Filomat 36:9 (2022), 3141–3154
https://doi.org/10.2298/FIL2209141L

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Weighted Composition Operators and Differences of Composition Operators Between Weighted Bergman Spaces on the Ball

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Abstract. In this paper, we estimate essential norms of weighted composition operators and differences of two composition operators on the weighted Bergman spaces in the unit ball.

1. Introduction

Let $\mathbb{B}_n$ denote the unit ball of the n-dimensional complex Euclidean space $\mathbb{C}^n$ and $H(\mathbb{B}_n)$ the space of functions analytic in $\mathbb{B}_n$. Every analytic self mapping of $\mathbb{B}_n$ can induce a *composition operator* C_φ on $H(\mathbb{B}_n)$ defined by

$$C_\varphi f = f \circ \varphi$$

for $f \in H(\mathbb{B}_n)$. Furthermore, if u is a function defined on $\mathbb{B}_n$, functions u and φ can induce a *weighted composition operator* uC_φ for which

$$uC_\varphi f = u \cdot f \circ \varphi$$

for $f \in H(\mathbb{B}_n)$.

Much effort has been expended on characterizing those analytic maps which induce bounded or compact composition operators. Readers interested in this topic can refer to the books [15] by Shapiro, [3] by Cowen and MacCluer, and [18, 19] by Zhu, which are excellent sources for the development of the theory of composition operators and function spaces. In series papers [4–6] by Cuckovic and Zhao, the essential norms of weighted composition operators acting on weighted Bergman spaces have been characterized, and [4, 5] for the weighted Bergman spaces on the disc and [6] for the unweighted Bergman spaces on the strictly pseudoconvex domain in $\mathbb{C}^n$.

An active topic is to study the compact differences of two non-compact composition operators acting on a given function space. In 1989, Shapiro and Sundberg [16] characterized the compact differences of composition operators $C_\varphi - C_\psi$ by the boundary conditions of φ and ψ. In 2005, Moorhouse [10] used pseudo-hyperbolic distance between $\varphi(z)$ and $\psi(z)$ to characterize compact difference for composition operators acting on $A^2_\lambda(\mathbb{D})$, $\lambda > -1$. Later, Kriete and Moorhouse [8] extended their study to general linear combinations. Recently, Saukko [13, 14] studied the differences of composition operators between weighted

2020 *Mathematics Subject Classification*. Primary 47B33; Secondary 32A36, 47B38
Keywords. weighted composition operators, differences; essential norm, Carleson measures
Received: 19 July 2019; Accepted: 29 August 2022
Communicated by Dragan S. Djordjević
Research supported by the National Natural Science Foundations of China (Grant No. 12171136), Natural Science Foundation of Hebei Province (Grant No. A2020202005), Natural Science Foundation of Tianjin City (Grant No. 20JCYBJC00750).
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Bergman spaces in the unit disk. The key method in [13, 14] relies on Carleson measures and interpolation sequence for the weighted Bergman spaces. Choe, Koo and Park [1, 2] extend the results in [10] to the unit polydisk and unit ball.

The subject of this paper is the weighted composition operators and differences of composition operators on the weighted Bergman spaces on the unit ball. We will use Carleson measures to estimate the norm and essential norm of the differences of composition operators. The motivation of this work is to extend results in [4–6, 9, 13] to the settings of open unit ball.

The paper is organized as follows: In section 2, some basic notions and tools are introduced including Bergman kernel functions, involutions, pseudohyperbolic and Bergman metric, Carleson measures and their norms. In section 3, the essential norms of weighted composition operators on the weighted Bergman spaces are characterized in terms of Carleson measures and reproducing kernel functions. In section 4, we discuss the differences of composition operators on the weighted Bergman spaces.

In the following context, we write $A \lesssim B$ if there exists an absolute constant $C > 0$ such that $A \leq C \cdot B$, and $A \approx B$ represents $A \lesssim B$ and $B \lesssim A$.

2. Preliminary

If μ is a positive measure on $\mathbb{B}_n$ and $p > 0$, we denote $L^p(\mu)$ the Lebesgue space over $\mathbb{B}_n$ with respect to the measure μ. That is, $L^p(\mu)$ consists of all functions f defined in $\mathbb{B}_n$ for which

$$\|f\|_{L^p(\mu)} := \left[\int_{\mathbb{B}_n} |f(z)|^p \mathrm{d}\mu(z)\right]^{1/p} < \infty.$$

When $p \geq 1$, $\|\cdot\|_{L^p(\mu)}$ defines a norm and $L^p(\mu)$ becomes a Banach space.

If $t > -1$ and $\mathrm{d}v$ denotes the normalized Lebesgue volume measure on $\mathbb{B}_n$, we will define the weighted volume measure $\mathrm{d}v_t$ as following:

$$\mathrm{d}v_t(z) = c_t(1 - |z|^2)^t \mathrm{d}v(z),$$

where the constant is chosen so that $\int_{\mathbb{B}_n} c_t(1 - |z|^2)^t \mathrm{d}v(z) = 1$. The Bergman space A_t^p is the subspace of $L_t^p := L^p(\mathrm{d}v_t)$ that consists of analytic functions in $\mathbb{B}_n$.

We denote by $K_a^t(z) = (1 - \langle z, a\rangle)^{-(n+1+t)}$ the reproducing kernel function of A_t^2. For $p > 0$ and $t > -1$, we define the normalized Bergman kernel function for A_t^p by

$$k_a^{p,t}(z) := \left(\frac{1 - |a|^2}{(1 - \langle z, a\rangle)^2}\right)^{(n+1+t)/p}.$$

Fix a point $a \in \mathbb{B}_n$ and let P_a be the orthogonal projection of $\mathbb{C}^n$ onto the subspace $[a] = \{\lambda a : \lambda \in \mathbb{C}^n\}$ generated by a. Thus $P_0(z) = 0$ and

$$P_a(z) = \frac{\langle z, a\rangle}{\langle a, a\rangle} a, \quad a \neq 0.$$

Let $Q_a(z) = z - P_a(z)$ be the projection onto the orthogonal complement of $[a]$, and let $s_a = (1 - |a|^2)^{1/2}$. Now define

$$\Phi_a(z) = \frac{a - P_a(z) - s_a Q_a(z)}{1 - \langle z, a\rangle}, \quad a \in \mathbb{B}_n.$$

It is well known that Φ_a is a biholomorphic mapping of $\mathbb{B}_n$ onto itself, also called an *involution* of $\mathbb{B}_n$, with the following properties (see [12]):

(*i*) $\Phi_a(0) = a, \Phi_a(a) = 0;$

(*ii*) $\Phi_a(\Phi_a(z)) = z;$

(*iii*) $1 - |\Phi_a(z)|^2 = \dfrac{(1 - |a|^2)(1 - |z|^2)}{|1 - \langle z, a\rangle|^2};$

