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On the Extension of Surjective Isometries whose Domain is the Unit Sphere of a Space of Compact Operators

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Abstract. We prove that every surjective isometry from the unit sphere of the space $K(H)$, of all compact operators on an arbitrary complex Hilbert space H, onto the unit sphere of an arbitrary real Banach space Y can be extended to a surjective real linear isometry from $K(H)$ onto Y. This is probably the first example of an infinite dimensional non-commutative C*-algebra containing no unitaries and satisfying the Mazur–Ulam property. We also prove that all compact C*-algebras and all weakly compact JB*-triples satisfy the Mazur–Ulam property.

1. Introduction

The problem of extending surjective isometries between the unit spheres of two Banach spaces has experienced a substantial turn with the introduction, by M. Mori and N. Ozawa, of the so-called strong Mankiewicz property. The celebrated Mazur–Ulam theorem has been a source of inspiration for many subsequent research. A key piece among the different generalizations that appeared later is due to P. Mankiewicz [33]. Let us recall that a convex subset K of a normed space X is called a *convex body* if it has non-empty interior in X. P. Mankiewicz proved in [33] that every surjective isometry between convex bodies in two arbitrary normed spaces can be uniquely extended to an affine function between the spaces. M. Mori and N. Ozawa introduced the strong Mankiewicz property in [35]. According to the just quoted paper, a convex subset K of a normed space X satisfies the *strong Mankiewicz property* if every surjective isometry Δ from K onto an arbitrary convex subset L in a normed space Y is affine. It is established by Mori and Ozawa that for a Banach space X satisfying that the closed convex hull of the extreme points, $\partial_e(\mathcal{B}_X)$, of the closed unit ball, $\mathcal{B}_X$, of X has non-empty interior in X, every convex body $K \subset X$ satisfies the strong Mankiewicz property (see [35, Theorem 2]). By the Russo-Dye theorem every convex body of a unital C*-algebra satisfies the strong Mankiewicz property, and the same property holds for convex bodies in real von Neumann algebras (see [35, Corollary 3]) and JBW*-triples (cf. [4, Corollary 2.2]).

Based on the strong Mankiewicz property, M. Mori and N. Ozawa proved that unital C*-algebras and real von Neumann algebras are among the spaces satisfying the Mazur–Ulam property, that is, every surjective

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isometry from the unit sphere, $S(A)$, of a unital C*-algebra, or of a real von Neumann algebra A, onto the unit sphere, $S(Y)$, of an arbitrary real Banach space Y, admits a unique extension to a surjective real linear isometry between the spaces (see [35]). J. Becerra Guerrero, M. Cueto-Avellaneda, F.J. Fernández-Polo and the author of this note showed that every JBW*-triple M which is not a Cartan factor of rank two always satisfies the Mazur–Ulam property (cf. [4]). In a recent collaboration with O.F.K. Kalenda we prove that every rank-2 Cartan factor satisfies the Mazur–Ulam property [29, Theorem 1.1], and consequently, every JBW*-triple enjoys the Mazur–Ulam property [29, Corollary 1.2]. These results simply are the most recent advances of a long list of papers studying the extension of isometries between the unit spheres of two Banach spaces (see, for example, [9, 11, 12, 19, 22–25, 28, 34, 35, 37, 38, 42, 43] and the surveys [36, 44]).

So the natural question is: what can we say in the case of Banach spaces or C*-algebras whose closed unit ball contains no extreme points? This is the case of the space $K(H)$, of all compact operators on an infinite dimensional complex Hilbert space H, where $\partial_e(\mathcal{B}_{K(H)}) = \emptyset$. The lacking of extreme points of the closed unit ball makes impossible a straight application of the arguments based on the strong Mankiewicz property.

In [38], R. Tanaka and the author of this note proved that every surjective isometry between the unit spheres of two compact C*-algebras (in particular between the unit spheres of two spaces of compact linear operators on a complex Hilbert space) extends (uniquely) to a surjective real linear isometry between the two C*-algebras. In collaboration with F.J. Fernández-Polo we showed that the same conclusion remains valid for a surjective isometry between the unit spheres of two complex Banach spaces in the strictly wider class of weakly compact JB*-triples (cf. [25]). The reader can get access to the concrete definitions of these objects in the subsequent section 2. In this paper we prove that weakly compact JB*-triples satisfy a much stronger property, namely, every weakly compact JB*-triple E satisfies the Mazur–Ulam property, in equivalent words, every surjective isometry from the unit sphere of E onto the unit sphere of an arbitrary Banach space Y extends to a surjective real linear isometry from E onto Y (cf. Theorem 5.4).

Our strategy to solve the problem determines the structure of this note. In section 3 we gather some new results derived from our knowledge on the facial structure of elementary JB*-triples. New technical properties of elementary JB*-triples, established in Propositions 3.5 and 3.7, are applied to deduce that every surjective isometry $\Delta : S(K) \to S(Y)$, where K is an elementary JB*-triple and Y is a real Banach space, is affine on every non-empty convex subset $C \subset S(K)$ (cf. Corollary 3.8).

Section 4 is aimed to prove that every elementary JB*-triple satisfies the Mazur–Ulam property (see Theorem 4.1). In particular, for any complex Hilbert space H, the space $K(H)$ satisfies the Mazur–Ulam property (cf. Corollary 4.2). This closes a natural question which remained open until now. Let us observe that in case that H is infinite dimensional the closed unit ball of $K(H)$ lacks of extreme points. As shown in [28], c_0 satisfies the Mazur–Ulam property. Probably, the results in this note show the first example of a non-commutative non-unital C*-algebra satisfying this property.

As we know from many other mathematical problems, certain questions are easier to answer from a more general point of view. This is the case of the Mazur–Ulam property for the space of compact operators; the arguments in the wider class of elementary and weakly compact JB*-triples are probably more abstract but offer an accesible proof.

Our goal in the second part of the paper is to prove that every weakly compact JB*-triple satisfies the Mazur–Ulam property (see Theorem 5.4). For this purpose, we shall first show that the closed unit ball of any weakly compact JB*-triple enjoys the strong Mankiewicz property (cf. Corollary 5.2). A consequence of this second main result, which is worth to be stated by its own importance, shows that every compact C*-algebra (that is, a C*-algebra which coincides with a c_0-sum of spaces of compact operators) also has the Mazur–Ulam property (cf. Corollary 5.5).

2. Basic background and references

The Riemann mapping theorem is one of the best known results in the theory of holomorphic functions of one variable. As it was already observed by H. Poincaré in 1907, if $\mathbb{C}$ is replaced with a higher dimensional

Banach space the conclusion of the Riemann mapping theorem is no longer valid, there are lots of simply connected domains which are not biholomorphic to the open unit ball. *Bounded symmetric domains* in finite dimensions were introduced and completely classified by E. Cartan. L. Harris proved in 1974 that the open unit ball of every C*-algebra is a bounded symmetric domain [27], a conclusion which remains true for JB*-algebras (see [5]). The most conclusive study was obtained by W. Kaup who proved that every bounded symmetric domain in a complex Banach space is biholomorphically equivalent to the open unit ball of a JB*-triple (cf. [30]).

