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Adjoint of a Weighted Composite Difference Operator on $L^2(\mathbb{R})$

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Abstract. A necessary and sufficient condition for a weighted composite difference operator to be bounded is investigated in this paper. The adjoint of a weighted composite difference operator is obtained.

1. Introduction and Preliminaries

Let $(\mathbb{R}, s, m)$ be a measurable space, where m is the Lebesgue measure and s is the $\sigma-$ algebra of Lebesgue measurable subsets of $\mathbb{R}$.
Let $L^2(\mathbb{R}, m) = \left\{ f | f : s \to \mathbb{C} \text{is measurable and} \int |f|^2 \, dm < \infty \right\}$.
Then $L^2(\mathbb{R}, m)$ is a Hilbert space under the inner product space $\langle f, g \rangle = \int f \bar{g} dm$. Suppose $T : \mathbb{R} \to \mathbb{R}$ is a non-singular measurable transformation, $f : \mathbb{R} \to \mathbb{C}$ is a measurable function. Then we can define a weighted composite transformation from $L^2(\mathbb{R}, m)$ into the space of complex valued functions defined on $\mathbb{R}$ by $\left(W^T_{\mu,D_a} f\right)(x) = \mu(x)\left[f(T(x)) + f(T(x) - a)\right]$ for $a \in \mathbb{R}$. If the range of W^T_{μ,D_a} is contained in $L^2(\mathbb{R}, m)$ and it is bounded we shall call it a weighted composite difference operator.

The difference operators on sequence spaces are studied by Akhmedov and Başar [1], Altay and Başar[2] and Chib and Komal[6]. These operators find wide applications in numerical and statistical methods. The difference operators are also used in Newton's forward and backward formula for interpolation.

a measurable transformation $T : (X, s) \to (X, s)$ from a measurable space is called non-singular if $m(E) = 0$ implies that $m(T^{-1}(E)) = 0$ for every $E \in s$. If T is non-singular, then the measurable mT^{-1} is absolutely continuous with respect to the measure m. By the Radon Nikodym theorem there exists a positive measurable function f_o such that

$$mT^{-1}(E) = \int f_o dm.$$

The function f_o is called Radon Nikodym derivative of the measurable mT^{-1} with respect to the measure m. Let s_o be the subsigma algebra of $\sigma-$ finite sigma algebra of s. Then the conditional expectation $E(.|s_o)$ is defined as a linear transformation from certain $s-$ measurable function spaces into their s_o measurable counterparts. In particular the conditional expectation is a bounded projection from $L_p(X, s, m)$ into $L_p(X, T^{-1}(s)), m)$. We denote it by E. We use the following properties of E

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(1) $E(f.goT) = E(f).goT$

(2) $E(f)$ has the form $E(f) = goT$ for exactly one measurable function g. In particular $g = E(f)oT^{-1}$ which is well defined measurable function.

(3) $\int_{T^{-1}(F)} E(f)dm = \int_{T^{-1}(F)} fdm$ for $f \in s$.

2. Bounded Weighted Composite Difference operators on $L^2\mathbb{R}$.

In this section we shall characterise bounded weighted composite difference operators on $L^2\mathbb{R}$.

Theorem 2.1. *Let $\mu : \mathbb{R} \to C$ be a measurable function and $T : \mathbb{R} \to \mathbb{R}$ be a measurable transformation. Then for $a \in \mathbb{R}$, $W^T_{\mu,D_a} : L^2\mathbb{R} \to L^2\mathbb{R}$ is a bounded operator if and only if $\exists\, M > 0$ such that $E(\left(|\mu|^2\right)oT^{-1})(x)f_o(x) \le M$ for $m-$ almost all $x \in \mathbb{R}$.*

Proof. Suppose the condition is true. Then for $f \in L^2\mathbb{R}$

$$\begin{aligned}
\left\|W^T_{\mu,D_a}f\right\|^2 &= \int_{\mathbb{R}} \left|\mu(x)\left[f(T(x)) - f(T(x)-a)\right]\right|^2 dm(x)\\
&= \int_{\mathbb{R}} |\mu(x)|^2 \left|f(T(x)) - (T_\alpha(f)(T(x))\right|^2 dm(x)\\
&= \int_{\mathbb{R}} E(|\mu(x)|^2 oT^{-1})(x)f_o(x) \left|f(x) - (T_\alpha(f))(x)\right|^2 dm(x)\\
&\le M \int_{\mathbb{R}} \left|f(x) - (T_\alpha f)(x)\right|^2 dm(x)\\
&\le 2M\Big(\int_{\mathbb{R}} |f(x)|^2 dm(x) + \int_{\mathbb{R}} |(T_\alpha f)(x)|^2 dm(x)\Big)\\
&= 2M\left(\|f\|^2 + \|T_\alpha f\|^2\right)\\
&= 2M\left(\|f\|^2 + \|f\|^2\right)\text{(because translation is an isometry)}\\
&= 4M\|f\|^2.
\end{aligned}$$

