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Fixed Point Results under Nonlinear Suzuki $(F, \mathcal{R}^{\neq})$-contractions with an Application

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Abstract. In this article, we introduce the idea of nonlinear Suzuki $(F, \mathcal{R}^{\neq})$-contractions, which is patterned after the contrctive conditions due to Suzuki [Nonlinear Anal. 71 (2009) 5313-5317] and Wardowski [Fixed Point Theory Appl. (2012) 94:6]. Utilizing our newly introduced contraction, we establish some relation-theoretic fixed point theorems. Furthermore, we adopt an example to highlight the genuineness of our newly proved results. Finally, we use our main results to establish the existence and uniqueness of solution for a nonlinear matrix equation.

1. Introduction

Banach fixed point theorem is very effective and applicable result of metric fixed point theory, which has been generalized and extended in various directions. Two relatively recent and novel contraction conditions are due to Suzuki [1] and Wardowski [2], which assert that a self-mapping $\mathcal{T}$ defined on a metric space $(\mathcal{M}, d)$ is said to be a Suzuki contraction if for all $u, v \in \mathcal{M}$ with $u \neq v$ and $\frac{1}{2}d(u, \mathcal{T}u) < d(u, v) \Rightarrow d(\mathcal{T}u, \mathcal{T}v) < d(u, v)$ while, a mapping $\mathcal{T} : \mathcal{M} \to \mathcal{M}$ is said to be F-contraction if there exist $\tau > 0$ and $F \in \mathcal{F}$ such that $\tau + F(d(\mathcal{T}u, \mathcal{T}v)) \leq F(d(u, v))$ with $d(\mathcal{T}u, \mathcal{T}v) > 0$ for all $u, v \in \mathcal{M}$, (where $\mathcal{F}$ is described in Definition 2.1 later) besides enlarging the class of underlying spaces along with the refinements of involved metrical terms on the lines ($cf.$ [4–6, 13]). Recently, Piri and Kumam [7] improved Wardowski types results for F-Suzuki-contraction, wherein the condition (F_3) of $\mathcal{F}$ is replaced by the continuity of auxiliary map F.

Ran and Reurings [8], and Nieto and Rodríguez-Lopéz [9] extended the Banach fixed point theorem to ordered metric spaces. Proceeding on the lines of [8, 9] and Durmaz *et al*. [10] extended this result employing F-contraction. On the other hand, Alam and Imdad [11] generalized the classical Banach contraction principle under an amorphous (arbitrary) binary relation. Very recently, Sawangsup *et al*. [12] using the notion of F-contraction improved the classical result on the complete metric space endowed with an amorphous binary relation.

Our aim in this article is to introduce the notion of "nonlinear Suzuki $(F, \mathcal{R}^{\neq})$-contraction" (nonlinear class of maps Φ will be defined later) and utilized the same to prove some fixed point results for nonlinear Suzuki $(F, \mathcal{R}^{\neq})$-contractions employing amorphous binary relation besides furnishing an example which

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demonstrate the worth of our newly proved results. Our newly proved result (combine) remains an improved version of novel results due to Suzuki [1] and Wardowski [2] under amorphous binary relation. Finally, we use our main results to establish the existence and uniqueness of solution for nonlinear matrix equation.

2. Preliminaries

Given a set $\mathcal{M} \neq \emptyset$, with a binary relation $\mathcal{R}$ on $\mathcal{M}$ which is subset of $\mathcal{M}^2$ and defined as $(u,v) \in \mathcal{R}$ instead of $u\mathcal{R}v$. For $\emptyset \neq W \subseteq X$, and a self mapping T on $\mathcal{M}$, the restriction of $\mathcal{R}$ to W, denoted by $\mathcal{R}|_W$, which is defined as $\mathcal{R}|_W = \mathcal{R} \cap W^2$ and $\mathcal{R}^{\neq} := \{(u,v) : Tu \neq Tv\}$ and $\mathcal{R}^s := \mathcal{R} \cup \mathcal{R}^{-1}$, (where $\mathcal{R}^{-1} := \{(y,x) \in \mathcal{M}^2 : (x,y) \in \mathcal{R}\}$). Throughout, $\mathcal{R}$ stands for a nonempty binary relation instead of 'binary relation'. Also, $\mathbb{N}$ stands for the set of natural numbers, while $\mathbb{N}_0$ (*i.e.*, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$), $\mathbf{R}$ and $\mathbf{R}^+$ stand for the set of whole, set of real and set of positive real numbers, respectively.

The following notion of $\mathcal{F}$-class and term of F-contraction were introduced by Wardowski, where F lies in $\mathcal{F}$.

Definition 2.1. *[2] Let $\mathcal{F}$ be the set of all mappings $F : \mathbf{R}^+ \to \mathbf{R}$ enjoying the following the properties:*

(F_1) *F is strictly increasing i.e., for all $\xi, \eta \in \mathbf{R}^+$ such that $\xi < \eta$ implies that $F(\xi) < F(\eta)$,*

(F_2) *for any sequence $\{\xi_n\} \subseteq \mathbf{R}^+$ with $\lim_{n\to\infty} \xi_n = 0$ if and only if $\lim_{n\to\infty} F(\xi_n) = -\infty$,*

(F_3) *there exists $k \in (0,1)$ such that $\lim_{\xi\to 0^+} \xi^k F(\xi) = 0$.*

Definition 2.2. *[2] A self-mapping $\mathcal{T}$ on a metric space $(\mathcal{M}, d)$ is said to be F-contraction if there exist $\tau > 0$ and $F \in \mathcal{F}$ such that*

$$d(\mathcal{T}u, \mathcal{T}v) > 0 \implies \tau + F(d(\mathcal{T}u, \mathcal{T}v)) \leq F(d(u,v)) \;\; \forall\, u, v \in \mathcal{M}.$$

The following well known functions are the examples of $\mathcal{F}$-class:

- $F_1(\xi) = ln\xi$ for all $\xi > 0$,
- $F_2(\xi) = \xi + ln\xi$ for all $\xi > 0$,
- $F_3(\xi) = -\frac{1}{\sqrt{\xi}}$ for all $\xi > 0$,
- $F_4(\xi) = ln(\xi + \xi^2)$ for all $\xi > 0$.