A complex Banach space E is called a *JB*-triple* if it can be equipped with a continuous triple product $\{.,.,.\} : E \times E \times E \to E$, which is conjugate linear in the middle variable and symmetric and bilinear in the outer variables satisfying the following axioms:

(a) (Jordan Identity) $L(a,b)L(x,y) - L(x,y)L(a,b) = L(L(a,b)x,y) - L(x,L(b,a)y)$, for all a,b,x,y, in E, where $L(x,y)$ is the linear operator defined by $L(a,b)(z) = \{a,b,z\}$ ($\forall z \in E$);
(b) The operator $L(a,a) : E \to E$ is hermitian and has non-negative spectrum;
(c) $\|\{a,a,a\}\| = \|a\|^3$, for every $a \in E$.

It can be found in the previously mentioned references that every C*-algebra A is a JB*-triple with respect to the triple product

$$(a,b,c) \mapsto \{a,b,c\} = 1/2(ab^*c + cb^*a), \quad (a,b,c \in A). \tag{1}$$

This triple product also induces a structure of JB*-triple for the space $B(H_1,H_2)$ of all bounded linear operators between two complex Hilbert spaces H_1 and H_2, and for every closed subspace of $B(H_1,H_2)$ which is closed for this triple product. In particular every complex Hilbert space and the space $K(H_1,H_2)$, of all compact linear operators from H_1 to H_2, are JB*-triples. The class of JB*-triples is also widen with all JB*-algebras when they are equipped with the triple product given by

$$\{a,b,c\} = (a \circ b^*) \circ c + (c \circ b^*) \circ a - (a \circ c) \circ b^*. \tag{2}$$

A subspace B of a JB*-triple E is a JB*-subtriple of E if $\{B,B,B\} \subseteq B$. A JB*-subtriple I of E is called an *inner ideal of* E if $\{I,E,I\} \subseteq I$. A subspace I of a C*-algebra A is called an *inner ideal* if $IAI \subseteq I$. For example, if p and q are projections in a C*-algebra A, the subspace $I = pAq$ is an inner ideal of A. Inner ideals of JB*-triples are studied and characterized in [15].

A *JBW*-triple* is a JB*-triple which is also a dual Banach space. Every von Neumann algebra is a JBW*-triple. The theory of JB*-triple runs in parallel to the theory of C*-algebras. For example, the second dual of a JB*-triple E is a JBW*-triple under a triple product extending the product of E [13]. It is also known that every JBW*-triple admits a unique isometric predual, and its triple product is separately weak* continuous [2].

Let us recall that an element e in a C*-algebra A is called a *partial isometry* if ee^* (equivalently, e^*e) is a projection. It is known that e is a partial isometry if and only if $ee^*e = e$. It is easy to see that, in terms of the triple product given in (1), an element $e \in A$ is a partial isometry if and only if $\{e,e,e\} = e$. In the wider framework of JB*-triples, elements satisfying $\{e,e,e\} = e$ are called *tripotents*. For each tripotent $e \in E$ the eigenvalues of the operator $L(e,e)$ are precisely $0, 1/2$ and 1. For $j \in \{0,1,2\}$, by denoting by $E_j(e)$ the $\frac{j}{2}$-eigenspace of $L(e,e)$, the JB*-triple E decomposes as the direct sum

$$E = E_2(e) \oplus E_1(e) \oplus E_0(e).$$

This decomposition is called the *Peirce decomposition* of E with respect to the tripotent e, and the projection of E onto $E_j(e)$, which is denoted by $P_j(e)$, is called the Peirce j-projection (see [32]). It is further known that Peirce projections are contractive (cf. [26]) and satisfy the following identities $P_2(e) = Q(e)^2$, $P_1(e) = 2(L(e,e) - Q(e)^2)$, and $P_0(e) = Id_E - 2L(e,e) + Q(e)^2$, where for each $a \in E$, $Q(a) : E \to E$ is the conjugate linear map given by $Q(a)(x) = \{a,x,a\}$. Consequently, in a JBW*-triple Peirce projections are weak* continuous and Peirce subspaces are weak* closed.

Triple products among Peirce subspaces satisfy certain rules, known as *Peirce rules* or *Peirce arithmetic*, asserting that, for $k, j, l \in \{0, 1, 2\}$ we have $\{E_0(e), E_2(e), E\} = \{E_2(e), E_0(e), E\} = \{0\}$,

$$\{E_k(e), E_j(e), E_l(e)\} \subseteq E_{k-j+l}(e), \text{ if } k - j + l \in \{0, 1, 2\}, \text{ and}$$
$$\{E_k(e), E_j(e), E_l(e)\} = \{0\} \text{ otherwise.}$$

A tripotent e in E is called *complete* (respectively, *unitary* or *minimal*) if $E_0(E) = \{0\}$ (respectively, $E_2(e) = E$ or $E_2(e) = \mathbb{C}e \neq \{0\}$).

Orthogonality in JB*-triples is another notion required in this note. Elements a, b in a JB*-triple are said to be *orthogonal* (written $a \perp b$) if $L(a, b) = 0$. It is known that $a \perp b$ if, and only if, $L(b, a) = 0$ if, and only if, $\{a, a, b\} = 0$ if, and only if, $\{b, b, a\} = 0$ (see [7, Lemma 1] for more equivalent reformulations). This notion is consistent with the usual concept of orthogonality in C*-algebras. Let a, b be elements in $B(H)$ (or in a general C*-algebra). We say that a and b are *orthogonal* if $ab^* = b^*a = 0$. It is well known that $\|a \pm b\| = \max\{\|a\|, \|b\|\}$ whenever $a \perp b$ in (see [26, Lemma 1.3(*a*)]). For each non-zero finite rank partial isometry $e \in K(H)$ there exists a finite family of mutually orthogonal minimal partial isometries $\{e_1, \ldots, e_m\}$ in $K(H)$ such that $e = e_1 + \ldots + e_m$. We say that a tripotent e in a JB*-triple E has *finite rank* if it can be written as a sum of finitely many mutually orthogonal minimal tripotents.

To make easier our subsequent arguments we remark the following property: let u and e be two orthogonal tripotents in a JB*-triple E. Clearly $e + u$ is a tripotent in E and we can easily deduce from Peirce arithmetic that

$$E_2(e + u) = E_2(e) \oplus E_2(u) \oplus E_1(e) \cap E_1(u). \tag{3}$$

A subset S of a JB*-triple E is called *orthogonal* if $0 \notin S$ and $x \perp y$ for every $x \neq y$ in S. The minimal cardinal number r satisfying $card(S) \leq r$ for every orthogonal subset $S \subseteq E$ is called the *rank* of E (cf. [31] and [3] for basic results on the rank of a Cartan factor and a JB*-triple).

