lim_{i\to\infty}\langle -x_{n_i}, x - \frac{u_{n_i} - x_{n_i}}{\beta_{n_i}} - x_{n_i}\rangle \\
&= \lim_{i\to\infty}\langle -x_{n_i} - (x - \frac{u_{n_i} - x_{n_i}}{\beta_{n_i}}) + (x - \frac{u_{n_i} - x_{n_i}}{\beta_n}), x - \frac{u_{n_i} - x_{n_i}}{\beta_{n_i}} - x_{n_i}\rangle \\
&= \lim_{i\to\infty}\Big[\langle x - \frac{u_{n_i} - x_{n_i}}{\beta_{n_i}} - x_{n_i}, x - \frac{u_{n_i} - x_{n_i}}{\beta_{n_i}} - x_{n_i}\rangle - \langle x - \frac{u_{n_i} - x_{n_i}}{\beta_{n_i}}, x - \frac{u_{n_i} - x_{n_i}}{\beta_{n_i}} - x_{n_i}\rangle\Big] \\
&= \lim_{i\to\infty}\Big[\langle x - \frac{u_{n_i} - x_{n_i}}{\beta_{n_i}} - x_{n_i}, x - \frac{u_{n_i} - x_{n_i}}{\beta_{n_i}} - x_{n_i}\rangle - \langle x, x - \frac{u_{n_i} - x_{n_i}}{\beta_{n_i}} - x_{n_i}\rangle \\
&\quad + \langle \frac{u_{n_i} - x_{n_i}}{\beta_{n_i}}, x - \frac{u_{n_i} - x_{n_i}}{\beta_{n_i}} - x_{n_i}\rangle\Big] \\
&= \|x - q\|^2 - \langle x, x - q\rangle. \qquad (37)
\end{aligned}$$

Since S is weakly continuous and $\frac{u_{n_i}-x_{n_i}}{\beta_{n_i}} + x_{n_i} \rightharpoonup q$, we obtain

$$\liminf_{n\to\infty}\langle Sx_n, x - \frac{u_n - x_n}{\beta_n} - x_n\rangle = \lim_{i\to\infty}\langle Sx_{n_i}, x - \frac{u_{n_i} - x_{n_i}}{\beta_{n_i}} - x_{n_i}\rangle = \langle Sq, x - q\rangle. \tag{38}$$

From (35), (37) and (38), we have

$$\begin{aligned}\langle Sq - q, x - q\rangle &= \langle Sq, x - q\rangle - \langle q, x - q\rangle \\ &= \langle Sq, x - q\rangle + \|x - q\|^2 - \langle x, x - q\rangle \\ &= \liminf_{n\to\infty}\Big[\langle Sx_n, x - \frac{u_n - x_n}{\beta_n} - x_n\rangle - \langle x_n, x - \frac{u_n - x_n}{\beta_n} - x_n\rangle\Big] \\ &= \liminf_{n\to\infty}\langle Sx_n - x_n, x - \frac{u_n - x_n}{\beta_n} - x_n\rangle \\ &\le \liminf_{n\to\infty} 3\frac{\|u_n - x_n\|}{\beta_n\sigma_n}\Big\|x - \frac{u_n - x_n}{\beta_n} - x_n\Big\| \\ &\le 0.\end{aligned}$$

Hence q solve Hierarchical fixed point problem, i.e., $q \in \Phi$.
From (33) and (37), we obtain $q \in \Gamma = \Omega \cap \Phi$.
From (32) and property of P_Γ, we have

$$\limsup_{n\to\infty}\langle z - x^*, x_n - x^*\rangle = \langle z - x^*, q - x^*\rangle \le 0.$$

where $x^* = P_\Gamma z$.
Step 5. We show that $\{x_n\}$ converges strongly to x^*, where $x^* = P_\Gamma z$.
From the definition of x_n and $x^* = P_\Gamma z$, we have