For the sake of completeness, we recall some noted results. The following recent core result due to Wardowski genuinely enriches the work of Edelstien on contractive mapping:

Theorem 2.3. *[2] Every F-contractive self-mapping $\mathcal{T}$ defined on a complete metric space $(\mathcal{M}, d)$ has a unique fixed point in $\mathcal{M}$. Moreover, for each $u_0 \in \mathcal{M}$, iterative sequense (Picard sequence) $\{\mathcal{T}^n u_0\}$ converges to the fixed point of $\mathcal{T}$.*

Suzuki [1] proved generalized versions of Edelstein's result on the compact metric space as follows:

Theorem 2.4. *[1] Let $(\mathcal{M}, d)$ be a compact metric space and $\mathcal{T} : \mathcal{M} \to \mathcal{M}$. Assume that if for all $u, v \in \mathcal{M}$ with $u \neq v$ and $\frac{1}{2}d(u, \mathcal{T}u) < d(u,v) \Rightarrow d(\mathcal{T}u, \mathcal{T}v) < d(u,v)$. Then $\mathcal{T}$ has a unique fixed point.*

3. Relevant Notions and Auxiliary Results

In sequel, we recall some relevant definitions and basic results for the use of our subsequent discussion:

Definition 3.1. *[11] Let $\mathcal{M}$ be a nonempty set, $\mathcal{R}$ a binary relation on $\mathcal{M}$ and T a self-mapping on $\mathcal{M}$. We say that "$\mathcal{R}$ is T-closed" if for any $u, v \in \mathcal{M}$,*

$$(Tu, Tv) \in \mathcal{R}, \quad \text{whenever } (u, v) \in \mathcal{R}.$$

Proposition 3.2. *[11] For a binary relation $\mathcal{R}$ defined on $\mathcal{M}$,*

$$(u, v) \in \mathcal{R}^s \Longleftrightarrow [u, v] \in \mathcal{R}.$$

Definition 3.3. *[11] Let $\mathcal{M}$ be a nonempty set and $\mathcal{R}$ a binary relation defined on $\mathcal{M}$. A sequence $\{u_n\} \subset \mathcal{M}$ is said to be an "$\mathcal{R}$-preserving" if*

$$(u_n, u_{n+1}) \in \mathcal{R} \;\; \forall\, n \in \mathbb{N}_0,$$

Proposition 3.4. *[14] Let $\mathcal{M}$ be a nonempty set, $\mathcal{R}$ a binary relation defined on $\mathcal{M}$ and T a self-mapping on $\mathcal{M}$, if "$\mathcal{R}$ is T-closed", then so is T^n (for all $n \in \mathbb{N}_0$,) where T^n denotes n^{th} iterate of T.*

Proposition 3.5. *[13] Let $\mathcal{M}$ be a nonempty set endowed with a binary relation $\mathcal{R}$ and T a self-mapping on $\mathcal{M}$ such that $\mathcal{R}$ is T-closed, then $\mathcal{R}^s$ is also T-closed.*

Definition 3.6. *[13] Let $(\mathcal{M}, d)$ be a metric space, and $\mathcal{R}$ a binary relation defined on $\mathcal{M}$. We say that $\mathcal{M}$ is "$\mathcal{R}$-complete" if every $\mathcal{R}$-preserving Cauchy sequence in $\mathcal{M}$ converges to a point in $\mathcal{M}$.*

Notice that notion of "$\mathcal{R}$-completeness" coincides with the usual "completeness" via universal relation.

Definition 3.7. *[13] Let $(\mathcal{M}, d)$ be a metric space equipped with a binary relation $\mathcal{R}$ and T a self-mapping on $\mathcal{M}$. Then T is said to be "$\mathcal{R}$-continuous" at $u \in X$ if for any $\mathcal{R}$-preserving sequence $\{u_n\}$ such that $u_n \xrightarrow{d} u$, we have $T(u_n) \xrightarrow{d} T(u)$. T is called $\mathcal{R}$-continuous if it is $\mathcal{R}$-continuous at each point of $\mathcal{M}$.*

The following notion of d-self-closedness due to Alam and Imdad [11] remains weaker over partial order relation ($\preceq$) contained in Turinici [15, 16]:

Definition 3.8. *[11] Let $(\mathcal{M}, d)$ be a metric space equipped with a binary relation $\mathcal{R}$. Then $\mathcal{R}$ is called "d-self-closed" if for any $\mathcal{R}$-preserving sequence $\{u_n\}$ such that $u_n \xrightarrow{d} u$, there exists a subsequence $\{u_{n_k}\}$* of $\{u_n\}$ with $[u_{n_k}, u] \in \mathcal{R} \;\; \forall\, k \in \mathbb{N}_0$.