Let B be a subset of a JB*-triple E. We shall denote by $B^\perp$ the *(orthogonal) annihilator of* B defined by $B^\perp = B^\perp_{E} := \{z \in E : z \perp x, \forall x \in B\}$. Given a tripotent e in E the inclusions

$$E_2(e) \oplus E_1(e) \supseteq \{e\}^{\perp\perp}_{E} = E_0(e)^\perp \supseteq E_2(e)$$

always hold (see [8, Proposition 3.3]). It is also known that the equality $\{e\}^{\perp\perp}_{E} = E_2(e)$ is not always true. The following counterexample can be found in [8, Remark 3.4]: suppose H_1 and H_2 are two infinite dimensional complex Hilbert spaces and p is a minimal projection in $B(H_1)$. If E denotes the orthogonal sum $pB(H_1) \oplus^\infty B(H_2)$ and we consider $e = p$ as an element in E, it can be checked that p is a non-complete tripotent in E, $\{e\}^\perp_{E} = B(H_2)$ and $\{e\}^{\perp\perp}_{E} = E_2(e) \oplus E_1(e) = pB(H_1) \neq \mathbb{C}p = E_2(e)$.

Despite of the previous counterexample, if we assume that E is a Cartan factor and e is a non-complete tripotent in E, then the equality $\{e\}^{\perp\perp} = E_0(e)^\perp = E_2(e)$ is always true (see [31, Lemma 5.6]).

This seems an appropriate moment to refresh the definition of Cartan factors. A JB*-triple is called a Cartan factor of type 1 if it is a JB*-triple of the form $B(H_1, H_2)$, where H_1 and H_2 are two complex Hilbert spaces. Let j be a conjugation on a complex Hilbert space H, and consider the linear involution $x \mapsto x^t := jx^*j$ on $B(H)$. A Cartan factor of type 2 (respectively, type 3) is a JB*-triple which coincides with the subtriple of $B(H)$ formed by the t-skew-symmetric (respectively, t-symmetric) operators. All we need to know in this note about types 4, 5 and 6 Cartan factors is that the first one is reflexive while the last two are finite dimensional (see [31] for more details).

According to [6], given a Cartan factor of type $j \in \{1, \ldots, 6\}$, the elementary JB*-triple K_j of type j is defined in the following terms: $K_1 = K(H_1, H_2)$; $K_i = C \cap K(H)$ when C is of type $i = 2, 3$, and $K_i = C$ if the latter is of type $4, 5$, or 6. Obviously, if K is an elementary JB*-triple of type j, its bidual is precisely a Cartan factor of j.

We establish next a version of [31, Lemma 5.6] for elementary JB*-triples.

Lemma 2.1. *Let K be an elementary JB*-triple. Suppose e is a non-complete tripotent in K. Then we have $\{e\}^{\perp\perp} = K_0(e)^{\perp} = K_2(e)$.*

Proof. If K is an elementary JB*-triple of type $j \in \{4,5,6\}$ we know that K is a Cartan factor of the same type and hence the conclusion follows from [31, Lemma 5.6].

Suppose K is an elementary JB*-triple of type $j \in \{1,2,3\}$. The corresponding bidual $K^{**} = C$ is a Cartan factor of type j. By Goldstine's theorem K is weak* dense in C. Since $e \in K$, and $P_0(e)$ is weak* continuous in C, we deduce that $K_0(e) = \{e\}^{\perp}_{_K}$ and $K_2(e)$ are weak* dense in $C_0(e) = \{e\}^{\perp}_{_C}$ and $C_2(e)$, respectively. Thus the desired conclusion also follows from [31, Lemma 5.6]. Namely, we know that $\{e\}^{\perp\perp}_{_K} = K_0(e)^{\perp} \supseteq K_2(e)$ [8, Proposition 3.3]. Let us take $x \in \{e\}^{\perp\perp}_{_K}$. In this case, $\{x,x,z\} = 0$ for all $z \in \{e\}^{\perp}_{_K} = K_0(e)$. It follows from the weak* density of $K_0(e)$ in $C_0(E)$ and the separate weak* continuity of the triple product of C that $\{x,x,a\} = 0$ for all $a \in \{e\}^{\perp}_{_C} = C_0(e)$. By [31, Lemma 5.6] we have $x \in \{e\}^{\perp\perp}_{_C} = C_0(e)^{\perp} = C_2(e)$, and consequently $x \in K_2(e) = C_2(e) \cap K$. □

According to [6], an element x in a JB*-triple E is called *weakly compact* (respectively, *compact*) if the operator $Q(x) : E \to E$ is weakly compact (respectively, compact). We say that E is *weakly compact* (respectively, *compact*) if every element in E is weakly compact (respectively, compact). If we denote by $K(E)$ the Banach subspace of E generated by its minimal tripotents, then $K(E)$ is a (norm closed) triple ideal of E and it coincides with the set of weakly compact elements of E (see Proposition 4.7 in [6]). It follows from [6, Lemma 3.3 and Theorem 3.4] that a JB* triple, E, is weakly compact if and only if one of the following statement holds:

a) $K(E^{**}) = K(E)$.

b) $K(E) = E$.

c) E is a c_0-sum of elementary JB*-triples.

Obviously each non-zero tripotent in a weakly compact JB*-triple is of finite rank. It was observed in [38, Corollary 2.5] that the results in [6] can be applied to deduce that a JB*-triple E is weakly compact if and only if it contains every tripotent of E^{**} which is compact relative to E in the sense of [16, 21].

It should be remarked here that weakly compact JB*-triples are the JB*-triple analogue of compact C*-algebras in the sense employed in [1, 45], with the exception that a C*-algebra is compact if, and only if, it is weakly compact (see [45]).

Corollary 2.2. *Let K be an elementary JB*-triple. Suppose e is a non-complete tripotent in K. Let x be an element in K satisfying $x \perp v$ for every minimal tripotent $v \in K$ with $v \perp e$. Then $x \in K_2(e)$.*

Proof. Let us observe that from Peirce arithmetic the Peirce 0-subspace $K_0(e) \neq \{0\}$ is an inner ideal of K. Corollary 3.5 in [6] together with the fact that K is a factor show that $K_0(e)$ must be a weakly compact JB*-triple. By Remark 4.6 in [6] every element in the weakly compact JB*-triple $K_0(e)$ can be approximated in norm by finite positive linear combinations of mutually orthogonal minimal tripotents in $K_0(e)$. Since every minimal tripotent in $K_0(e)$ is a minimal tripotent in K which is orthogonal to e, it follows from the hypothesis that $x \perp K_0(e)^{\perp} = \{e\}^{\perp\perp}$. Lemma 2.1 implies that $x \in K_2(e)$. □

Let us observe that Corollary 2.2 can be also proved by a direct argument in the case in which $K = K(H)$ where H is a complex Hilbert space.

We are actually interested in finding conditions on a tripotent e in a weakly compact JB*-triple E to guarantee that the equality $\{e\}^{\perp\perp} = K_0(e)^{\perp} = K_2(e)$ holds (compare Lemma 2.1 and the counterexample in page 3078).