(*iv*) $1 - \langle \Phi_a(z), \Phi_a(w)\rangle = \dfrac{(1 - \langle a, a\rangle)(1 - \langle z, w\rangle)}{(1 - \langle z, a\rangle)(1 - \langle a, w\rangle)}.$

Recall that the pseudo-hyperbolic metric $\rho : \mathbb{B}_n \times \mathbb{B}_n \to [0, 1)$ is defined by

$$\rho(z, w) = |\Phi_z(w)|$$

for $z, w \in \mathbb{B}_n$. We denote the pseudohyperbolic ball by

$$B_\rho(a, r) = \{z \in \mathbb{B}_n : \rho(z, a) < r\}.$$

It is also well known that the pseudohyperbolic metric of $\mathbb{B}_n$ has the following properties (see [7]): for $z, w, a \in \mathbb{B}_n$ and the unitary matrix U, we have

$$\rho(U(z), U(w)) = \rho(z, w), \text{ and}$$

$$\rho(\Phi_a(z), \Phi_a(w)) = \rho(z, w), \text{ and}$$

$$\frac{|\rho(z, a) - \rho(a, w)|}{1 - \rho(z, a)\rho(a, w)} \leq \rho(z, w) \leq \frac{\rho(z, a) + \rho(a, w)}{1 + \rho(z, a)\rho(a, w)}. \tag{1}$$

We can define the so-called Bergman metric, β on $\mathbb{B}_n$, by:

$$\beta(z, w) = \frac{1}{2} \log \frac{1 + \rho(z, w)}{1 - \rho(z, w)}.$$

Let $B_\beta(z, r)$ be the ball in the Bergman metric of radius r centered at z. It is well-known that for $w \in B_\beta(z, \mathrm{arctanh} r)$ (equivalently $w \in B_\rho(z, r)$) there holds:

$$\mathrm{vol}_t(B_\rho(z, r)) \simeq |1 - \langle z, w\rangle|^{n+1+t} \simeq \left(1 - |z|^2\right)^{n+1+t} \simeq \left(1 - |w|^2\right)^{n+1+t}, \tag{2}$$

where the constants depend only on r. (See [19].) We will make heavy use of these estimates.

The following class of measures is useful in the study of many different operators. Similar classes have been defined and studied intensively in various analytic functions spaces.

Definition 2.1. *Let $p, q > 0$ and $t > -1$. A positive Borel measure μ on $\mathbb{B}_n$ is a (A_t^p, q)-Carleson measure if the inclusion map*

$$I : A_t^p \to L^q(\mu)$$

is bounded.

We introduce the Carleson measure theorem for A_t^p as the following theorem, see Theorem A in [11] and Theorem 50 in [17].

Theorem 2.2. *For a positive Borel measure μ on $\mathbb{B}_n$, $0 < p \leq q < \infty$, and $t > -1$, the following quantities are equivalent:*

(i) $\|\mu\|_{Oper}^q := \|I\|^q_{A_t^p \to L^q(d\mu)}$;

(ii) $\|\mu\|_{r,Geo} := \sup_{z\in\mathbb{B}_n} \frac{\mu(B_\rho(z,r))}{(1-|z|^2)^{(n+1+t)q/p}}$, *for $r > 0$;*

(iii) $\|\mu\|_{RK}^q := \sup_{a\in\mathbb{B}_n} \int_{\mathbb{B}_n} \left|\frac{(1-|a|^2)}{(1-\langle z,a\rangle)^2}\right|^{q(n+1+t)/p} d\mu(z) = \sup\limits_{a\in\mathbb{B}_n} \|k_a^{p,t}\|^q_{L^q(d\mu)}$.

Here, $\|\mu\|_{Oper}, \|\mu\|_{r,Geo}$ and $\|\mu\|_{RK}$ refer to the operator norm, geometric norm and reproducing kernel norm of μ respectively.

Carleson measures play an important role in the study of weighted composition operators. Suppose $u : \mathbb{B}_n \to \mathbb{C}$ is a measurable function and φ is an analytic self mapping of $\mathbb{B}_n$. Define measure $\mu_{u,\varphi}^{q,t}$ in $\mathbb{B}_n$ by

$$\mu_{u,\varphi}^{q,t}(E) = \int\limits_{\varphi^{-1}(E)} |u(z)|^q \mathrm{d}v_t(z)$$

for all Borel set $E \subset \mathbb{B}_n$.

The following result is a direct consequence of Theorem 2.2.

Theorem 2.3. *Suppose* $0 < p \le q < \infty$, $0 < r < 1$ *and* $s, t > -1$. *Let* u *be a measurable function on* $\mathbb{B}_n$ *and* φ *an analytic self mapping of* $\mathbb{B}_n$. *Then the following are equivalent:*

(i) *The weighted composition operator* $uC_\varphi : A_t^p \to L_s^q$ *is bounded.*

(ii) $\left\|\mu_{u,\varphi}^{q,s}\right\|_{r,Geo} < \infty$.

(iii) $\sup\limits_{a \in \mathbb{B}_n} \|uC_\varphi k_a^{p,t}\|_{L_s^q}^q < \infty$.

Furthermore, $\|uC_\varphi\|_{A_t^p \to L_s^q}^q$ *and* $\left\|\mu_{u,\varphi}^{q,s}\right\|_{r,Geo}$ *and the quantity in **(iii)** are all comparable.*

3. Weighted composition operators

In this section, we will discuss the essential norm of weighted composition operators. The results described are the generalization of those in [9] and [6]. For any bounded linear operator $T : X \to Y$, the essential norm of T is defined by

$$\|T\|_{X \to Y, e} = \inf\{\|T - \mathcal{K}\|_{X \to Y} : \mathcal{K} : X \to Y \text{ is compact.}\}.$$

It is clearly that $\|T\|_{X \to Y, e} = 0$ if and only if T is compact from X to Y.

Let $N \in \mathbb{N}$. Define the partial sum operator $S_N : H(\mathbb{B}_n) \to H(\mathbb{B}_n)$ by

$$S_N\left(\sum_\alpha c_\alpha z^\alpha\right) = \sum_{|\alpha| \le N} c_\alpha z^\alpha,$$

where $f = \sum_\alpha c_\alpha z^\alpha \in H(\mathbb{B}_n)$. Define also $R_N = I - S_N$. It is easy to see that these operators are uniformly bounded on A_t^p when $p > 1$. Furthermore, S_N is clearly compact. We denote $\mu_\delta(E) := \mu((\mathbb{B}_n \setminus \delta\mathbb{B}_n) \cap E)$ for $\delta \in (0, 1)$ and positive Borel measure μ. It is clear that μ_δ is a Carleson measure if μ is a Carleson measure.