Hence W^T_{μ,D_a} is a bounded operator. Conversely suppose that W_{μ,D_a} is a bounded operator. We shall prove that the condition holds.
For, if the condition is false, then for every n, there exists a measurable set E_n of positive measure such that

$$E((|u|^2)oT^{-1})(x) \ge n$$

for almost every $x \in E_n$ For each $p \in \mathbb{Z}$, let $F_p = \left[p\frac{a}{4}, (p+1)\frac{a}{4}\right]$. Then $\mathbb{R} = \bigcup_{p\in Z} F_p$ and $F_p \cap F_k = \phi$ for $p \neq k$. Since $E_n = E_n \bigcap \mathbb{R} = \bigcup_{p\in Z}(E_n \bigcap F_p) > 0$ for some $p \in \mathbb{Z}$ say $p = p(n)$. Write $G_n = E_n \bigcap F_{p(n)}$. Clearly $0 < m(G_n) < \infty$ for $n \in \mathbb{N}$.
Consider

$$
\begin{aligned}
\left\|W_\mu^T D_a \chi G_n\right\|^2 &= \int \left|\mu(x)\left[\chi G_n(T(x)) - \chi G_n(T(x)-a)\right]\right|^2 dm(x)\\
&= \int \left|\mu(x)\left[\chi G_n(T(x)) - \chi G_n(T_a(T(x))\right]\right|^2 dm(x)\\
&= \int E(\left|\mu\right|^2 oT^{-1})(x) f_o(x) \left|\chi_{G_n}(x) - (\chi G_n o T_a)(x)\right|^2 dm(x)\\
&= \int E(\left|\mu\right|^2 oT^{-1})(x) f_o(x) \left|\chi_{G_n}(x) - (\chi G_n o T_a)(x)\right|^2 dm(x)\\
&= \int E(\left|\mu\right|^2 oT^{-1})(x) f_o(x) \left[\left|\chi_{G_n}(x)\right|^2 + |\chi(G_n)(T_a(x))|^2\right] dm(x)\\
&\geq \int E(\left|\mu\right|^2 oT^{-1})(x) f_o(x) \left|\chi_{G_n}(x)\right|^2 dm(x)\\
&\geq n \int \left|\chi_{G_n}(x)\right|^2 dm(x)\\
&= n \left\|\chi G_n\right\|^2 .
\end{aligned}
$$

This is a contradiction. Hence the condition must hold.This completes the proof of the theorem. □

Example 2.2. *Let $\mu : \mathbb{R} \to \mathbb{C}$ be defined by $\mu(x) = e^{-4x^2}$ and let $T : \mathbb{R} \to \mathbb{R}$ be defined as $T(x) = 2x$ for every $x \in \mathbb{R}$. Then $T^{-1}(x) = \frac{x}{2}$ and so $f_o(x) = \frac{dmT^{-1}(x)}{dm(x)} = \frac{1}{2}$. Also $T^{-1}(s) = s$. Therefore $E(\left|\mu\right|^2 oT^{-1})(x) f_o(x) = (\left|\mu\right|^2 oT^{-1})(x) f_o(x) = e^{-2x^2}$ which is a bounded function.*
Hence $W_\mu^T D_a$ is a bounded operator for every difference interval a.

3. Adjoint of a Weighted Composite Difference Operator

In this section we obtain the adjoint of a weighted composite difference operator we know that if

$$(D_a f)(x) = f(x) - f(x-a),$$

then

$$(D_a^* f)(x) = f(x) - f(x+a).$$

Theorem 3.1. *Let $W_\mu^T D_a \in B(L^2\mathbb{R})$. Then*

$$(W_\mu^T D_a)^* g = \overline{D_a^* \left(E(\mu, \overline{g})\, oT^{-1} f_o\right)}$$

for $g \in L^2\mathbb{R}$.