Lemma 2.3. *Let $E = \bigoplus_{i\in I}^{c_0} K_i$ be a weakly compact JB*-triple, where each K_i is an elementary JB*-triple. For each $i \in I$, let π_i denote the projection of E onto K_i. Suppose e is a tripotent in E such that $\pi_i(e)$ is a non-complete tripotent in K_i for every $i \in I$. Then the identity $\{e\}^{\perp\perp}_{_E} = E_0(e)^{\perp} = E_2(e)$ holds.*

Proof. For each $x \in E$, we shall write $x = (x_i)_{i\in I}$ with $x_i = \pi_i(x)$. Since e is a tripotent in E the set $I_1 := \{i \in I : \pi_i(e) \neq 0\}$ must be finite. We observe that

$$\{e\}_E^{\perp} = \left\{x \in E \ : \ x_i \in \{e_i\}_{K_i}^{\perp} \text{ for all } i \in I_1\right\},$$

and hence

$$\{e\}_E^{\perp\perp} = \left\{x \in E \ : \ x_i \in \{e_i\}_{K_i}^{\perp\perp} \text{ for all } i \in I_1, \ x_i = 0 \text{ for all } i \in I\backslash I_1\right\}.$$

Since for $i \in I_1$, e_i is a non-complete tripotent in K_i it follows from Lemma 2.1 that $\{e_i\}_{K_i}^{\perp\perp} = (K_i)_2(e_i)$, and consequently $\{e\}_E^{\perp\perp} = K_2(e)$. □

Let us observe that the conclusion in the previous Lemma 2.3 remains true when E is replaced with an ℓ_∞-sum of Cartan factors.

3. New properties derived from the facial structure of an elementary JB*-triple

Along this paper, given a Banach space X, the symbols $\mathcal{B}_X$ and $S(X)$ will stand for the closed unit ball and the unit sphere of X, respectively.

The main goal of this paper is to prove that every weakly compact JB*-triple satisfies the Mazur-Ulam property. In a first step we shall study this property in the case of an elementary JB*-triple K. If K is reflexive it follows that $K = K^{**}$ is a Cartan factor, and hence the desired property follows from [4, Theorem 4.14, Proposition 4.15 and Remark 4.16] when K has rank one or rank bigger than or equal to three and from [29, Theorem 1.1.] in the remaining cases. We shall therefore restrict our study to the case in which K is a non-reflexive elementary JB*-triple (equivalently, an infinite dimensional elementary JB*-triple of type 1, 2 or 3).

As in many previous studies, the facial structure of the closed unit ball of a Banach space X is a key tool to determine if X satisfies the Mazur–Ulam property. The main reason being the fact that a surjective isometry Δ between the unit spheres of two Banach spaces X and Y maps maximal proper faces of $\mathcal{B}_X$ to maximal proper faces of $\mathcal{B}_Y$ (cf. [10, Lemma 5.1], [39, Lemma 3.5] and [40, Lemma 3.3]).

We recall that a convex subset F of a convex set C is called a *face* of C if for every $x \in F$ and every $y, z \in C$ such that $x = ty + (1-t)z$ for some $t \in [0, 1]$, we have $y, z \in F$. Let us observe that every proper (i.e., non-empty and non-total) face of the closed unit ball of a Banach space X is contained in $S(X)$. Following the notation in [35], a closed face $F \subseteq S(X)$ is called an *intersection face* if

$$F = \bigcap \left\{E \ : \ E \subseteq S(X) \text{ a maximal face containing } F\right\}.$$

If X is a complex C*-algebra or the the predual of a von Neumann algebra, or more generally, a JB*-triple or the predual of a JBW*-triple, every proper norm closed face of $\mathcal{B}_X$ is an intersection face (see [41, Corollary 3.4] and [18, Proof of Proposition 2.4 and comments after and before Corollary 2.5]). It should be remarked that this conclusion can be also derived from the main results in [14]. These facts together with [35, Lemma 8] are employed in the next result which was already stated in [29, Lemma 2.2].

Lemma 3.1. ([29, Lemma 2.2], [35, Lemma 8], [18, Proposition 2.4], [14, Corollary 3.11]) *Let $\Delta : S(E) \to S(Y)$ be a surjective isometry where E is a JB*-triple and Y is a real Banach space. Then Δ maps proper norm closed faces of $\mathcal{B}_E$ to intersection faces in $S(Y)$. Furthermore, if F is a proper norm closed face of $\mathcal{B}_E$ then $\Delta(-F) = -\Delta(F)$.*

The structure of all norm closed faces of the closed unit ball of a JB*-triple E was completely determined in [14], where it is shown that each norm closed face of $\mathcal{B}_E$ is univocally given by a tripotent in E^{**} which is compact relative to E (see [14, Theorem 3.10 and Corollary 3.12] and the concrete definitions therein). As we observed before, each weakly compact JB*-triple E contains all tripotents in E^{**} which are compact relative to E (see [38, Corollary 2.5]). Therefore, the norm closed faces of the closed unit ball of a weakly compact JB*-triple E are completely determined by the tripotents in E, consequently Theorem 3.10 in [14] in this concrete setting assures that for each proper norm closed face F of $\mathcal{B}_E$ there exists a non-zero finite rank tripotent $e \in E$ such that

$$F = F_e = e + \mathcal{B}_{E_0(e)} = (e + E_0(e)) \cap \mathcal{B}_E. \tag{4}$$

Let us comment some of the difficulties we can find when applying our current knowledge. Suppose $\Delta : S(K(H)) \to S(Y)$ is a surjective isometry, where H is an infinite dimensional complex Hilbert space and Y is a real Banach space. For each non-zero partial isometry $e \in K(H)$, Lemma 3.1 shows that $\Delta(F_e)$ is an intersection face in $S(Y)$ and the restriction $\Delta|_{F_e} : F_e \to \Delta(F_e)$ is a surjective isometry too. Henceforth, given an element x_0 in a Banach space X, we shall write $\mathcal{T}_{x_0} : X \to X$ for the translation mapping defined by $\mathcal{T}_{x_0}(x) = x + x_0$ ($x \in X$). By considering the commutative diagram

$$\begin{array}{ccc} F_e & \xrightarrow{\ \Delta|_{F_e}\ } & \Delta(F_e) \\ \downarrow{\scriptstyle \mathcal{T}_{-e}} & \nearrow{\scriptstyle \Delta_e} & \\ (1-ee^*)\mathcal{B}_{K(H)}(1-e^*e) & & \end{array}$$

we realize that if we could prove that $\mathcal{B}_{(1-ee^*)K(H)(1-e^*e)}$ satisfied the strong Mankiewicz property, we could get some progress to determine the behavior of Δ on F_e. However, $(1-ee^*)K(H)(1-e^*e)$ is a JB*-triple whose closed unit ball contains no extreme points (let us observe that H is infinite dimensional with ee^* and e^*e finite rank projections). So, our current technology is not enough to attack the problem from this perspective. We shall develop a new facial argument not contained in the available literature.

The following result is implicit in [20, Remark 20] and a detailed explanation can be found in [38, Lemma 3.3] from where it has been taken.

Lemma 3.2. *[38, Lemma 3.3] Let e be a tripotent in a JB*-triple E. Suppose x is an element in $\mathcal{B}_E$ satisfying $\|e \pm x\| = 1$. Then $x \perp e$.*

We shall need the next consequence.