Definition 3.9. *[17] Let $\mathcal{M}$ be a nonempty set and $\mathcal{R}$ a binary relation defined on $\mathcal{M}$. Then a subset W of $\mathcal{M}$ is said to be "$\mathcal{R}$-directed" if for each $u, v \in W$, there exists $w \in \mathcal{M}$ such that $(u, w) \in \mathcal{R}$ and $(v, w) \in \mathcal{R}$.*

Definition 3.10. *[18] Let $\mathcal{M}$ be a non-empty set endowed with a binary realtion $\mathcal{R}$. Then $\mathcal{R}$ is called complete if for all u, v in $\mathcal{M}$, either $(u, v) \in \mathcal{R}$ or $(u, v) \in \mathcal{R}$ which is denoted by $[u, v] \in \mathcal{R}$.*

Definition 3.11. *[18] Let $\mathcal{M}$ be a nonempty set and $\mathcal{R}$ a binary relation defined on $\mathcal{M}$. For any $u, v \in \mathcal{M}$, there is a finite sequence $\{\xi_0, \xi_1, \xi_2, ..., \xi_l\} \subset \mathcal{M}$ such that:*

(i) $\xi_0 = u$ and $\xi_l = v$,

(ii) $(\xi_i, \xi_{i+1}) \in \mathcal{R}$ for each i $(0 \leq i \leq l-1)$.

Then this sequence is called a "path of length l" (where l is a natural number) from u to v in $\mathcal{R}$. Here a path of length l involves $l+1$ elements of $\mathcal{M}$ also, they need not be distinct.

Definition 3.12. *[13] Let $\mathcal{M}$ nonempty set and $\mathcal{R}$ a binary relation defined on $\mathcal{M}$. A subset W of $\mathcal{M}$ is called "$\mathcal{R}$-connected" if for each pair $u, v \in W$, there exists a path in $\mathcal{R}$ from u to v.*

Firstly, we define the nonlinear class as follows:

$$\Phi = \{\phi : (0,\infty) \to (0,\infty) : \liminf_{r\to t^+} \phi(r) > 0 \text{ for each } t \geq 0\}.$$

Suppose T be a self-mapping on a metric space $(\mathcal{M}, d)$ and $\mathcal{R}$ a binary relation defined on $\mathcal{M}$. Then T is said to be "nonlinear Suzuki $(F,\mathcal{R}^{\neq})$-contraction", if there exist $\phi \in \Phi$ and $F \in \mathcal{F}$ such that $u, v \in \mathcal{M}$ with $(u, v) \in \mathcal{R}^{\neq}$,

$$\frac{1}{2}d(u, Tu) < d(u, v) \Longrightarrow \phi(d(u, v)) + F(d(Tu, Tv)) \leq F(d(u, v)).$$

In view of symmetry of d, we can deduce the following results.

Proposition 3.13. *Let $(\mathcal{M}, d)$ be a metric space, $\mathcal{R}$ a binary relation on $\mathcal{M}$, T a self-mapping on $\mathcal{M}$, $\phi \in \Phi$ and $F \in \mathcal{F}$ then the following contractivity conditions are equivalent:*

(I) $\frac{1}{2}d(u, Tu) < d(u, v) \Longrightarrow \phi(d(u, v)) + F(d(Tu, Tv)) \leq F(d(u, v)) \;\; \forall\, u, v \in X$ *with* $(u, v) \in \mathcal{R}^{\neq}$,
(II) $\frac{1}{2}d(u, Tu) < d(u, v) \Longrightarrow \phi(d(u, v)) + F(d(Tu, Tv)) \leq F(d(u, v)) \;\; \forall\, u, v \in X$ *with* $[u, v] \in \mathcal{R}^{\neq}$.

A proof of above proposition can be completed on the lines of the proof of Proposition 2.22 in [21].

For the subsequent discussion, we denote the following notations:[Given a self-mapping T on a nonempty set $\mathcal{M}$ together with a binary relation $\mathcal{R}$, we denote the following notations]:

- $F(T)$: the class of all fixed points of T;
- $\mathcal{M}(T, \mathcal{R})$: the class of all points of $\mathcal{M}$ such that $(u, Tu) \in \mathcal{R}$.

4. Main Results

We establish a result on the existence of fixed points for nonlinear Suzuki $(F,\mathcal{R}^{\neq})$-contraction (via the class Φ) employing arbitrary binary relation, which runs as follows:

Theorem 4.1. *Suppose T be a self-mapping on a metric space $(\mathcal{M}, d)$ and $\mathcal{R}$ a binary relation defined on $\mathcal{M}$. Assume that the following conditions hold:*

(a) *$(\mathcal{M}, d)$ is $\mathcal{R}$-complete;*
(b) *$\mathcal{M}(T, \mathcal{R})$ is non-empty;*
(c) *$\mathcal{R}$ is T-closed;*
(d) *T is $\mathcal{R}$-continuous or $\mathcal{R}$ is d-self-closed;*
(e) *there exist $\phi \in \Phi$ and $F \in \mathcal{F}$ such that T is nonlinear Suzuki $(F,\mathcal{R}^{\neq})$-contraction.*

Then T has a fixed point.

Proof. Since $\mathcal{M}(T, \mathcal{R})$ is non-empty, we can choose $u_0 \in \mathcal{M}(T, \mathcal{R})$ such that $u_1 = Tu_0$, continuing in such a way we can construct a sequence $\{u_n\}$ in $\mathcal{M}$ such that

$$u_n = T^n(u_0). \tag{1}$$

As $\mathcal{R}$ is T-closed and Proposition 3.4, we get

$$(T^n u_0, T^{n+1} u_0) \in \mathcal{R}$$

so that

$$(u_n, u_{n+1}) \in \mathcal{R} \;\; \forall\, n \in \mathbb{N}_0. \tag{2}$$

So the sequence $\{u_n\}$ is $\mathcal{R}$-preserving. Now, if $d(u_{n_0+1}, u_{n_0}) = 0$ for some $n_0 \in \mathbb{N}_0$, then in view of (1), we have $T(u_{n_0}) = u_{n_0}$ so that u_{n_0} is a fixed point of T and hence the proof is over.