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \|\alpha_n(z - x^*) + (1 - \alpha_n)(Qu_n - x^*)\|^2 \\ &\le (1 - \alpha_n)^2\|Qu_n - x^*\|^2 + 2\alpha_n\langle z - x^*, x_{n+1} - x^*\rangle \\ &\le (1 - \alpha_n)^2\|u_n - x^*\|^2 + 2\alpha_n\langle z - x^*, x_{n+1} - x^*\rangle \\ &\le (1 - \alpha_n)[\|x_n - x^*\|^2 + \beta_n\sigma_n\|Sx_n - x^*\|^2] + 2\alpha_n\langle z - x^*, x_{n+1} - x^*\rangle \\ &\le (1 - \alpha_n)\|x_n - x^*\|^2 + \beta_n\sigma_n\|Sx_n - x^*\|^2 + 2\alpha_n\langle z - x^*, x_{n+1} - x^*\rangle\end{aligned}$$

From step 4. , condition (ii) and lemma 2.4 (b), we conclude that $\{x_n\}$ converges strongly to $x^* = P_\Gamma z$. This completes this proof. □

Remark 3.2. *if $F(T) \cap F(S) \neq \emptyset$, we don't need the condition (iv) $\lim_{n\to\infty} \frac{\|x_n-u_n\|}{\beta_n\sigma_n} = 0$ in Theorem (3.1) to prove strong convergence of the sequence acquired from the proposed iterative methods.*

4. Application

In 2013, Kangtunyakarn [10] introduced a modification of system of variational inequalities as follows: finding $(x^*, z^*) \in C \times C$ such that

$$\begin{cases}\langle x^* - (I - \lambda_1 D_1)(ax^* + (1 - a)z^*), x - x^*\rangle \ge 0, & \forall x \in C,\\ \langle z^* - (I - \lambda_2 D_2)x^*, x - z^*\rangle \ge 0, & \forall x \in C,\end{cases} \tag{39}$$

where $D_1, D_2 : C \to H$ be two mappings, for every $\lambda_1, \lambda_2 \ge 0$ and $a \in [0, 1]$.

Let h be a proper lower semicontinuous convex function of H into $(-\infty, +\infty]$. The subdifferential ∂h of h defined by

$$\partial h(x) = \{z \in H : h(x) + \langle z, u - x\rangle \le h(u), \forall u \in H\}$$

for all $x \in H$. From Rockafellar [8], we get that ∂h is a maximal monotone operator. Let C be a nonempty closed convex subset of H and i_C be the indicator function of C, i.e.,

$$i_C = \begin{cases} 0 & ; if x \in C \\ +\infty & ; if x \notin C \end{cases}$$

Then, i_C is a proper, lower semicontinuous and convex function on H and so the subdifferential ∂i_C of i_C is a maximal monotone operator. The resolvent operator $J_r^{\partial i_C}$ of i_C for $\lambda > 0$, can be defined by $J_r^{\partial i_C}(x) = (I + \lambda \partial i_C)^{-1}(x), x \in H$. we have that $J_r^{\partial i_C}(x) = P_C x$, for all $x \in H$ and $\lambda > 0$. As special case, if $M_A = M_B = \partial i_C$ in Lemma 2.8, we find that $J_{\lambda_A}^{M_A} = J_{\lambda_B}^{M_B} = P_C$. So we obtain the following result.

Lemma 4.1. *[10] Let C be a nonempty closed convex subset of a real Hilbert space H and let $D_1, D_2 : C \to H$ be mappings.for every $\lambda_1, \lambda_2 > 0$ and $a \in [0,1]$, the following statements are equivalent:*
(a) $(x^*, z^*) \in C \times C$ is a solution of problem (39),
(b) x^* is a fixed point of mapping $G : C \to C$, i.e., $x^* \in F(G)$, defined by

$$G(x) = P_C(I - \lambda_1 D_1)(ax + (1-a)P_C(I - \lambda_2 D_2)x), \tag{40}$$

where $z^* = P_C(I - \lambda_2 D_2)x^*$

Theorem 4.2. *Let H be a real Hilbert space. Let $S,T : H \to H$ be two nonexpansive mapping. Let $D_1, D_2 : H \to H$ be α and β - inverse strongly monotone mappings with $\eta = min\{\alpha, \beta\}$. Define the mapping $G : H \to H$ by (40) for all $x \in H$, $a \in (0,1)$ and $\lambda_1, \lambda_2 > 0$. Assume that $\Gamma = \Phi \cap F(G) \neq \emptyset$. Let $\{u_n\}$ and $\{x_n\}$ be generated by iterative algorithm :*