Lemma 3.3. *Let e be a tripotent in a JB*-triple E. Suppose x is an element in $S(E)$ satisfying $\|e \pm x\| \leq 1$. Then $x \perp e$.*

Proof. Since $S(E) \ni x = \frac{1}{2}(x+e) + \frac{1}{2}(x-e)$, we deduce from $\|e \pm x\| \leq 1$ that $\|e \pm x\| = 1$. Lemma 3.2 implies that $x \perp e$ as desired. □

The next result has been borrowed from [29].

Lemma 3.4. *[29, Corollary 2.4] Let $\Delta : S(E) \to S(Y)$ be a surjective isometry where E is a JB*-triple and Y is a real Banach space. Suppose e is a non-zero tripotent in E, then $\Delta(-e) = -\Delta(e)$.*

We are now in position to establish a new geometric result, based on the facial structure of $\mathcal{B}_{K(H)}$, which provides a new tool to prove the Mazur–Ulam property in the case of elementary JB*-triples.

Proposition 3.5. *Let $\Delta : S(K) \to S(Y)$ be a surjective isometry, where K is an elementary JB*-triple and Y is a real Banach space. Then for each tripotent $e \in K$ and each minimal tripotent $u \in K$ with $e \perp u$ the set $\Delta(u + \mathcal{B}_{K_2(e)})$ is convex and the restriction $\Delta|_{u+\mathcal{B}_{K_2(e)}} : u + \mathcal{B}_{K_2(e)} \to \Delta(u + \mathcal{B}_{K_2(e)})$ is an affine mapping. Consequently, there exists a real linear isometry T_e^u from $K_2(e)$ onto a norm closed subspace of Y satisfying $\Delta(\mathcal{T}_u(x)) = \Delta(u + x) = T_e^u(x) + \Delta(u)$ for all $x \in \mathcal{B}_{K_2(e)}$.*

Proof. As we commented in page 3080, by [4, Theorem 4.14, Proposition 4.15 and Remark 4.16] and [29, Theorem 1.1.], we can assume that K is non-reflexive (i.e. an infinite dimensional elementary JB*-triple of type 1, 2 or 3), and hence every tripotent in K is non-complete and of finite rank and K has infinite rank.

Let e and u be non-zero tripotents in K such that u is minimal and $u \perp e$. The conclusion clearly holds for $e = 0$, we can therefore assume that e is a non-zero finite rank tripotent. Set $w = e + u$. Keeping the notation in (4) for each non-zero tripotent $v \in K$ we write $F_v = v + \mathcal{B}_{K_0(v)}$ for the proper norm closed face of $\mathcal{B}_K$ associated with v.

Let $\mathcal{D}_0 = \Delta(F_u)$. Lemma 3.1 implies that $\mathcal{D}_0$ is an intersection face in $S(Y)$. Let us fix an arbitrary minimal tripotent $v \in K$ such that $v \perp w$. We set

$$\mathcal{D}_1^v := \{y \in \mathcal{D}_0 \ : \ \|y \pm \Delta(v)\| \leq 1\}.$$

We claim that $\mathcal{D}_1^v$ is norm closed and convex. Namely, given $y_1, y_2 \in \mathcal{D}_1^v$ and $t \in [0,1]$ the convex combination $ty_1 + (1-t)y_2 \in \mathcal{D}_0$ because $\mathcal{D}_0$ is convex. Furthermore

$$\|ty_1 + (1-t)y_2 \pm \Delta(v)\| \leq t\, \|y_1 \pm \Delta(v)\| + (1-t)\, \|y_2 \pm \Delta(v)\| \leq 1,$$

witnessing that $ty_1 + (1-t)y_2 \in \mathcal{D}_1^v$. Clearly $\mathcal{D}_1^v$ is norm closed.

Let us consider the inner ideal $K_2(e)$. Clearly $K_2(e)$ is a finite dimensional JBW*-triple. We shall next prove that

$$\Delta\left(\mathcal{B}_{K_2(e)} + u\right) = \bigcap \left\{\mathcal{D}_1^v : v \text{ a minimal tripotent in } K \text{ with } v \perp w\right\}. \tag{5}$$

($\subseteq$) Let $v \in K$ be a minimal tripotent with $v \perp w$. We take an element $u + z \in u + \mathcal{B}_{K_2(e)}$. Since $u + z \perp v$, $\Delta(-v) = -\Delta(v)$ (see Lemma 3.4), and Δ is an isometry we have

$$\|\Delta(u+z) \pm \Delta(v)\| = \|z + u \pm v\| = \max\{\|z+u\|, \|v\|\} = 1.$$

($\supseteq$) Suppose next that $y \in \mathcal{D}_1^v$ for every minimal tripotent v in K with $v \perp w$. It follows from the definition that $y \in \mathcal{D}_0 = \Delta(F_u)$, and hence there exists $x \in F_u$ satisfying $\Delta(x) = y$. Thus, by Lemma 3.4 and the assumptions, we have $1 \geq \|y \pm \Delta(v)\| = \|x \pm v\|$, for every v as above. Lemma 3.3 implies that $x \perp v$ for every minimal tripotent $v \in K$ with $v \perp w$. It follows from Corollary 2.2 that $x \in K_2(w)$, and since $x \in F_u = u + \mathcal{B}_{K_0(u)}$ we can easily see, for example from (3), that $x \in u + \mathcal{B}_{K_2(e)}$, as desired.

We consider next the following commutative diagram

$$\begin{array}{ccc} u + \mathcal{B}_{K_2(e)} & \xrightarrow{\ \Delta\ } & \Delta\left(u + \mathcal{B}_{K_2(e)}\right) \\ \tau_{-u}\big\downarrow & & \big\uparrow \tau_{\Delta(u)} \\ \mathcal{B}_{K_2(e)} & \overset{\Delta_e^u}{\dashrightarrow} & \Delta\left(u + \mathcal{B}_{K_2(e)}\right) - \Delta(u) \end{array} \tag{6}$$

It follows from (5) that $\Delta\left(u + \mathcal{B}_{K_2(e)}\right)$, and hence $\Delta\left(u + \mathcal{B}_{K_2(e)}\right) - \Delta(u)$, is a norm closed convex subset of $S(Y)$. Since $K_2(e)$ is a finite dimensional JBW*-triple, it follows from [4, Corollary 2.2] that $\mathcal{B}_{K_2(e)}$ satisfies the strong Mankiewicz property. Therefore, by the strong Mankiewicz property there exists a surjective real linear isometry T_e^u from $K_2(e)$ onto a norm closed subspace of Y whose restriction to $\mathcal{B}_{K_2(e)}$ is Δ_e^u, that is

$$\Delta(u + x) = \Delta(u) + \Delta_e^u(x) = \Delta(u) + T_e^u(x),$$

for all $x \in \mathcal{B}_{K_2(e)}$. $\square$

Let us remark a consequence of the previous Lemma 3.3 and Corollary 2.2. The statement has been actually outlined in the proof of the previous proposition.