Otherwise, if $d(u_{n+1}, u_n) > 0\ \forall\ n \in \mathbb{N}_0$, so the inequality holds $\frac{1}{2}d(u_n, u_{n+1}) < d(u_n, u_{n+1})\ \forall\ n \in \mathbb{N}_0$, then utilizing (1) and the contractivity condition (e) to (2), we deduce, for all $n \in \mathbb{N}_0$ that

$$\begin{aligned} \phi(d(u_n, u_{n+1})) + F(d(Tu_n, Tu_{n+1})) &\leq F(d(u_n, u_{n+1})) \\ \phi(d(u_n, u_{n+1})) + F(d(u_{n+1}, u_{n+2})) &\leq F(d(u_n, u_{n+1})) \end{aligned}$$

or,

$$F(d(u_{n+1}, u_{n+2})) \leq F(d(u_n, u_{n+1})) - \phi(d(u_n, u_{n+1})) \tag{3}$$

Denote $\delta_n := d(u_n, u_{n+1})$ for all $n \in \mathbb{N}_0$. Due to (3) and the property (F_1) of $\mathcal{F}$, $\{\delta_n\}$ is decreasing sequence of numbers. Therefore, there exists $\rho \geq 0$ such that $\lim_{n\to\infty} \delta_n = \rho$. We claim that

$$\lim_{n\to\infty} \delta_n = 0. \tag{4}$$

Let if possible $\rho > 0$. Using (3), the following holds:

$$\begin{aligned} F(d(u_{n+1}, u_{n+2})) &\leq F(d(u_0, u_1)) - \phi(d(u_n, u_{n+1})) - \phi(d(u_{n-1}, u_n)) - \cdots - \phi(d(u_0, u_1)) \\ &= F(d(u_0, u_1)) - \phi(\delta_n) - \phi(\delta_{n-1}) - \cdots - \phi(\delta_0). \end{aligned} \tag{5}$$

Set $\phi(\delta_{p_n}) := \min\{\phi(\delta_n), \phi(\delta_{n-1}), \cdots, \phi(\delta_0)\}$ for all $n \in \mathbb{N}_0$. Therefore inequality (5), reduces to

$$F(d(u_{n+1}, u_{n+2}) \leq F(d(u_0, u_1)) - n\phi(\delta_{p_n}). \tag{6}$$

In the similar fashion from (6), we can obtain

$$F(d(u_n, u_{n+1})) \leq F(d(u_0, u_1)) - n\phi(\delta_{p_n}). \tag{7}$$

Consider the sequence $\{\phi(\delta_{p_n})\}$, then there are two cases arise. Case (i): For each $n \in \mathbb{N}$, there is $m \in \mathbb{N}$ with $m > n$, such that $\phi(\delta_{p_n}) > \phi(\delta_{p_m})$. Thus, we can obtain subsequence $\{\delta_{p_{n_k}}\}$ of $\{\delta_{p_n}\}$ with $\phi(\delta_{p_{n_k}}) > \phi(\delta_{p_{n_{k+1}}})$ for all $k \in \mathbb{N}$. As $\delta_{p_{n_k}} \to \rho^+$, when $k \to \infty$. Applying the definition of Φ, we get

$$\liminf_{k\to\infty} \phi(\delta_{p_{n_k}}) > 0.$$

Therefore (7), yields that $F(\delta_{n_k}) \leq F(\delta_0) - n_k\phi(\delta_{p_{n_k}})$ for all $k \in \mathbb{N}$, hence by (F_2), $\lim_{k\to\infty} F(\delta_{n_k}) = -\infty$ and again by (F_2), we obtain $\lim_{k\to\infty} \delta_{p_{n_k}} = 0$, which contradicts that $\delta_{p_{n_k}} \to \rho^+$. Thus $\rho = 0$.
Case (ii): There is $n_0 \in \mathbb{N}$ such that $\phi(\delta_{p_{n_0}}) = \phi(\delta_{p_m})$ for all $m > n_0$. Then $F(\delta_m) \leq F(\delta_0) - m\phi(\delta_{p_m}))$ for all $m > n_0$. Hence, in view of (F_2) $\lim_{k\to\infty} \delta_{p_m} = 0$, which contradicts that $\rho > 0$. Thus in all $\rho = 0$. In view of (F_3) there exists $k \in (0, 1)$ such that

$$\lim_{n\to\infty} \delta_n^k F(\delta_n) = 0. \tag{8}$$

In view of (7), the following holds for all $n \in \mathbb{N}$,

$$\delta_n^k F(\delta_n) - \delta_n^k F(\delta_0) \leq -n\delta_n^k \phi(\delta_n)) \leq 0. \tag{9}$$

Making $n \to \infty$ in (9) and using (8) along with $\lim_{n\to\infty} \delta_n^k = 0$, give rise

$$\lim_{n\to\infty} n\delta_n^k = 0. \tag{10}$$

In view of (10), there exists $n_0 \in \mathbb{N}$ such that $n\delta_n^k < 1$ for all $n \geq n_{n_0}$, therefore

$$\delta_n < \frac{1}{n^{\frac{1}{k}}}. \tag{11}$$

We require to show that $\{u_n\}$ is a Cauchy sequence. To substantiate this, in view of triangle inequality of d and (11), (for all $m, n \in \mathbb{N}_0$ with $m > n$), we have

$$\begin{aligned} d(u_n, u_m) &\leq d(u_n, u_{n+1}) + d(u_{n+1}, u_{n+2}) + \cdots + d(u_{m-1}, u_m) \\ &= \sum_{j=n}^{m-1} d(u_j, u_{j+1}) \\ &\leq \sum_{j\geq n} d(u_j, u_{j+1}) \\ &\leq \sum_{j\geq n} \frac{1}{j^{\frac{1}{k}}} \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

Hence $\{u_n\}$ is "$\mathcal{R}$-preserving Cauchy sequence" in $\mathcal{M}$. As $(\mathcal{M}, d)$ is $\mathcal{R}$-complete so that there exists $u \in \mathcal{M}$, such that $u_n \xrightarrow{d} u$ as $n \to \infty$. Next, we assert that $u = T(u)$. In order to prove the assertion, suppose that T is $\mathcal{R}$-continuous. As $\{u_n\}$ is $\mathcal{R}$-preserving with $u_n \xrightarrow{d} u$, $\mathcal{R}$-continuity of T implies that $u_{n+1} = T(u_n) \xrightarrow{d} T(u)$. In view of uniqueness of limit, we obtain $T(u) = u$.