Lemma 3.1. *Suppose* $1 < p \le q$, $t, s > -1$, $u \in L_s^q$ *and* φ *is an analytic self mapping of* $\mathbb{B}_n$ *such that the operator* $uC_\varphi : A_t^p \to L_s^q$ *is bounded. Then we have*

(i)

$$\lim_{N \to \infty} \|(uC_\varphi)R_N\|_{A_t^p \to L_s^q}^q \lesssim \lim_{\delta \to 1} \sup_{\|f\|_{A_t^p} \le 1} \int_{\varphi^{-1}(\mathbb{B}_n \setminus \delta\mathbb{B}_n)} |u(z) f(\varphi(z))|^q dv_s(z).$$

(ii) *For every* $0 < \delta < 1$,

$$\lim_{N \to \infty} \sup_{\|f\|_{A_t^p} \le 1} \int_{\varphi^{-1}(\delta\mathbb{B}_n)} \left|(uC_\varphi \circ R_N f)(z)\right|^q dv_s(z) = 0.$$

Proof. Let $0 < \delta < 1$ be fixed. We assume that $f(z) = \sum_\alpha c_\alpha z^\alpha \in A_t^p$ with norm 1. We are going to prove **(i)**. The proof of **(ii)** is similar and we omit it.

Using the reproducing property of the Bergman kernel we get

$$\begin{aligned} \left|uC_\varphi \circ R_N f(z)\right| &= \left|uC_\varphi\left(\sum_{|\alpha|>N} c_\alpha z^\alpha\right)\right| = \left|u(z)\left(\sum_{|\alpha|>N} c_\alpha \varphi(z)^\alpha\right)\right| \\ &= \left|u(z)\left\langle R_N f, K^t_{\varphi(z)}\right\rangle_{L^2_t}\right| = \left|u(z)\left\langle f, R_N K^t_{\varphi(z)}\right\rangle_{L^2_t}\right| \\ &\leq |u(z)|\|f\|_{A^p_t}\left\|R_N K^t_{\varphi(z)}\right\|_{L^{p'}_t}, \end{aligned}$$

where p' represents the Hölder conjugate of p. Denote $l_j = \frac{\Gamma(n+1+t+j)}{j!\Gamma(n+1+t)}$. Then for all $z \in \varphi^{-1}(\delta\mathbb{B}_n)$,

$$\left|R_N K^t_{\varphi(z)}(w)\right| = \left|\sum_{j=N+1}^{\infty} l_j(\langle\varphi(z), w\rangle)^j\right| \leq \sum_{j=N+1}^{\infty} l_j\delta^j.$$

Thus

$$\sup_{\|f\|_{A^p_t}\leq 1}\int_{\varphi^{-1}(\delta\mathbb{B}_n)} |(uC_\varphi \circ R_N f)(z)|^q \mathrm{d}v_s(z) \leq \|u\|^q_{L^q_s}\left[\sum_{j=N+1}^{\infty} l_j\delta^j\right]^q \to 0$$

as $N \to \infty$. Therefore we can estimate

$$\begin{aligned} \lim_{N\to\infty}\|(uC_\varphi)R_N\|^q_{A^p_t\to L^q_s} &\leq \lim_{N\to\infty}\sup_{\|f\|_{A^p_t}\leq 1}\int_{\varphi^{-1}(\mathbb{B}_n\setminus\delta\mathbb{B}_n)} |uC_\varphi \circ R_N f(z)|^q \mathrm{d}v_s(z) \\ &\lesssim \sup_{\|f\|_{A^p_t}\leq 1}\int_{\varphi^{-1}(\mathbb{B}_n\setminus\delta\mathbb{B}_n)} |uC_\varphi f(z)|^q \mathrm{d}v_s(z) \end{aligned}$$

where the last inequality comes from the fact that R_N is uniformly bounded. Finally letting $\delta \to 1^-$ yields the claim. □

We can immediately get the following estimate for the essential norm.

Lemma 3.2. *Let the assumptions be as those in the previous lemma. Then*

$$\|uC_\varphi\|^q_{A^p_t\to L^q_s,e} \lesssim \lim_{\delta\to 1}\sup_{\|f\|_{A^p_t}\leq 1}\int_{\varphi^{-1}(\mathbb{B}_n\setminus\delta\mathbb{B}_n)} |u(z)f(\varphi(z))|^q dv_s(z).$$

Proof. Noting that

$$\|uC_\varphi\|_{A^p_t\to L^q_s,e} \leq \|uC_\varphi - uC_\varphi S_N\|_{A^p_t\to L^q_s} = \|uC_\varphi R_N\|_{A^p_t\to L^q_s},$$

we get the desired estimate by Lemma 3.1.

□

To control the essential norm of weighted composition operators by the pullback measures, we need the following lemma stated in [19].

Lemma 3.3. *Suppose $r > 0$, $p > 0$, and $t > -1$. Then there exists a constant $C > 0$ such that*

$$|f(z)|^p \leq \frac{C}{(1-|z|^2)^{n+1+t}}\int_{B_\rho(z,r)} |f(w)|^p dv_t(w) \tag{3}$$

for $f \in H(\mathbb{B}_n)$ and $z \in \mathbb{B}_n$.

Theorem 3.4. *Suppose $1 < p \leq q < \infty$, $0 < r < 1$ and $t, s > -1$. Let u be a measurable function on $\mathbb{B}_n$ and φ an analytic self mapping of $\mathbb{B}_n$ such that the operator $uC_\varphi : A_t^p \to L_s^q$ is bounded. Then*

$$\|uC_\varphi\|^q_{A_t^p \to L_s^q, e} \lesssim \lim_{\delta \to 1^-} \left\| \left(\mu_{u,\varphi}^{q,s}\right)_\delta \right\|_{r,Geo}.$$