Lemma 3.6. *Let e be a non-complete tripotent in an elementary JB*-triple K. Let $\{e\}^{\perp}_{min} = \{v$ minimal tripotent in K with $v \perp e\}$. Then*

$$\mathcal{B}_{K_2(e)} = \{x \in \mathcal{B}_K : \|x - v\| \leq 1 \text{ for all } v \in \{e\}^{\perp}_{min}\}.$$

Proof. ($\subseteq$) Take $x \in \mathcal{B}_{K_2(e)}$. It follows from Peirce arithmetic that $x \perp v$ for all $v \in \{e\}^{\perp}_{min}$, and thus $\|x - v\| = \max\{\|x\|, \|v\|\} \leq 1$.

($\supseteq$) Assume now that $x \in \mathcal{B}_K$ with $\|x - v\| \leq 1$ for all $v \in \{e\}^{\perp}_{min}$. We observe that $-v \in \{e\}^{\perp}_{min}$ for all $v \in \{e\}^{\perp}_{min}$. Therefore $\|x \pm v\| \leq 1$ for all $v \in \{e\}^{\perp}_{min}$. Lemma 3.3 implies that $x \perp v$ for all $v \in \{e\}^{\perp}_{min}$. Corollary 2.2 proves that $x \in K_2(e)$ as desired. □

In the following proposition we explore the properties of the real linear isometries T^u_e given by Proposition 3.5.

Proposition 3.7. *Let $\Delta : S(K) \to S(Y)$ be a surjective isometry, where K is an elementary JB*-triple and Y is a real Banach space. Suppose e and v are tripotents in K with v minimal and $v \perp e$. Let $T^v_e : K_2(e) \to Y$ be the real linear isometry given by Proposition 3.5. Then the following statements hold:*

(*a*) $T^v_e = T^{-v}_{-e} = T^{-v}_e = T^v_{-e}$;
(*b*) *Suppose u is a minimal tripotent with $u \perp v$. Then $\Delta(u) = T^v_u(u)$;*
(*c*) *Suppose u is a minimal tripotent in $K_2(e)$. Then $\Delta(u) = T^v_e(u)$;*
(*d*) *For each $x \in S(K_2(e))$ we have $T^v_e(x) = \Delta(x)$;*
(*e*) *If w is another minimal tripotent with $w \perp e$, the real linear isometries T^v_e and T^w_e coincide;*

Proof. If K is reflexive, the desired conclusion is an easy consequence of [4, Theorem 4.14, Proposition 4.15 and Remark 4.16] and [29, Theorem 1.1.] because the latter results show that Δ admits an extension to a surjective real linear isometry. We shall therefore assume that K is non-reflexive. As we commented before, under our hypotheses, $K_2(e)$ is a (weakly compact) finite dimensional JBW*-triple, and thus every element in $K_2(e)$ can be written as a finite positive combination of mutually orthogonal minimal projections in K.

(*a*) It follows from the arguments in the previous paragraph that it suffices to show that $T^v_e(u) = T^{-v}_{-e}(u) = T^{-u}_e(u) = T^u_{-e}(u)$ for every minimal tripotent u in $K_2(e) = K_2(-e)$. Fix a minimal tripotent $u \in K_2(e)$. By Lemma 3.4 and Proposition 3.5 we have

$$\Delta(v+u) = \Delta(v) + T^v_e(u) = -\Delta(-v-u) = \Delta(v) - T^{-v}_e(-u) = \Delta(v) + T^{-v}_e(u),$$

$$\Delta(v-u) = \Delta(v) - T^v_e(u) = -\Delta(-v+u) = -\Delta(-v) - T^{-v}_{-e}(u) = \Delta(v) - T^{-v}_{-e}(u),$$

witnessing the desired equalities.

(*b*) The elements $u \pm v$ belong to the convex set $u + \mathcal{B}_{K_2(v)}$. By Proposition 3.5 Δ is affine on $u + \mathcal{B}_{K_2(v)}$. Therefore $\Delta(u) = \frac{1}{2}\Delta(u+v) + \frac{1}{2}\Delta(u-v)$. Let us observe that $\pm v + u \in \pm v + \mathcal{B}_{K_2(u)}$. By applying Proposition 3.5, Lemma 3.4 and (*a*) we deduce that

$$\Delta(u) = \frac{1}{2}\Delta(u+v) + \frac{1}{2}\Delta(u-v) = \frac{1}{2}\left(\Delta(v) + T^v_u(u) + \Delta(-v) + T^{-v}_u(u)\right) = T^v_u(u).$$

(*c*) By Proposition 3.5 and statement (*b*) we get

$$\Delta(v) + T^v_e(u) = \Delta(v+u) = \Delta(v) + T^v_u(u) = \Delta(v) + \Delta(u),$$

which proves the desired identity.

(*d*) Let us take $x \in S(K_2(e))$. Having in mind the arguments at the beginning of this proof, we can find mutually orthogonal minimal tripotents $e_1, \ldots, e_m$ such that $x = e_1 + \sum_{k=2}^{m} t_k e_k$, where $t_2, \ldots, t_m \in (0, 1]$. Set $w = e_2 + \ldots + e_m$. Proposition 3.5 applied to w and e_1 guarantees that

$$\Delta(x) = \Delta(e_1) + \sum_{k=2}^{m} t_k T^{e_1}_w(e_k) = \Delta(e_1) + \sum_{k=2}^{m} t_k \Delta(e_k)$$

$$= T_e^v(e_1) + \sum_{k=2}^{m} t_k T_e^v(e_k) = T_e^v\left(e_1 + \sum_{k=2}^{m} t_k e_k\right) = T_e^v(x),$$

where in the second and third equalities we applied (c).

(e) This is a trivial consequence of (d) because by this statement the real linear isometries T_e^v and T_e^w coincide with Δ on $S(K_2(e))$. □

We can establish next a version of [29, Corollary 2.7] within the framework of elementary JB*-triples.

Corollary 3.8. *Let $\Delta : S(K) \to S(Y)$ be a surjective isometry, where K is an elementary JB*-triple and Y is a real Banach space. Let $C \subset S(K)$ be a non-empty convex subset. Then $\Delta|_C$ is an affine mapping.*

Proof. If K is reflexive, then the conclusion follows from [29, Corollary 2.7]. Let us assume that K is non-reflexive. Let $C \subset S(K)$ be a non-empty convex subset, $x, y \in C$ and $t \in [0, 1]$. By the structure of elementary JB*-triples, for each $0 < \varepsilon < \frac{1}{2}$, there exists a (finite rank) tripotent e in K and elements $x_1, y_1 \in S(K_2(e))$ satisfying $\|x - x_1\|, \|y - y_1\| < \varepsilon$ (cf. [6, Remark 4.6]). A standard argument shows that

$$\left\|tx + (1-t)y - (tx_1 + (1-t)y_1)\right\| < \varepsilon.$$

Since $C \subset S(K)$ is a convex subset we have $\|tx + (1-t)y\| = 1$,

$$\left|1 - \left\|(tx_1 + (1-t)y_1)\right\|\right| \leq \left\|tx + (1-t)y - (tx_1 + (1-t)y_1)\right\| < \varepsilon,$$

and

$$\left\|\frac{tx_1 + (1-t)y_1}{\|tx_1 + (1-t)y_1\|} - (tx_1 + (1-t)y_1)\right\| = \left|1 - \left\|(tx_1 + (1-t)y_1)\right\|\right| < \varepsilon.$$