Alternately, assume that $\mathcal{R}$ is d-self-closed. Since $\{u_n\}$ is $\mathcal{R}$-preserving such that $u_n \xrightarrow{d} u$, the d-self-closedness of $\mathcal{R}$ implies that the existence of a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ with $[u_{n_k}, u] \in \mathcal{R}$ $(\forall\, k \in \mathbb{N}_0)$. We claim that

$$d(u_{n_k+1}, Tu) \leq d(u_{n_k}, u) \ \forall\, k \in \mathbb{N}. \tag{12}$$

There are two cases arise, let us consider a partition of $\mathbb{N}$ *i.e.*, $\mathbb{N}^0 \cup \mathbb{N}^+ = \mathbb{N}$ and $\mathbb{N}^0 \cap \mathbb{N}^+ = \emptyset$ verifying that

(*i*) $d(u_{n_k}, u) = 0 \ \forall\, k \in \mathbb{N}^0$,
(*ii*) $d(u_{n_k}, u) > 0 \ \forall\, k \in \mathbb{N}^+$.

In case (*i*), we have $d(Tu_{n_k}, Tu) = 0 \leq d(u_{n_k}, u)\ \forall\, k \in \mathbb{N}^0$. In case (*ii*), $\frac{1}{2}d(u, u_{n_k}) < d(u, u_{n_k})$ for all $k \in \mathbb{N}^+$, which on employing assumption (*e*), Proposition 3.13 and $[u_{n_k}, u] \in \mathcal{R}$, we get

$$\begin{aligned} F(d(u_{n_k+1}, Tu)) &\leq F(d(u_{n_k}, u)) - \phi(d(u_{n_k}, u)) \\ F(d(u_{n_k+1}, Tu)) &< F(d(u_{n_k}, u)) \end{aligned} \tag{13}$$

In view of (F_1), we deduce that $d(u_{n_k+1}, Tu) < d(u_{n_k}, u)$ for all $k \in \mathbb{N}^+$. Thus in all (12) is proved. Letting $n \to \infty$ in (12) and using the fact that $u_{n_k} \xrightarrow{d} u$ as $k \to \infty$, yields that $u_{n_k+1} \xrightarrow{d} T(u)$. Now, utilizing the fact of uniqueness of a limit, we have $T(u) = u$ so that u is a fixed point of T. Thus the proof is completed. □

Now, we prove a uniqueness theorem corresponding to Theorem 4.1.

Theorem 4.2. *Suppose all the hypotheses of Theorem 4.1 together with the following condition holds:*

(*v*) *: $T(\mathcal{M})$ is $\mathcal{R}^s$-connected.*

Then T has a unique fixed point.

Proof. By Theorem 4.1, $F(T) \neq \emptyset$. Choose $u, v \in F(T)$, then for all $n \in \mathbb{N}_0$, we get

$$T^n(u) = u \text{ and } T^n(v) = v. \tag{14}$$

In lieu of assumption (*v*), there exists a path (say $\{\xi_0, \xi_1, \xi_2, ..., \xi_l\}$) of some finite length l in $\mathcal{R}^s$ from u to v so that

$$\xi_0 = u,\ \xi_l = v \text{ and } [\xi_i, \xi_{i+1}] \in \mathcal{R} \text{ for each } i\ (0 \leq i \leq l-1). \tag{15}$$

Since $\mathcal{R}$ is T-closed, utilizing Propositions 3.4-3.5 and (15), we obtain

$$[T^n\xi_i, T^n\xi_{i+1}] \in \mathcal{R} \text{ for each } i\ (0 \le i \le l-1) \text{ and for each } n \in \mathbb{N}_0. \tag{16}$$

Now, for each $n \in \mathbb{N}_0$ and for each i $(0 \le i \le l-1)$, write $\delta^i_n := d(T^n\xi_i, T^n\xi_{i+1})$.
We assert that

$$\lim_{n\to\infty} \delta^i_n = 0 \quad \text{for each } i\ (0 \le i \le l-1). \tag{17}$$

For fix i, we distinguish two cases. Firstly, consider that $\delta^i_{n_0} = d(T^{n_0}\xi_i, T^{n_0}\xi_{i+1}) = 0$ for some $n_0 \in \mathbb{N}_0$, *i.e.*, $T^{n_0}(\xi_i) = T^{n_0}(\xi_{i+1})$, which implies that $T^{n_0+1}(\xi_i) = T^{n_0+1}(\xi_{i+1})$. Consequently, we get $\delta^i_{n_0+1} = d(T^{n_0+1}\xi_i, T^{n_0+1}\xi_{i+1}) = 0$. Thus by induction on n, we obtain $\delta^i_n = 0\ \forall\ n \ge n_0$, so that $\lim_{n\to\infty} \delta^i_n = 0$. Secondly, suppose that $\delta^i_n > 0\ \forall\ n \in \mathbb{N}_0$, $\frac{1}{2}d(T^n\xi_i, T^n\xi_{i+1}) < d(T^n\xi_i, T^n\xi_{i+1})$ then applying (14), the contractivity condition (e), and Proposition 3.13, we deduce, (for all $n \in \mathbb{N}^+$)

$$\begin{aligned} \phi(d(T^n\xi_i, T^n\xi_{i+1})) + F(d(T^{n+1}\xi_i, T^{n+1}\xi_{i+1})) &\le F(d(T^n\xi_i, T^n\xi_{i+1}))\ or, \\ F(d(T^{n+1}\xi_i, T^{n+1}\xi_{i+1})) &\le F(d(T^n\xi_i, T^n\xi_{i+1})) - \phi(d(T^n\xi_i, T^n\xi_{i+1})) \end{aligned} \tag{18}$$