Proof. Since uC_φ is bounded from A_t^p to L_s^q, the pullback measure $\mu_{u,\varphi}^{q,s}$ is a Carleson measure, and so is $\left(\mu_{u,\varphi}^{q,s}\right)_\delta$ for any $\delta \in (0, 1)$. Combining this fact with Lemma 3.3 and Fubini's Theorem, we have

$$\begin{aligned}
& \int_{\varphi^{-1}(\mathbb{B}_n \setminus \delta\mathbb{B}_n)} |u(z) f(\varphi(z))|^q \mathrm{d}v_s(z) \\
\lesssim \; & \int_{\mathbb{B}_n} \mathbb{1}_{\varphi^{-1}(\mathbb{B}_n \setminus \delta\mathbb{B}_n)}(z) |u(z)|^q \frac{\int_{B_\rho(\varphi(z), r)} |f(w)|^q \mathrm{d}v_t(w)}{(1 - |\varphi(z)|^2)^{n+1+t}} \mathrm{d}v_s(z) \\
\simeq \; & \int_{\mathbb{B}_n} |f(w)|^q \frac{\int_{\varphi^{-1}\left(B_\rho(w,r) \cap (\mathbb{B}_n \setminus \delta\mathbb{B}_n)\right)} |u(z)|^q \mathrm{d}v_s(z)}{(1 - |w|^2)^{n+1+t}} \mathrm{d}v_t(w) \\
= \; & \int_{\mathbb{B}_n} |f(w)|^q \frac{\left(\mu_{u,\varphi}^{q,s}\right)_\delta (B_\rho(w, r))}{\left[\mathrm{vol}_t(B_\rho(w, r))\right]^{q/p}} \left[\mathrm{vol}_t(B_\rho(w, r))\right]^{\frac{q-p}{p}} \mathrm{d}v_t(w) \\
\leq \; & \left\| \left(\mu_{u,\varphi}^{q,s}\right)_\delta \right\|_{r,Geo} \int_{\mathbb{B}_n} |f(w)|^p \left[|f(w)| \mathrm{vol}_t(B_\rho(w, r))^{1/p}\right]^{q-p} \mathrm{d}v_t(w) \\
\leq \; & \left\| \left(\mu_{u,\varphi}^{q,s}\right)_\delta \right\|_{r,Geo} \|f\|^p_{A_t^p} \cdot \|f\|^{q-p}_{A_t^p} \\
= \; & \left\| \left(\mu_{u,\varphi}^{q,s}\right)_\delta \right\|_{r,Geo} \|f\|^q_{A_t^p}.
\end{aligned}$$

The proof will be completed by following Lemma 3.2. □

In the rest of this section, we are going to characterize the essential norm of weighted composition operators in terms of reproducing kernels.

Lemma 3.5. *Let $0 < p \leq q < \infty$ and $0 < \delta < 1$. Suppose that a positive measure μ on $\mathbb{B}_n$ is an (A_t^p, q)-Carleson measure. Then μ_δ is also an (A_t^p, q)-Carleson measure. Moreover, for any fixed $0 < \epsilon < 1$, we have*

$$\|\mu_\delta\|^q_{Oper} \lesssim \sup_{w \in \mathbb{B}_n \setminus (1-\epsilon)\delta\mathbb{B}_n} \int_{\mathbb{B}_n} \left| k_w^{p,t}(z) \right|^q d\mu(z),$$

where the constant involved depends on ϵ.

Proof. For $z \in \mathbb{B}_n \setminus \delta\mathbb{B}_n$, we have $\rho(z, 0) = |z| \geq \delta$. For $0 < \epsilon < 1$ fixed, $\rho(w, z) < \epsilon\delta$ implies that

$$\rho(w, 0) \geq \rho(z, 0) - \rho(w, z) > \delta(1 - \epsilon).$$

Keeping this fact in mind, we have that

$$\begin{aligned}
\|\mu_\delta\|_{Oper}^q &= \sup_{\|f\|_{A_t^p}\le 1}\int_{\mathbb{B}_n}|f(z)|^q \mathrm{d}\mu_\delta(z)\\
&\lesssim \sup_{\|f\|_{A_t^p}\le 1}\int_{\mathbb{B}_n}\left(\frac{1}{\mathrm{vol}_t(B_\rho(z,\delta\epsilon))}\int_{B_\rho(z,\delta\epsilon)}|f(w)|^p \mathrm{d}v_t(w)\right)^{q/p}\mathrm{d}\mu_\delta(z)\\
&\simeq \sup_{\|f\|_{A_t^p}\le 1}\int_{\mathbb{B}_n}\left(\int_{\mathbb{B}_n}\mathbb{1}_{B_\rho(z,\delta\epsilon)}(w)\left|k_w^{p,t}(z)\right|^p|f(w)|^p \mathrm{d}v_t(w)\right)^{q/p}\mathrm{d}\mu_\delta(z)\\
&\le \sup_{\|f\|_{A_t^p}\le 1}\left[\int_{\mathbb{B}_n}\left(\int_{\mathbb{B}_n}\mathbb{1}_{B_\rho(z,\delta\epsilon)}(w)\left|k_w^{p,t}(z)\right|^q|f(w)|^q \mathrm{d}\mu_\delta(z)\right)^{p/q}\mathrm{d}v_t(w)\right]^{q/p}\\
&\le \sup_{\|f\|_{A_t^p}\le 1}\left(\sup_{w\in\mathbb{B}_n\setminus(1-\epsilon)\delta\mathbb{B}_n}\int_{\mathbb{B}_n}\left|k_w^{p,t}(z)\right|^q \mathrm{d}\mu(z)\right)\cdot\|f\|_{A_t^p}^q,
\end{aligned}$$

where the first inequality follows from (3), and the second inequality follows from Minkowski's inequality for integrals. That completes the proof. □

The Bergman projection operator $P_t : L_t^p \to A_t^p$ for $t > -1$ is defined by

$$P_t f(z) = \int_{\mathbb{B}_n}\frac{f(w)}{(1-\langle z,w\rangle)^{n+1+t}}\mathrm{d}v_t(w),\quad f\in L_t^1.$$

By Theorem 2.10 in [19], P_t is bounded from L_t^p into A_t^p.

Lemma 3.6. *Let $1 < p < \infty$. Let $f \in A_t^p$. Then as $\delta \to 1^-$,*

$$\sup_{\|f\|_{A_t^p}\le 1}|P_t(f-\mathbb{1}_{\delta\mathbb{B}_n}f)(w)| \to 0$$

uniformly on compact subsets of $\mathbb{B}_n$.

Proof. Using Hölder's inequality for any $f \in A_t^p$ with $\|f\|_{A_t^p} \le 1$, we have

$$\begin{aligned}
|P_t(f-\mathbb{1}_{\delta\mathbb{B}_n}f)(w)| &= \left|\int_{\mathbb{B}_n\setminus\delta\mathbb{B}_n}\frac{f(z)}{(1-\langle w,z\rangle)^{n+1+t}}\mathrm{d}v_t(z)\right|\\
&\le \left(\int_{\mathbb{B}_n\setminus\delta\mathbb{B}_n}|f(z)|^p \mathrm{d}v_t(z)\right)^{1/p}\cdot\left(\int_{\mathbb{B}_n\setminus\delta\mathbb{B}_n}|1-\langle w,z\rangle|^{-(n+1+t)p'}\mathrm{d}v_t(z)\right)^{1/p'}\\
&\le \left(\int_{\mathbb{B}_n\setminus\delta\mathbb{B}_n}|1-\langle w,z\rangle|^{-(n+1+t)p'}\mathrm{d}v_t(z)\right)^{1/p'},
\end{aligned}$$

where p' represents the Hölder conjugate of p. Let E be a compact subset of $\mathbb{B}_n$. It is clear that $|1 - \langle w,z\rangle|^{-(n+1+t)p'}$ is uniformly bounded for all $w \in E$ and $z \in \mathbb{B}_n$. Thus as $\delta \to 1^-$,

$$\begin{aligned}
\sup_{\|f\|_{A_t^p}\le 1}|P_t(f-\mathbb{1}_{\delta\mathbb{B}_n}f)(w)| &\le \left(\int_{\mathbb{B}_n\setminus\delta\mathbb{B}_n}|1-\langle w,z\rangle|^{-(n+1+t)p'}\mathrm{d}v_t(z)\right)^{1/p'}\\
&\lesssim \mathrm{vol}_t(\mathbb{B}_n\setminus\delta\mathbb{B}_n)^{1/p'} \to 0.
\end{aligned}$$

That completes the proof. □

Now we can estimate the essential norm in terms of reproducing kernel.