Consequently,

$$\left\|\frac{tx_1 + (1-t)y_1}{\|tx_1 + (1-t)y_1\|} - (tx + (1-t)y)\right\| < 2\varepsilon.$$

Since K has infinite rank, we can find a minimal tripotent v in K which is orthogonal to e. Let $T_e^v : K_2(e) \to Y$ be the real linear isometry given by Proposition 3.5. Having in mind that $x_1, y_1 \in K_2(e)$, it follows from the just quoted proposition, the hypothesis on Δ and Proposition 3.7(d) that

$$\begin{aligned}
&\left\|t\Delta(x) + (1-t)\Delta(y) - \Delta(tx + (1-t)y)\right\| \\
&\leq \left\|t\Delta(x) + (1-t)\Delta(y) - t\Delta(x_1) - (1-t)\Delta(y_1)\right\| \\
&+ \left\|tT_e^v(x_1) + (1-t)T_e^v(y_1) - T_e^v(tx_1 + (1-t)y_1)\right\| \\
&+ \left\|T_e^v(tx_1 + (1-t)y_1) - T_e^v\left(\frac{tx_1 + (1-t)y_1}{\|tx_1 + (1-t)y_1\|}\right)\right\| \\
&+ \left\|\Delta\left(\frac{tx_1 + (1-t)y_1}{\|tx_1 + (1-t)y_1\|}\right) - \Delta(tx + (1-t)y)\right\| < 3\varepsilon.
\end{aligned}$$

The arbitrariness of $0 < \varepsilon < \frac{1}{2}$ implies that $t\Delta(x) + (1-t)\Delta(y) = \Delta(tx + (1-t)y)$ as desired. □

4. Elementary JB*-triples satisfy the Mazur–Ulam property

Our goal in this section is to prove that every weakly compact JB*-triple satisfies the Mazur–Ulam property. By this result we shall exhibit an example of a non-unital and non-commutative C*-algebra containing no unitaries but satisfying the Mazur–Ulam property. The new geometric properties related to the facial structure of elementary JB*-triples developed in the previous section will be the germ to prove our result. Compared with the techniques in [4, 12, 35] and [29], we shall not base our proof on [35, Lemma 6] nor on [17, Lemma 2.1].

Theorem 4.1. *Every elementary JB*-triple K satisfies the Mazur–Ulam property, that is, for every real Banach space Y, every surjective isometry $\Delta : S(K) \to S(Y)$ admits a (unique) extension to a surjective real linear isometry from K onto Y.*

Proof. As we already commented in previous sections, if K is reflexive the conclusion follows from [4, Theorem 4.14, Proposition 4.15 and Remark 4.16] and [29, Theorem 1.1 and Corollary 1.2]. As in the proof of Proposition 3.5 we shall assume that K is non-reflexive (i.e. an infinite dimensional elementary JB*-triple of type 1, 2 or 3), and hence every tripotent in K is non-complete and of finite rank and K has infinite rank.

Let $\mathcal{F} = \mathcal{F}(K)$ denote the linear subspace of K generated by all minimal tripotents in K. In our case $\mathcal{F}$ is a normed subspace of K which is norm dense but non-closed. We shall define a mapping $T : \mathcal{F} \to Y$. By definition every element x in $\mathcal{F}$ can be written as a finite positive combination of mutually orthogonal minimal tripotents in K. We can therefore find a tripotent e in K such that $x \in K_2(e)$. Since K has infinite rank we can always find a minimal tripotent $v \in K$ with $v \perp e$. We set $T(x) := T_e^v(x)$. Clearly $T(0) = 0$.

We claim that T is well defined. Pick $0 \neq x \in \mathcal{F}$. Suppose e_1, e_2 are tripotents in K with $x \in K_2(e_j)$ for $j = 1, 2$ and v_1, v_2 are two minimal tripotents in K with $v_j \perp e_j$ for $j = 1, 2$. Proposition 3.7(d) implies that

$$T_{e_1}^{v_1}\left(\frac{x}{\|x\|}\right) = \Delta\left(\frac{x}{\|x\|}\right) = T_{e_2}^{v_2}\left(\frac{x}{\|x\|}\right),$$

and hence T is well defined.

We shall next show that T is linear. For each $x \in \mathcal{F}$, each pair of tripotents e, v in K with v minimal, $v \perp e$ and $x \in K_2(e)$, and each real number α, we have $\alpha x \in K_2(e)$ and $T(\alpha x) = T_e^v(\alpha x) = \alpha T_e^v(x) = \alpha T(x)$. Let us now take $x, y \in \mathcal{F}$. We can find a tripotent $e \in K$ such that $x, y, x + y \in K_2(e)$. Let $v \in K$ be a minimal tripotent with $v \perp e$. By definition

$$T(x + y) = T_e^v(x + y) = T_e^v(x) + T_e^v(y) = T(x) + T(y).$$

Since $T : \mathcal{F} \to Y$ is a real linear isometry and $\mathcal{F}$ is norm dense in K, we can find a unique extension of T to a surjective real linear isometry from K to Y which will be denoted by the same symbol T.

We shall finally show that T coincides with Δ on $S(K)$. By continuity and norm density of $\mathcal{F}$, it suffices to prove that T coincides with Δ on $S(\mathcal{F})$. By definition, given $x \in S(\mathcal{F})$, tripotents e, v in K with $v \perp e$ and $x \in K_2(e)$, Proposition 3.7(d) assures that $T(x) = T_e^v(x) = \Delta(x)$, which concludes the proof. □

The main results in [25] prove that every surjective isometry between the unit spheres of two elementary JB*-triples K_1 and K_2 can be extended to a surjective real linear isometry between K_1 and K_2 (see [25, Theorems 4.4-4.7 and 4.9] and some precedents in [38]). Our previous Theorem 4.1 offers an strengthened conclusion by showing that every surjective isometry from the unit sphere of an elementary JB*-triple onto the unit sphere of any real Banach space extends to a surjective real linear isometry.

The following straightforward consequence is interesting by itself.

Corollary 4.2. *For each complex Hilbert space H, the space $K(H)$ of all compact linear operators on H satisfies the Mazur–Ulam property.*

5. More on the strong Mankiewicz property

In this section we pursue a result showing that every weakly compact JB*-triple satisfies the strong Mankiewicz property. We begin by establishing a technical result which is valid for an abstract class of Banach spaces including all weakly compact JB*-triples.