Due to (18) and property (F_1), $\{\delta^i_n\}$ (for each i, $0 \le i \le l-1$) is decreasing sequence. Therefore, there exists $\delta^i \ge 0$ such that $\lim_{n\to\infty} \delta^i_n = \delta^i$ (for each i, $0 \le i \le l-1$). Let if possible $\delta^i > 0$. Utilizing the (3) (on the similar pattern), the following holds:

$$\begin{aligned} F(d(T^{n+1}\xi_i, T^{n+1}\xi_{i+1})) &\le F(d(\xi_i, \xi_{i+1})) - \phi(d(T^n\xi_i, T^n\xi_{i+1})) - \phi(d(T^{n-1}\xi_i, T^{n-1}\xi_{i+1})) - \cdots - \phi(d(\xi_i, \xi_{i+1})) \\ &= F(d(\xi_i, \xi_{i+1})) - \phi(\delta^i_n) - \phi(\delta^i_{n-1}) - \cdots - \phi(\delta^i_0) \end{aligned} \tag{19}$$

On setting $\phi(\delta^i_{p_n}) := \min\{\phi(\delta^i_n), \phi(\delta^i_{n-1}), \cdots, \phi(\delta^i_0)\}$ for all $n \in \mathbb{N}_0$. Therefore inequality (19), reduces to

$$F(d(T^{n+1}\xi_i, T^{n+1}\xi_{i+1})) \le F(d(\xi_i, \xi_{i+1})) - n\phi(\delta^i_{p_n}). \tag{20}$$

Now, on the lines similar to proof of (4), we can deduce that $\lim_{n\to\infty} \delta^i_n = 0$ for each i $(0 \le i \le l-1)$. Making use of (14), (16), (17) and the triangular inequality, we have

$$d(u, v) = d(T^n\xi_0, T^n\xi_l) \le \delta^0_n + \delta^1_n + \cdots + \delta^{l-1}_n \to 0\ \ as\ \ n \to \infty$$

so that $u = v$. Hence T has a unique fixed point. □

Corollary 4.3. *Conclusions of Theorem 4.1 remains true if the condition* (v) *is replaced by one of the following conditions (besides retaining rest of the hypotheses):*

(u') $\mathcal{R}|_{T(\mathcal{M})}$ *is complete,*
(u'') $T(\mathcal{M})$ *is* $\mathcal{R}^s$*-directed.*

A sketch of the proof of above corollary can be attempted on the lines of the proof of Corollary 3.4 contained in [14].

The following example is given to show the worth of Theorems 4.1 and 4.2 over corresponding earlier known results.

Example 4.4. *Let* $\mathcal{M} = [0, 4]$ *equipped with usual metric "d" and a self-map T on* $\mathcal{M}$ *defined by*

$$T(u) = \begin{cases} 4, & u = 0, \\ 1, & u \in (0, 2], \\ 2, & otherwise. \end{cases}$$

Let $\mathcal{R} := \{(0,1), (1,3), (4,1), (1,2), (4,2), (2,2), (1,1), (3,3)\}$ *be a binary relation on* $\mathcal{M}$*, then* $\mathcal{R}^{\neq} = \{(0,1), (1,3), (4,1), (4,2)\}$ *(owing to use of self-mapping T). Notice that,* $\mathcal{M}$ *is* $\mathcal{R}$*-complete and* $\mathcal{R}$ *is T-closed. Consider the functions* ϕ *and F* *defined by* $\phi(t) = \frac{1}{3t+1}$ *and* $F(t) = -\frac{1}{\sqrt{t}}$ $\forall\ t \in (0,\infty)$. *Clearly,* $\phi \in \Phi$ *and* $F \in \mathcal{F}$ *and* $\mathcal{M}(T,\mathcal{R}) \neq \emptyset$ *(as* $(1,T1) \in \mathcal{R}$*). Now, for all* $(u,v) \in \mathcal{R}^{\neq}$ *such that* $\frac{1}{2}d(u,Tu) < d(u,v)$ *holds only for* $(u,v) \in \{(1,3),(4,1),(4,2)\}$*, we have*

$$\phi(d(1,3)) + F(d(T1,T3)) \leq F(d(1,3)) \Rightarrow \frac{1}{7} - 1 \leq -\frac{1}{\sqrt{2}}$$
$$\phi(d(4,1)) + F(d(T4,T1)) \leq F(d(4,1)) \Rightarrow \frac{1}{9} - 1 \leq -\frac{1}{\sqrt{3}}$$
and
$$\phi(d(4,2)) + F(d(T4,T2)) \leq F(d(4,2)) \Rightarrow \frac{1}{7} - 1 \leq -\frac{1}{\sqrt{2}},$$