Theorem 3.7. *Let $1 < p \leq q < \infty$. Suppose uC_φ is bounded from A^p_t into L^q_s. Then we have*

$$\|uC_\varphi\|_{A^p_t \to L^q_s, e} \simeq \limsup_{|z|\to 1^-} \left\| uC_\varphi(k^{p,t}_z) \right\|_{L^q_s}$$

Proof. It is well known that $k^{p,t}_z \in A^p_t$ is unimodular and converges to 0 uniformly on compact subsets of $\mathbb{B}_n$ as $|z| \to 1$. For any compact operator $\mathcal{K}$ from A^p_t into L^q_s, one has $\|\mathcal{K}k^{p,t}_z\|_{L^q_s} \to 0$ as $|z| \to 1$. Therefore,

$$\begin{aligned}
\|uC_\varphi - \mathcal{K}\|_{A^p_t \to L^q_s} &\geq \limsup_{|z|\to 1} \|(uC_\varphi - \mathcal{K})k^{p,t}_z\|_{L^q_s} \\
&\geq \limsup_{|z|\to 1} \left(\|(uC_\varphi)k^{p,t}_z\|_{L^q_s} - \|(\mathcal{K})k^{p,t}_z\|_{L^q_s} \right) \\
&= \limsup_{|z|\to 1} \|(uC_\varphi)k^{p,t}_z\|_{L^q_s}.
\end{aligned}$$

Hence

$$\|uC_\varphi\|_{A^p_t \to L^q_s, e} = \inf_{\mathcal{K}} \|uC_\varphi - \mathcal{K}\|_{A^p_t \to L^q_s} \geq \limsup_{|z|\to 1} \|(uC_\varphi)k^{p,t}_z\|_{L^q_s}.$$

To prove the contrary inequality, we let

$$T_k(\cdot) = uC_\varphi P_t \left(\mathbb{1}_{\frac{k-1}{k}\mathbb{B}_n} \cdot \right) \text{ for } k = 1, 2, \ldots$$

It is easy to verify that the operator $(\mathbb{1}_{r\mathbb{B}_n} \cdot) : A^p_t \to L^p_t$ ($r \in (0,1)$) is compact. We just note that $\|\mathbb{1}_{r\mathbb{B}_n} f_j\|_{L^p_t} \to 0$ as $j \to \infty$ whenever $\{f_j\}$ is bounded in A^p_t and converges to 0 uniformly on compact subsets of $\mathbb{B}_n$. Thus $\{T_k\}$ are compact operators. For any $f \in A^p_t$ with norm 1, we have

$$\begin{aligned}
&\|(uC_\varphi - T_k)f\|^q_{L^q_s} \\
= &\int_{\mathbb{B}_n} \left| uC_\varphi f(w) - uC_\varphi P_t \left(\mathbb{1}_{\frac{k-1}{k}\mathbb{B}_n}(w) f(w) \right) \right|^q \mathrm{d}v_s(w) \\
= &\int_{\mathbb{B}_n} \left| f(w) - P_t \left(\mathbb{1}_{\frac{k-1}{k}\mathbb{B}_n} f \right)(w) \right|^q \mathrm{d}\mu^{q,s}_{u,\varphi}(w) \\
= &\left(\int_{\delta\mathbb{B}_n} + \int_{\mathbb{B}_n \setminus \delta\mathbb{B}_n} \right) \left| P_t \left[f - \mathbb{1}_{\frac{k-1}{k}\mathbb{B}_n} f \right](w) \right|^q \mathrm{d}\mu^{q,s}_{u,\varphi}(w) \\
= &I_1 + I_2
\end{aligned}$$

where $0 < \delta < 1$. By Lemma 3.6, for any $\epsilon > 0$, there exists a $k_0 > 0$ so that for any $k \geq k_0$,

$$\sup_{\|f\|_{A^p_t} \leq 1} I_1 < \epsilon.$$

For I_2, since $\left(\mu^{q,s}_{u,\varphi}\right)_\delta$ is also an (A^p_t, q)-Carleson measure, we have

$$\begin{aligned}
I_2 &= \int_{\mathbb{B}_n} \left| P_t \left[f - \mathbb{1}_{\frac{k-1}{k}\mathbb{B}_n} f \right](w) \right|^q \mathrm{d}\left(\mu^{q,s}_{u,\varphi}\right)_\delta(w) \\
&\leq \left\| \left(\mu^{q,s}_{u,\varphi}\right)_\delta \right\|^q_{Oper} \cdot \left\| P_t \left[f - \mathbb{1}_{\frac{k-1}{k}\mathbb{B}_n} f \right] \right\|^q_{A^p_t} \\
&\leq \left\| \left(\mu^{q,s}_{u,\varphi}\right)_\delta \right\|^q_{Oper} \cdot \left\| \left(1 - \mathbb{1}_{\frac{k-1}{k}\mathbb{B}_n} \right) f \right\|^q_{A^p_t} \cdot \|P_t\|^q_{L^p_t \to A^p_t} \\
&\leq \left\| \left(\mu^{q,s}_{u,\varphi}\right)_\delta \right\|^q_{Oper} \|P_t\|^q_{L^p_t \to A^p_t}.
\end{aligned}$$

Combining the estimates above, for any fixed $0 < \delta < 1$, whenever $k \geq k_0$, we can get

$$\begin{aligned} \|uC_\varphi - T_k\|^q_{A^p_t \to L^q_s} &= \sup_{\|f\|_{A^p_t} \leq 1} \|(uC_\varphi - T_k)f\|^q_{L^q_s} \\ &\leq \epsilon + \left\|\left(\mu^{q,s}_{u,\varphi}\right)_\delta\right\|^q_{Oper} \|P_t\|^q_{L^p_t \to A^p_t}. \end{aligned}$$