$$\begin{cases} x_0 \in H; \\ u_n = (1-\beta_n)x_n + \beta_n(\sigma_n S x_n + (1-\sigma_n)T x_n) \\ x_{n+1} = \alpha_n z + (1-\alpha_n)G u_n \end{cases} \tag{41}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\sigma_n\} \in (0,\ 1)$ and $0 < a < \beta_n < b < 1$, for some $a, b > 0$.
Suppose the following conditions hold:
(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$
(ii) $\sum_{n=0}^{\infty} \sigma_n < \infty$
(iii) $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$,$\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.
Then $\{x_n\}$ converges strongly to $x^* \in \Gamma$ where $x^* = P_\Gamma z$.

Proof. Taking $J_{\lambda_A}^{M_A} = J_{\lambda_B}^{M_B} = P_C$ in Theorem 3.1, we obtain the desired conclusion. □

In order to apply our main result, we give the following Lemma.

Lemma 4.3. *[10] Let C be a nonempty closed convex subset of real Hilbert space H. Let $T, S : C \to C$ be nonexpansive mappings. Define a mapping $B^A : C \to C$ by $B^A x = T(aI + (1a)S)x$ for every $x \in C$ and $a \in (0,1)$. Then $F(B^A) = F(T) \cap F(S)$ and B^A is a nonexpansive mapping.*

We apply our Theorem 3.1, by using with Lemma 4.3 , to find a solution of the variational inclusion problem.

Lemma 4.4. *Let H be a real Hilbert space and let $A, B : H \to H$ be α and β-inverse strongly monotone mappings with $\eta = min\{\alpha, \beta\}$. Let $M_A, M_B : H \to 2^H$ be multivalue maximal monotone mappings with $VI(H, A, M_A) \cap VI(H, B, M_B) \neq \emptyset$. Define a mapping $Q : H \to H$ as in Lemma 2.8 for all $x \in H$, $a \in (0,1)$ and $\lambda_A, \lambda_B \in (0, 2\eta)$. Then $F(Q) = VI(H, A, M_A) \cap VI(H, B, M_B)$.*

Proof. Let $x, y \in C$. From Lemma 2.8, we have Q is nonexpansive and $J_{M_A,\lambda_A}(I-\lambda_A A)$ and $J_{M_B,\lambda_B}(I-\lambda_B B)$ are nonexpansives. Since

$$Q(x) = J_{M_A,\lambda_A}(I-\lambda_A A)(ax + (1-a)J_{M_B,\lambda_B}(I-\lambda_B B)x),$$

and Lemma 4.3, we have

$$F(Q) = F(J_{M_A,\lambda_A}(I-\lambda_A A)) \cap F(J_{M_B,\lambda_B}(I-\lambda_B B)).$$

By Lemma 2.7, we have

$$F(Q) = VI(H, A, M_A) \cap VI(H, B, M_B).$$

□

Theorem 4.5. *Let H be a real Hilbert space. Let $S,T : H \to H$ be two nonexpansive mappings. Let $A,B : H \to H$ be α and β - inverse strongly monotone mappings with $\eta = min\{\alpha, \beta\}$. Define the mapping $Q : H \to H$ by $Q(x) = J_{M_A,\lambda_A}(I-\lambda_A A)(ax + (1-a)J_{M_B,\lambda_B}(I-\lambda_B B)x)$ for all $x \in H$, $a \in (0, 1)$ and $\lambda_A, \lambda_B \in (0, 2\eta)$. Assume that $\Gamma = VI(H, A, M_A) \cap VI(H, B, M_B) \neq \emptyset$. Let $\{u_n\}$ and $\{x_n\}$ be generated by iterative algorithm :*

$$\begin{cases} x_0 \in H; \\ u_n = (1-\beta_n)x_n + \beta_n(\sigma_n S x_n + (1-\sigma_n)T x_n) \\ x_{n+1} = \alpha_n z + (1-\alpha_n)Q u_n \end{cases} \tag{42}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\sigma_n\} \in (0, 1)$ and $0 < a < \beta_n < b < 1$, for some $a, b > 0$.
Suppose the following conditions hold:
(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$
(ii) $\sum_{n=0}^{\infty} \sigma_n < \infty$
(iii) $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$,$\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$
(iv) $\lim_{n\to\infty} \frac{\|x_n - u_n\|}{\beta_n \sigma_n} = 0$.
Then $\{x_n\}$ converges strongly to $x^* \in \Gamma$ where $x^* = P_\Gamma z$.