which shows that T is nonlinear Suzuki $(F,\mathcal{R}^{\neq})$*-contraction. We may choose* $\mathcal{R}$*-preserving sequence* $\{u_n\}$ *such that* $u_n \xrightarrow{d} u$. *Since* $(u_n,u_{n+1}) \in \mathcal{R}$*, for all* $n \in \mathbb{N}$*, there exists* $N \in \mathbb{N}$ *such that* $u_n = u \in \{1,2,3\}$, for all $n \geq N$. *Hence, we are able to find a subsequence* $\{u_{n_k}\}$ *of a sequence* $\{u_n\}$ *such that* $u_{n_k} = u$*, for all* $k \in \mathbb{N}$*, which amounts to say that* $[u_{n_k},u] \in \mathcal{R}$*, for all* $k \in \mathbb{N}$*. Thus,* $\mathcal{R}$ *is d-self-closed. Further, remaining hypotheses of Theorem 4.2 can be easily verified. Thus, we observe that T has a unique fixed point (namely:* $u = 1$*). As* $(0,1) \in \mathbb{R}$*, but*

$$\tau + F(d(T0,T1)) > F(d(0,1)) \ \ for\ \ all\ \ \tau \in \mathbb{R}^{+} and\ \ F \in \mathcal{F},$$

which shows that T is not $F_{\mathcal{R}}$*-contraction. Furthermore, since* $0,1 \in \mathcal{M}$ *and* $\frac{1}{2}d(1,T1) < d(1,0)$ *does not imply*

$$3 = d(T0,T1) \leq d(0,1) = 1,$$

which means that T is not Suzuki contraction. Hence Theorems 2.3 (due to Wardowski [2]) and 2.4 (due to Suzuki [1]) are not applicable. Thus this example demonstrate the worth of our newly proved results.

5. Application

Finally, in last section, we establish the guaranty of existence of solution of a noted matrix equation with the help of our main results.

Firstly, we recall the some notations: Let $\mathcal{M}(n)$ be the set of all complex matrices of oder n (*i.e.*, order n means $n \times n$ matricx), $\mathcal{H}(n) \subseteq \mathcal{M}(n)$, $\mathcal{P}(n) \subseteq \mathcal{H}(n)$, and $\mathcal{H}^{+}(n) \subseteq \mathcal{H}(n)$, respectively stand the classes of Hermitian matrices, positive definite matrices and positive semidefinite matrices all of order n. For $U \in \mathcal{H}^{+}(n)$ we denote $U \succeq O$ and similarly $U \in \mathcal{P}(n)$ means $U \succ O$. Now, we relate some notations for $U - V \succeq O$ and $U - V \succ O$ respectively implies that $U \succeq V$ and $U \succ V$. Now, for any $A, B \in \mathcal{H}(n)$, there is a greatest lower bound and least upper bound (see [20]). Here notation $\|.\|$ stands the spectral norm of a matrix A *i.e.*, $\|A\| = \sqrt{\lambda^{+}(A^{*}A)}$ such that $\lambda^{+}(A^{*}A)$ the largest eigenvalue of $A^{*}A$, where A^{*} is the conjugate transpose of A. We utilize the metric induced by the trace norm $\|.\|_{tr}$ defined by $\|A\|_{tr} = \sum_{i=1}^{n} S_{i(A)}$, $S_i(A)$, $i = 1,2,\cdots,n$ are the singular values of $A \in \mathcal{M}(n)$. Since $\mathcal{H}(n)$ equipped with this norm is complete matic space (see [8, 19, 20]). Further, it is to see that is partially ordered set with partial order $\succeq$, defined as $U \preceq V \iff V \succeq U$. Now, we discuss on the nonlinear matrix equation:

$$U = Q + \sum_{i=1}^{n} A_i^{*}\mathcal{L}(U)A_i, \tag{21}$$

where A_i are arbitrary matrices of order n, Q is a Hermitian positive definite matrix and $\mathcal{L}$ is continous order preserving map from $\mathcal{H}(n)$ to $\mathcal{P}(n)$ with $\mathcal{L}(0) = 0$.

For the use of our subsequent discussion, we require following lemmas:

Lemma 5.1. *[8] Let $A \geq O$ and $B \geq O$ be the matrices of oder n, then $0 \leq tr(AB) \leq \|A\| tr(B)$.*

Lemma 5.2. *[22] Given $A \in \mathcal{H}(n)$ satisfies $A \prec I_n$, then $\|A\| < 1$.*

Theorem 5.3. *Consider the equation described in (21), suppose that there exist a positive constant k and $\phi \in \Phi$ such that:*

(a) For any $U, V \in \mathcal{H}(n)$ with $U \preceq V$ such that $\sum_{i=1}^{n} A_i^* \mathcal{L}(U) A_i \neq \sum_{i=1}^{n} A_i^* \mathcal{L}(V) A_i$ and $\frac{1}{2}|tr(U - Q - \sum_{i=1}^{n} A_i^* \mathcal{L}(U) A_i)| < |tr(U - V)|$ implies that

$$|tr(\mathcal{L}V) - tr(\mathcal{L}U)| \leq \frac{|tr(V-U)|}{k\left(1 + \frac{1}{1+|tr(V-U)|}\sqrt{tr(V-U)}\right)^2}$$

(b) $\sum_{i=1}^{l} (A_i A_i^*) \leq k I_n$ and $\sum_{i=1}^{l} A_i^* \mathcal{L}(U) A_i \prec O$.

Then the system (21) has a unique solution. Furthermore, the iteration

$$U_n = Q + \sum_{i=1}^{n} A_i^* \mathcal{L}(U_{n-1}) A_i \tag{22}$$

where $U_0 \in \mathcal{H}(n)$ comparable with $U_0 \preceq Q + \sum_{i=1}^{n} A_i^* \mathcal{L}(U) A_i$ and converges with respect to trace norm $\|.\|_{tr}$ to the solution of matrix equation (21).