Since ϵ is arbitrary,

$$\|uC_\varphi\|_{A^p_t \to L^q_s, e} \leq \inf_k \|uC_\varphi - T_k\|_{A^p_t \to L^q_s} \leq \left\|\left(\mu^{q,s}_{u,\varphi}\right)_\delta\right\|_{Oper} \|P_t\|_{L^p_t \to A^p_t}$$

for any $0 < \delta < 1$. Letting $\delta \to 1^-$ we obtain that

$$\|uC_\varphi\|_{A^p_t \to L^q_s, e} \leq \|P_t\|_{L^p_t \to A^p_t} \cdot \limsup_{\delta \to 1^-} \left\|\left(\mu^{q,s}_{u,\varphi}\right)_\delta\right\|_{Oper}.$$

To complete the proof, we use Lemma 3.5 to conclude that

$$\begin{aligned} \|uC_\varphi\|^q_{A^p_t \to L^q_s, e} &\lesssim \limsup_{|z| \to 1^-} \int_{\mathbb{B}_n} \left|k^{p,t}_z(w)\right|^q \mathrm{d}\mu^{q,s}_{u,\varphi}(w) \\ &= \limsup_{|z| \to 1^-} \int_{\mathbb{B}_n} |u(w)|^q \left|k^{p,t}_z(\varphi(w))\right|^q \mathrm{d}v_s(w) \\ &= \limsup_{|z| \to 1^-} \left\|uC_\varphi(k^{p,t}_z)\right\|^q_{L^q_s}. \end{aligned}$$

□

4. Differences of composition operators

In this section, we will estimate the essential norm of differences of composition operators from A^p_t to L^q_s when $1 < p \leq q < \infty$. The following lemma can be verified by calculation directly, and the proof is omitted.

Lemma 4.1. *The function* $h : (-1, 1) \times (-1, 1) \to \mathbb{R}$ *is defined as*

$$h(x, y) := \frac{x + y}{1 + xy}.$$

Then $h < 1$, *h increases strictly in x and y. Moreover*

$$\lim_{x \to 1^-} h(x, y) = 1 \quad \text{and} \quad \lim_{y \to 1^-} h(x, y) = 1.$$

The following lemma will play a role in the proof of our main result in this section. The statement is reasonable, but we did not find any proof in detail, hence we include the proof below for integrity.

Lemma 4.2. *Let* $0 < p < \infty$, $t > -1$ *and* $0 < r < 1$. *Fix* $z \in \mathbb{B}_n$. *Then for all* $a \in \mathbb{B}_n$ *with* $\rho(z, a) < r$ *there are* $0 < R < 1$ *and* $C > 0$ *such that*

$$|f(z) - f(a)|^p \leq C\rho(z, a)^p \frac{\int_{B_\rho(a,R)} |f(w)|^p dv_t(w)}{(1 - |a|^2)^{n+1+t}}$$

where $f \in A^p_t$ *with* $\|f\|_{A^p_t} \leq 1$.

Proof. Let $g = f \circ \Phi_a$. Hence $f = g \circ \Phi_a$. We have

$$|f(z) - f(a)| = |g(\Phi_a(z)) - g(0)| \leq |\Phi_a(z)| \sup_{|\xi|<|\Phi_a(z)|} |\nabla g(\xi)|,$$

where

$$\nabla g(\xi) = \left(\frac{\partial}{\partial z_1} g, \ldots, \frac{\partial}{\partial z_n} g\right)(\xi).$$

Let $R_1 = \frac{1+r}{2}$. According to Theorem 2.2 in [19],

$$g(R_1\xi) = \int_{\mathbb{B}_n} \frac{g(R_1\eta)}{(1 - \langle \xi, \eta \rangle)^{n+1}} \mathrm{d}v(\eta), \quad \forall \xi \in \mathbb{B}_n.$$

Changing variables gives

$$g(\xi) = \int_{\mathbb{B}_n} \frac{g(R_1\eta)}{\left(1 - \langle \frac{\xi}{R_1}, \eta \rangle\right)^{n+1}} \mathrm{d}v(\eta) = R_1^2 \int_{R_1\mathbb{B}_n} \frac{g(\eta)}{\left(R_1^2 - \langle \xi, \eta \rangle\right)^{n+1}} \mathrm{d}v(\eta).$$

Then we have

$$\begin{aligned} \left|\frac{\partial}{\partial z_j} g(\xi)\right| &= \left|R_1^2 \int_{R_1\mathbb{B}_n} \frac{g(\eta)(n+1)\bar{\eta}_j}{\left(R_1^2 - \langle \xi, \eta \rangle\right)^{n+2}} \mathrm{d}v(\eta)\right| \\ &\leq (n+1)\left(\frac{1+r}{2}\right)^2 \left(\frac{4}{1-r^2}\right)^{n+2} \int_{R_1\mathbb{B}_n} |g(\eta)| \mathrm{d}v(\eta) \\ &\leq C_r \sup_{|\eta|<R_1} |g(\eta)|. \end{aligned}$$

It follows that

$$|\nabla g(\xi)|^p \leq n^{p/2} C_r \sup_{|\eta|<R_1} |g(\eta)|^p$$

for every $|\xi| < r$. Hence we have

$$|f(z) - f(a)|^p \leq |\Phi_a(z)|^p \sup_{|\xi|<|\Phi_a(z)|} |\nabla g(\xi)|^p \tag{4}$$

$$\leq n^{p/2} C_r |\Phi_a(z)|^p \sup_{|\eta|<R_1} |g(\eta)|^p = n^{p/2} C_r |\Phi_a(z)|^p \sup_{|\Phi_a(\zeta)|<R_1} |f(\zeta)|^p. \tag{5}$$

For every $\zeta \in \mathbb{B}_n$ with $\rho(a, \zeta) < R_1 = (1+r)/2$, we let $R = \frac{1+3r}{2+r+r^2}$. According to the *strong triangle inequality* (1), conditions $\rho(\zeta, a) < (1+r)/2$ and $\rho(\zeta, \omega) < r$ imply that $\rho(a, \omega) < \frac{(1+r)/2+r}{1+r(1+r)/2} = R$. By Lemma 3.3 and (2), we have

$$|f(\zeta)|^p \lesssim \frac{1}{(1 - |\zeta|^2)^{n+1+t}} \int_{\rho(\zeta,\omega)<r} |f(\omega)|^p \mathrm{d}v_t(\omega) \tag{6}$$

$$\lesssim \frac{1}{(1 - |a|^2)^{n+1+t}} \int_{\rho(a,\omega)<R} |f(\omega)|^p \mathrm{d}v_t(\omega). \tag{7}$$

The inequality we need can be obtained by plugging (6) into (4). □

Now we can prove the upper bound of the essential norm of differences.