Proof. From Lemma 4.4, and Theorem 3.1, we obtain the desired conclusion. □

Remark 4.6. *If $VI(H, A, M_A) \cap VI(H, B, M_B) \neq \emptyset$, then observe that $VI(H, A, M_A) \cap VI(H, B, M_B) = \Omega$.*

5. Example and numerical results

In this section, we give an example supporting Theorem 3.1.

Example 5.1. *Let $\mathbb{R}$ be the set of real numbers and A, B: $\mathbb{R} \to \mathbb{R}$ be defined by $Ax = x - 2$ and $Bx = \frac{x}{2} - \frac{13}{2}$, for all $x \in \mathbb{R}$. Let $M_A : \mathbb{R} \to 2^{\mathbb{R}}$ be defined by $M_A x = \{2x - 1\}$ for all $x \in \mathbb{R}$ and $M_B : \mathbb{R} \to 2^{\mathbb{R}}$ defined by $M_B = \{4x + 2\}$ for all $x \in \mathbb{R}$, let $Q : H \to H$ be defined by*

$$Q(x) = J_{M_A,\frac{1}{2}}(I - \frac{1}{2}A)(0.25x + 0.75J_{M_B,1}(I - 1B)x).$$

where $a = 0.25$, $\lambda_A = \frac{1}{2}$ and $\lambda_B = 1$. Let S, T: $\mathbb{R} \to \mathbb{R}$ be defined by $Sx = \frac{x}{2}$ and $Tx = sin(\frac{\pi x}{2})$. Let $x_0 \in H$, $\{u_n\}$ and $\{x_n\}$ generated by (3.1) where $\alpha_n = \frac{1}{3n}$, $\beta_n = \frac{1}{2} - \frac{1}{n+2}$ and $\sigma_n = \frac{1}{n^3}$. By the definitions of S, T, A, B, M_A and M_B we have $1 \in \Gamma = \Phi \cap \Omega$. From Theorem 3.1., we can conclude that the sequences $\{u_n\}$ and $\{x_n\}$ converge strongly to 1. We can rewrite (3.1) as follows:

$$\begin{cases} u_n = (1 - \frac{2}{n})x_n + \frac{2}{n}(\frac{1}{n^3} S x_n + (1 - \frac{1}{n^3})T x_n) \\ x_{n+1} = \frac{1}{3n} z + (1 - \frac{1}{3n})Q u_n \end{cases} \tag{43}$$

n	$\mathbf{x}_n$	$\mathbf{u}_n$	ϵ_t
0	2.000000	1.833333	100.000000
1	1.378472	1.236611	37.847222
2	1.182687	1.111527	18.268722
3	1.119166	1.071418	11.916584
⋮	⋮	⋮	⋮
50	1.007104	1.003657	0.710414
⋮	⋮	⋮	⋮
98	1.003585	1.001821	0.358539
99	1.003549	1.001802	0.354877
100	1.003513	1.001783	0.351290

Table 1: The values of $\mathbf{x}_n$, $\mathbf{u}_n$ and ϵ_t (Relative Error) with $\mathbf{x}_0 = z = \mathbf{2}$ and $N = 100$ of the iterative (43).

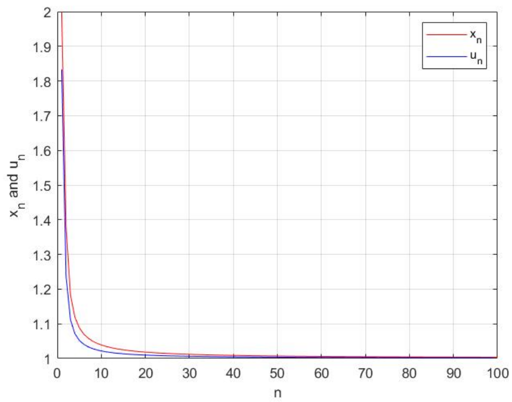

Figure 1: The convergence of $\mathbf{x}_n$ and $\mathbf{u}_n$ with $\mathbf{x}_0 = z = \mathbf{2}$ and $N = 100$.

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