Proof. On $\mathcal{H}(n)$, a self-mapping $\mathcal{T}$ define by

$$\mathcal{T}(U) = Q + \sum_{i=1}^{l} A_i^* \mathcal{L}(U) A_i \quad for \ \ all \ \ U \in \mathcal{H}(n). \tag{23}$$

Let the mappings $\phi : (0, \infty) \to (0, \infty)$ and $F : \mathbf{R}^+ \to \mathbf{R}$ be defined by

$$\phi(t) = \frac{1}{1+t} \quad \text{and} \quad F(a) = -\frac{1}{\sqrt{a}} \quad \text{for all} \ \ a \in \mathbf{R}^+$$

respectively. Clearly, $\phi \in \Phi$ and $F \in \mathcal{F}$. Now, we define a binary relation $\mathcal{R} := \{(U, V) : (U, V) \in \mathcal{H}(n) \times \mathcal{H}(n), U \preceq V\}$. Let $U, V \in \mathcal{R}^{\neq} = \{(U, V) \in \mathcal{R} : \mathcal{T}(U) \neq \mathcal{T}(V)\}$. This means $\mathcal{L}(U) \prec \mathcal{L}(V)$. Notice that $\mathcal{L}$ is an order preserving mapping, which gives rise, $U \prec V$. Clearly, $\mathcal{T}$ is well defined, $(\mathcal{H}(n), \|\cdot\|_{tr})$ is $\mathcal{R}$-complete, $\mathcal{R}$ on $\mathcal{H}(n)$ is $\mathcal{T}$-closed and $\mathcal{T}$ is $\mathcal{R}$-continuous. As $\sum_{i=1}^{n} A_i^* \mathcal{L}(Q) A_i \succ 0$, for some $Q \in \mathcal{H}(n)$, we have

$Q \in \mathcal{H}(n)(\mathcal{T},\mathcal{R})$. Hence $\mathcal{H}(n)(\mathcal{T},\mathcal{R}) \neq \emptyset$. Now, any $(U,V) \in \mathcal{R}^{\neq}$ such that $\frac{1}{2}\|U-\mathcal{T}(U)\|_{tr} < \|U-V\|_{tr}$, then

$$\begin{aligned}
\|\mathcal{T}(V)-\mathcal{T}(U)\|_{tr} &= tr(\mathcal{T}(V)-\mathcal{T}(U)) \\
&= tr\Big(\sum_{i=1}^{l} A_i^*(\mathcal{L}(V)-\mathcal{L}(U))A_i\Big) \\
&= \sum_{i=1}^{l} tr(A_i^*(\mathcal{L}(V)-\mathcal{L}(U))A_i) \\
&= \sum_{i=1}^{l} tr(A_iA_i^*(\mathcal{L}(V)-\mathcal{L}(U)) \\
&= tr\Big(\Big(\sum_{i=1}^{l} A_iA_i^*\Big)(\mathcal{L}(V)-\mathcal{L}(U)\Big) \\
&\leq \|\sum_{i=1}^{l} A_iA_i^*\| \cdot \|(\mathcal{L}(V)-\mathcal{L}(U)\| \\
&\leq \frac{\|\sum_{i=1}^{l} A_iA_i^*\|}{k}\left(\frac{\|V-U\|}{\left(1+\frac{1}{1+|tr(V-U)|}\sqrt{\|V-U\|_{tr}}\right)^2}\right) \\
&< \frac{\|V-U\|_{tr}}{(1+\phi(|tr(V-U)|)(\sqrt{\|V-U\|_{tr}}))^2}
\end{aligned}$$

so that

$$\frac{\left(1+\phi(|tr(V-U)|)(\sqrt{\|V-U\|_{tr}})\right)^2}{\|V-U\|_{tr}} \leq \frac{1}{\|\mathcal{T}(V)-\mathcal{T}(Q)\|_{tr}}.$$

This implies that

$$\begin{aligned}
\frac{\left(1+\phi(|tr(V-U))|(\sqrt{\|V-U\|_{tr}})\right)^2}{\|V-U\|_{tr}} &\leq \frac{1}{\|\mathcal{T}(V)-\mathcal{T}(Q)\|_{tr}}. \\
\left(\phi(|tr(V-U))|)+\frac{1}{\sqrt{\|V-U\|_{tr}}}\right)^2 &\leq \frac{1}{\|\mathcal{T}(V)-\mathcal{T}(U)\|_{tr}} \\
\phi(|tr(V-U))|)+\frac{1}{\sqrt{\|V-U\|_{tr}}} &\leq \frac{1}{\sqrt{\|\mathcal{T}(V)-\mathcal{T}(U)\|_{tr}}}
\end{aligned}$$

so that

$$\phi(|tr(V-U))|)-\frac{1}{\sqrt{\|\mathcal{T}(V)-\mathcal{T}(U)\|_{tr}}} \leq -\frac{1}{\sqrt{\|V-U\|_{tr}}}$$

which yields that

$$\phi(\|V-U\|_{tr})+F(\|\mathcal{T}(V)-\mathcal{T}(U)\|_{tr}) \leq F(\|V-U\|_{tr}).$$

Hence $\mathcal{T}$ is nonlinear Suzuki $(F,\mathcal{R}^{\neq})$-contraction. Thus all the conditions of Theorem 4.2 are verified. Due to Theorem 4.2, there is $\hat{X} \in \mathcal{H}(n)$ such that $\mathcal{T}(\hat{X}) = \hat{X}$. Furthermore, due to availability of least upper bound and greatest lower bound for each $U,V \in \mathcal{H}(n)$, we have $\mathcal{T}(\mathcal{H}(n))$ is $\mathcal{R}^s$-connected. Hence system (21) admits a unique solution in $\mathcal{H}(n)$ (due to Theorem 4.1) (or, Theorem 4.2), this ends the proof. □

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