Theorem 4.3. *Let $0 < p \leq q < \infty$ and $t, s > -1$. Suppose φ and ψ are analytic self mappings of $\mathbb{B}_n$ and $\sigma := \rho(\varphi, \psi)$ such that the operators σC_φ and σC_ψ map A_t^p into L_s^q. Then the following holds:*

(i) *The difference operator $C_\varphi - C_\psi$ maps A_t^p into A_s^q and*

$$\|C_\varphi - C_\psi\|^q_{A_t^p \to A_s^q} \lesssim \max\left\{\left\|\mu^{q,s}_{\sigma,\varphi}\right\|_{r,Geo}, \left\|\mu^{q,s}_{\sigma,\psi}\right\|_{r,Geo}\right\}$$

where the involved constant depends only on t, s, p and q.

(ii) *If $p > 1$, then*

$$\|C_\varphi - C_\psi\|^q_{A_t^p \to A_s^q, e} \lesssim \max\left\{\lim_{\delta\to 1}\left\|\left(\mu^{q,s}_{\sigma,\varphi}\right)_\delta\right\|_{r,Geo}, \lim_{\delta\to 1}\left\|\left(\mu^{q,s}_{\sigma,\psi}\right)_\delta\right\|_{r,Geo}\right\}$$

Proof. Similarly to Theorem 3.2 in [13], we just sketch the proof of **(ii)**. Since the partial sum operator S_N is compact for each $N \in \mathbb{N}$ we can estimate

$$\|C_\varphi - C_\psi\|_{A_t^p \to A_s^q, e} \le \limsup_{N\to\infty}\left\|\left(C_\varphi - C_\psi\right)R_N\right\|_{A_t^p \to A_s^q}.$$

Denote $E = \{z \in \mathbb{B}_n : \sigma(z) \ge r\}$ and $E' = \mathbb{B}_n \setminus E$. Then for $f \in A_t^p$

$$\left\|\left(C_\varphi - C_\psi\right)R_N f\right\|^q_{A_t^p \to A_s^q} = \left(\int_E + \int_{E'}\right)^q := \left(\mathcal{E}_N + \mathcal{E}'_N\right)^q.$$

We firstly estimate $\mathcal{E}_N$ as follows

$$\begin{aligned}
\mathcal{E}_N &= \int_E |(C_\varphi - C_\psi)\circ R_N f(z)|^q \mathrm{d}v_t(z) \\
&\lesssim \int_E |(\sigma C_\varphi)\circ R_N f(z)|^q \mathrm{d}v_t(z) + \int_E |(\sigma C_\psi)\circ R_N f(z)|^q \mathrm{d}v_t(z) \\
&\le \|(\sigma C_\varphi)R_N\|^q_{A_t^p \to L_s^q} + \|(\sigma C_\psi)R_N\|^q_{A_t^p \to L_s^q}
\end{aligned}$$

whenever $\|f\|_{A_t^p} \le 1$, $N \in \mathbb{N}$. Thus by **(i)** in Lemma 3.1,

$$\limsup_{N\to\infty}\sup_{\|f\|_{A_t^p}\le 1}\mathcal{E}_N \lesssim \max\left\{\lim_{\delta\to 1}\left\|(\mu^{q,s}_{\sigma,\varphi})_\delta\right\|_{r,Geo}, \lim_{\delta\to 1}\left\|(\mu^{q,s}_{\sigma,\psi})_\delta\right\|_{r,Geo}\right\}.$$

To estimate $\mathcal{E}'_N$, we let $0 < \delta < 1$ be arbitrary. Suppose $z \in E' \cap \varphi^{-1}(\delta\mathbb{B}_n)$. By the strong triangle inequality (1) of the pseudohyperbolic metric, we can find $\delta' \in (0,1)$ such that $E' \cap \varphi^{-1}(\delta\mathbb{B}_n) \subset \psi^{-1}(\delta'\mathbb{B}_n)$. Thus by **(ii)** in Lemma 3.1,

$$\lim_{N\to\infty}\sup_{\|f\|_{A_t^p}\le 1}\int_{E'\cap\varphi^{-1}(\delta\mathbb{B}_n)}\left|(C_\varphi \circ R_N f)(z)\right|^q \mathrm{d}v_s(z) = 0$$

and

$$\lim_{N\to\infty}\sup_{\|f\|_{A_t^p}\le 1}\int_{E'\cap\varphi^{-1}(\delta\mathbb{B}_n)}\left|(C_\psi \circ R_N f)(z)\right|^q \mathrm{d}v_s(z) = 0.$$

Let $F = E' \cap \varphi^{-1}(\mathbb{B}_n \setminus \delta\mathbb{B}_n)$. Since $\{R_N\}$ are uniformly bounded with respect to N, we have

$$\limsup_{N\to\infty}\sup_{\|f\|_{A_t^p}\le 1}\mathcal{E}'_N \lesssim \sup_{\|f\|_{A_t^p}\le 1}\int_F\left|(C_\varphi - C_\psi)f(z)\right|^q \mathrm{d}v_s(z).$$

Following Lemma 4.2 and Fubini's theorem we have

$$\begin{aligned}
&\int_F \left|(C_\varphi - C_\psi)f(z)\right|^q \,\mathrm{d}v_s(z)\\
\lesssim\ & \int_F |\sigma(z)|^q \frac{\int_{\rho(\varphi(z),w)<R} |f(w)|^p \mathrm{d}v_t(w)}{(1-|\varphi(z)|^2)^{(n+1+t)q/p}} \mathrm{d}v_s(z)\\
\lesssim\ & \int_{\mathbb{B}_n} |f(w)|^p \frac{\int_{\varphi^{-1}(\{\rho(z,w)<R\})\cap F} |\sigma(z)|^q \mathrm{d}v_s(z)}{(1-|w|^2)^{(n+1+t)q/p}} \mathrm{d}v_t(w)\\
\le\ & \int_{\mathbb{B}_n} |f(w)|^p \frac{\int_{\varphi^{-1}(\{\rho(z,w)<R\})\cap(\mathbb{B}_n\setminus\delta\mathbb{B}_n)} |\sigma(z)|^q \mathrm{d}v_s(z)}{(1-|w|^2)^{(n+1+t)q/p}} \mathrm{d}v_t(w)\\
\le\ & \|f\|^p_{A^p_t} \left\|(\mu^{q,s}_{\sigma,\varphi})_\delta\right\|_{r,Geo}.
\end{aligned}$$

Letting $\delta \to 1$ and using the above estimates we get

$$\begin{aligned}
&\left\|C_\varphi - C_\psi\right\|^q_{A^p_t\to A^q_s,e} \le \limsup_{N\to\infty} \sup_{\|f\|_{A^p_t}<1} \mathcal{E}_N + \limsup_{N\to\infty} \sup_{\|f\|_{A^p_t}<1} \mathcal{E}'_N\\
\lesssim\ & \max\left\{\lim_{\delta\to1} \left\|(\mu^{q,s}_{\sigma,\varphi})_\delta\right\|_{r,Geo}, \lim_{\delta\to1} \left\|(\mu^{q,s}_{\sigma,\psi})_\delta\right\|_{r,Geo}\right\}
\end{aligned}$$

This proves **(ii)**. □

Next we will prove the lower bound for the essential norm. The following lemma is well known. See [13].