Proof. Let a be the length of the difference interval. For $f, g \in L^2(\mathbb{R})$, Consider

$$
\begin{aligned}
\left\langle W_{\mu}^{T} D_a f, g\right\rangle &= \int \mu(x)\left[f(T(x)) - f(T(x)-a)\right]\overline{g}(x)dm(x)\\
&= \int \mu(x)\overline{g}(x)\left[f(T(x)) - f(T_{\alpha}(T(x)))\right]dm(x)\\
&= \int E(u.\overline{g})oT^{-1}(x)\left[f(x) - f(T_{\alpha}(x))\right]dmT^{-1}(x)\\
&= \int E(u.\overline{g})oT^{-1}(x)f_o(x)\left[f(x) - f(T_{\alpha}(x))\right]dm(x)\\
&= \int E(u.\overline{g})oT^{-1}(x)f_o(x)F(x)dm(x) - \int E(u.\overline{g})oT^{-1}(x)f_o(x)f(x-a)dm(x)\\
&= \int E(u.\overline{g})oT^{-1}(x)f_o(x)f(x)dm(x) - \int E(u.\overline{g})oT^{-1}(x+a)f_o(x+a)f(x)dm(x)\\
&= \int \left[E(u.\overline{g})oT^{-1}(x)f_o(x) - E(u.\overline{g})(x+a)f_o(x+a)\right]f(x)dm(x)\\
&= \int \overline{D_a^*}\left(E(u.\overline{g})oT^{-1}.f_o(x)f(x)\right)dm(x)\\
&= \left\langle f, \overline{D_a^*((E(u.\overline{g})\ o\ T^{-1}.f_o)}\right\rangle\\
&= \left\langle f, W_{\mu,D_a}^{T*} g\right\rangle.
\end{aligned}
$$

□

Example 3.2. *Choose $a, b \in \mathbb{R}$ such that $0 < b < a$. Let $T : \mathbb{R} \to \mathbb{R}$ be defined by $T(x) = ax + b$. Then $T^{-1}(x) = \frac{x-b}{a}$ and $f_o(x) = \frac{dmT^{-1}(x)}{dm(x)} = \frac{1}{a}$.*
Let $\mu : \mathbb{R} \to \mathbb{R}$ be defined as

$$
\mu(x) = \begin{cases} x^2, & b-a < x < b+a \\ 0, & elsewhere \end{cases}
$$

By property of projection E on $L^2(\mathbb{R}, T^{-1}(s), \mu)$there exists a measurable function $g \in L^2(\mathbb{R}, s, \mu)$ such that $E(\left|\mu\right|^2) = g\ o\ T$ i.e, $E(\left|\mu\right|^2)\ o\ T^{-1} = g$. Consider

$$
\begin{aligned}
\int_{\frac{-b}{a}}^{\frac{x-b}{a}} \left|\mu(x)\right|^2 dm(x) &= \int_{T^{-1}(T[\frac{-b}{a}, \frac{x-b}{a}])} \left|\mu\right|^2 dm\\
&= \int_{T^{-1}[0,x]} \left|\mu\right|^2 dm\\
&= \int_{T^{-1}[0,x]} E(\left|\mu\right|^2)dm\\
&= \int_{T^{-1}[0,x]} g\ o\ Tdm\\
&= \int_{T^{-1}[0,x]} \left|\mu\right|^2 dm\\
&= \int_{[0,x]} g.f_o dm
\end{aligned}
$$

Differentiating with respect to , x we get

$\left|\mu\left(\frac{x-b}{a}\right)\right|^2 . \frac{1}{a} = g(x) f_o(x) = g(x) . \frac{1}{a}$

$$org(x) = \left|\mu\left(\frac{x-b}{a}\right)\right|^2 = \begin{cases} \left(\frac{x-b}{a}\right)^4, & for \quad |x-(1+a)b| < a^2 \\ 0, & elsewhere \end{cases}$$

$$orE\left(\left|\mu^2\right| oT^{-1}\right)(x) = \begin{cases} \left|\frac{x-b}{a}\right|^4, & for \quad |x-(1+a)b| < a^2 \\ 0, & elsewhere \end{cases}$$

This shows that $E\left(\left|\mu^2\right| oT^{-1}\right)$ *is a bounded function. Hence*

$$\left((W_\mu^T D_a)^* g\right)(x) = \overline{D_a^*\left(E(\mu, \overline{g})\, oT^{-1} f_o\right)}$$

$$= \overline{E(\mu\overline{g}) oT^{-1}(x)} - \overline{E(\mu\overline{g} oT^{-1})(x+a)}$$

$$= \overline{\mu\left(\frac{x-b}{a}\right)\overline{g}\left(\frac{x-b}{a}\right)} - \overline{\mu\left(\frac{x+a-b}{a}\right) g\left(\frac{x+a-b}{a}\right)}$$

$$= \overline{\mu}\left(\frac{x-b}{a}\right) g\left(\frac{x-b}{a}\right) - \overline{\mu}\left(\frac{x-b+a}{a}\right) g\left(\frac{x-b+a}{a}\right).$$

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