Lemma 4.4. *Let X and Y be Banach spaces and $T : X \to Y$ a bounded linear operator. Let $\{x_n\}$ be any sequence in X such that $\|x_n\|_X = 1$ for every n and $x_n \to 0$ weakly when $n \to \infty$. Then*

$$\|T\|_{X\to Y,e} \ge \limsup_{n\to\infty} \|T(x_n)\|_Y.$$

Lemma 4.5. *If $0 < r < 1$, then*

$$\left|1 - \frac{1-\langle z,a\rangle}{1-\langle w,a\rangle}\right| \gtrsim |a|\rho(z,w)$$

whenever $a, w \in \mathbb{B}_n$ and $\rho(z,a) < r$. The constant involved depends only on r.

Proof. Let $a, z \in \mathbb{B}_n$ such that $\rho(a,z) < r$. For every $w \in \mathbb{B}_n$ we have that

$$\begin{aligned}
|z-w|^2 &= |z-(P_z(w)+Q_z(w))|^2 = |z-P_z(w)|^2 + |Q_z(w)|^2 &(8)\\
&\ge |z-P_z(w)|^2 + |s_zQ_z(w)|^2 = |z-P_z(w)-s_zQ_z(w)|^2 &(9)
\end{aligned}$$

where $s_z = (1-|z|^2)^{1/2} < 1$. We can obtain

$$\left|1 - \frac{1-\langle z,a\rangle}{1-\langle w,a\rangle}\right| = |a|\frac{|z-w|}{|1-\langle z,w\rangle|}\left|\frac{1-\langle w,z\rangle}{1-\langle w,a\rangle}\right| \gtrsim |a|\rho(z,w)$$

where the last inequality follows from the estimate above and the remark below Lemma 2.27 in [19]. □

Applying Lemma 4.5, we can obtain the following lemma by the same method as Lemma 4.4 in [13].

Lemma 4.6. *If* $0 < r < 1$, $\gamma > 0$, *then*

$$\left| \left(\frac{1-|a|^2}{(1-\langle z,a\rangle)^2} \right)^{\gamma} - \left(\frac{1-|a|^2}{(1-\langle w,a\rangle)^2} \right)^{\gamma} \right| \gtrsim |a| \frac{\rho(z,w)}{(1-|a|^2)^{\gamma}}$$

for $a, w \in \mathbb{B}_n$ *and* $\rho(z,a) < r$. *The constant involved depends only on* r *and* γ.

Theorem 4.7. *Let* $0 < p \le q < \infty$ *and* $t, s > -1$. *Suppose* φ *and* ψ *are holomorphic self mappings of* $\mathbb{B}_n$ *such that* $C_\varphi - C_\psi$ *is bounded from* A_t^p *to* A_s^q. *Then*

(i) *the operators* σC_φ *and* σC_ψ *map* A_t^p *into* L_s^q *and*

$$\sup_{a\in\mathbb{B}_n} \left\| (C_\varphi - C_\psi) k_a^{p,t} \right\|_{A_s^q}^q \gtrsim \max\left\{ \left\| \mu_{\sigma,\varphi}^{q,s} \right\|_{r,Geo}, \left\| \mu_{\sigma,\psi}^{q,s} \right\|_{r,Geo} \right\}$$

(ii)

$$\limsup_{|a|\to 1} \left\| (C_\varphi - C_\psi) k_a^{p,t} \right\|_{A_s^q}^q \gtrsim \max\left\{ \lim_{\delta\to 1} \left\| (\mu_{\sigma,\varphi}^{q,s})_\delta \right\|_{r,Geo}, \lim_{\delta\to 1} \left\| (\mu_{\sigma,\psi}^{q,s})_\delta \right\|_{r,Geo} \right\}.$$

The constants involved above depend on r.

Proof. We prove **(ii)** firstly. It is enough to show that

$$\limsup_{|a|\to 1} \left\| (C_\varphi - C_\psi) k_a^{p,t} \right\|_{A_s^q}^q \gtrsim \lim_{\delta\to 1} \left\| (\mu_{\sigma,\varphi}^{q,s})_\delta \right\|_{r,Geo}.$$

By Lemma 4.6, Lemma 4.1 and the definition of geometric norm of pullback measures, we can calculate that

$$\begin{aligned}
& \limsup_{|a|\to 1} \left\| (C_\varphi - C_\psi) k_a^{p,t} \right\|_{A_s^q}^q \\
= \; & \limsup_{|a|\to 1} \int_{\mathbb{B}_n} \left| \left(\frac{1-|a|^2}{(1-\langle \varphi(z),a\rangle)^2} \right)^{\frac{n+1+t}{p}} - \left(\frac{1-|a|^2}{(1-\langle \psi(z),a\rangle)^2} \right)^{\frac{n+1+t}{p}} \right|^q dv_s(z) \\
\gtrsim \; & \limsup_{|a|\to 1} \int_{\varphi^{-1}(B_\rho(a,r))} \frac{\sigma^q(w)}{(1-|a|^2)^{\frac{(n+1+t)q}{p}}} dv_s(w) \quad \text{(by Lemma 4.6)} \\
= \; & \limsup_{|a|\to 1} \frac{\mu_{\sigma,\varphi}^{q,s}(B_\rho(a,r))}{(1-|a|^2)^{\frac{(n+1+t)q}{p}}} \\
= \; & \lim_{\delta\to 1} \sup_{|a|>\frac{\delta-r}{1-\delta r}} \frac{\mu_{\sigma,\varphi}^{q,s}(B_\rho(a,r))}{(1-|a|^2)^{\frac{(n+1+t)q}{p}}} \quad \text{(by Lemma 4.1)} \\
\ge \; & \lim_{\delta\to 1} \sup_{|a|>\frac{\delta-r}{1-\delta r}} \frac{\mu_{\sigma,\varphi}^{q,s}(B_\rho(a,r)\cap(\mathbb{B}_n\setminus\delta\mathbb{B}_n))}{(1-|a|^2)^{\frac{(n+1+t)q}{p}}} \\
= \; & \lim_{\delta\to 1} \left\| \left(\mu_{\sigma,\varphi}^{q,s} \right)_\delta \right\|_{r,Geo}.
\end{aligned}$$

To prove **(i)**, one can use the same method. The proof is completed. □

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