Proposition 5.1. *Let $(X_i)_{i\in I}$ be a family of Banach spaces, and let $X = \bigoplus_{i\in I}^{c_0} X_i$. Suppose that the following hypotheses hold: for each $x, y \in X$ and each $\varepsilon > 0$ there exist a finite subset $F \subseteq I$ (depending on x, y and $\varepsilon > 0$), closed subspaces $Z_i \subseteq X_i$ and elements $a_i, b_i \in Z_i$ ($i \in F$) such that*

(1) *For each* $i \in F$ *there exists a subset* $M_i \subseteq \mathcal{B}_{X_i}$ *satisfying*

$$\mathcal{B}_{Z_i} = \{x_i \in \mathcal{B}_{X_i} : \|x_i - m_i\| \leq 1 \text{ for all } m_i \in M_i\};$$

(2) $\|x - (a_i)_{i\in F}\|, \|y - (b_i)_{i\in F}\| < \varepsilon$ $\Big($*we obviously regard* $\bigoplus_{i\in F}^{\ell_\infty} Z_i$ *as a closed subspace of* $X\Big)$*;*

(3) *The closed unit ball of the space* $\bigoplus_{i\in F}^{\ell_\infty} Z_i$ *satisfies the strong Mankiewicz property.*

Then every convex body $K \subset X$ *satisfies the strong Mankiewicz property.*

Proof. By [35, Lemma 4] it suffices to prove that the closed unit ball of X satisfies the strong Mankiewicz property. To this end let $\Delta : \mathcal{B}_X \to L$ be a surjective isometry, where L is a convex subset in a normed space Y. We shall show that Δ is affine.

Let us take $x, y \in \mathcal{B}_X$ and $t \in (0, 1)$. Fix an arbitrary $\varepsilon > 0$. It follows from our hypotheses that there exist a finite set $F \subseteq I$, closed subspaces $Z_i \subseteq X_i$ and elements $a_i, b_i \in \mathcal{B}_{Z_i}$ $(i \in F)$ such that the closed unit ball of the space $Z = \bigoplus_{i\in F}^{\ell_\infty} Z_i$ satisfies the strong Mankiewicz property, $\|x - (a_i)_{i\in F}\| < \varepsilon$ and $\|y - (b_i)_{i\in F}\| < \varepsilon$.

We also know the existence of subsets $M_i \subseteq \mathcal{B}_{X_i}$ $(i \in I)$ satisfying (1). Let $M \subseteq X$ denote the set given by

$$M := \Big\{x = (x_i)_i \in \mathcal{B}_X : x_i \in M_i \text{ for all } i \in F,\ x_j \in \mathcal{B}_{X_j} \text{ for } j \in I\backslash F\Big\}.$$

We also set

$$L_1 := \{y \in L : \|y - \Delta(m)\| \leq 1 \text{ for all } m \in M\}.$$

Having in mind that L is convex, it is easy to see that L_1 also is convex.

We claim that

$$\Delta(\mathcal{B}_Z) = L_1. \tag{7}$$

Indeed, by the assumptions each $z \in \mathcal{B}_Z$ satisfies that $\|z-m\| \leq 1$ for all $m \in M$, and hence $\|\Delta(z)-\Delta(m)\| \leq 1$ for all $m \in M$. This proves that $\Delta(\mathcal{B}_Z) \subseteq L_1$.

Take now, $y \in L_1$. Since Δ is surjective there exists (a unique) $z = (z_i)_i \in \mathcal{B}_X$ with $\Delta(z) = y$. Since Δ is an isometry and $y \in L_1$ we deduce that

$$\|z - m\| = \|\Delta(z) - \Delta(m)\| \leq 1, \text{ for all } m \in M,$$

in particular,

$$\|z_i - m_i\| \leq 1, \text{ for all } m_i \in M_i \text{ and for all } i \in F$$

and

$$\|z_i - w_i\| \leq 1, \text{ for all } w_i \in \mathcal{B}_{X_i} \text{ and all } i \in I\backslash F.$$

It follows from the hypothesis (1) that $z_i \in \mathcal{B}_{Z_i}$ for all $i \in F$, and clearly $z_i = 0$ for all $i \in I\backslash F$. Therefore $z \in \mathcal{B}_Z$ which finishes the proof of (7).

We deduce from (7) and the preceding comments that $\Delta|_{\mathcal{B}_Z} : \mathcal{B}_Z \to L_1$ is a surjective isometry and L_1 is a convex subset of Y. We apply now that the closed unit ball of the space Z satisfies the strong Mankiewicz property to deduce that $\Delta|_{\mathcal{B}_Z}$ is affine, and thus

$$\Delta\left(t(a_i)_{i\in F} + (1-t)(b_i)_{i\in F}\right) = t\Delta\left((a_i)_{i\in F}\right) + (1-t)\Delta\left((b_i)_{i\in F}\right),$$

and consequently

$$\begin{aligned} &\|\Delta(tx + (1-t)y) - (t\Delta(x) + (1-t)\Delta(y))\| \\ &\leq \|\Delta(tx + (1-t)y) - \Delta\left(t(a_i)_{i\in F} + (1-t)(b_i)_{i\in F}\right)\| \\ &+ t\|\Delta\left((a_i)_{i\in F}\right) - \Delta(x)\| + (1-t)\|\Delta\left((b_i)_{i\in F}\right) - \Delta(y)\| < 2\varepsilon. \end{aligned}$$

The arbitrariness of $\varepsilon > 0$ implies that $\Delta(tx+(1-t)y) = t\Delta(x)+(1-t)\Delta(y)$, which concludes the proof. $\square$

An appropriate framework to apply the previous proposition is provided by weakly compact JB*-triples.

Corollary 5.2. *Every weakly compact JB*-triple E satisfies the hypotheses of the previous Proposition 5.1. Consequently every convex body $K \subset E$ satisfies the strong Mankiewicz property.*

Proof. As we commented in subsection 2 every weakly compact JB*-triple coincides with a c_0-sum of a family $\{K_i : i \in I\}$ of elementary JB*-triples (cf. [6]). Given $x, y \in E$ and $\varepsilon > 0$ there exist a finite subset $F \subseteq I$ satisfying $\|x - (x_i)_{i\in F}\|, \|y - (y_i)_{i\in F}\| < \frac{\varepsilon}{2}$.

Fix an arbitrary $i \in F$. If K_i is reflexive (i.e. it is finite dimensional or an elementary JB*-triple of type 4), we know that K_i is a JBW*-triple. In this case we take $Z_i = K_i$, $a_i = x_i$, $b_i = y_i$, and $M_i = \{0\}$. Clearly,

$$\mathcal{B}_{K_i} = \{x_i \in K_i : \|x_i - 0\| \leq 1\}.$$

Suppose next that K_i is not reflexive (i.e. an infinite dimensional elementary JB*-triple of type 1, 2 or 3). By [6, Remark 4.6] and/or basic theory of compact operators we can find a finite rank (and hence non-complete) tripotent e_i (i.e. a tripotent which is a finite sum of mutually orthogonal minimal tripotents in K_i) such that $\|x_i - P_2(e_i)(x_i)\|, \|y_i - P_2(e_i)(y_i)\| < \frac{\varepsilon}{2}$. The subtriple $Z_i = (K_i)_2(e_i)$ is finite dimensional and thus a JBW*-triple. Take $a_i = P_2(e_i)(x_i)$, $b_i = P_2(e_i)(y_i) \in Z_i$. Set $M_i := \{e_i\}^{\perp}_{min} \cap K_i = \{v \text{ minimal tripotent in } K_i : v \perp e_i\} \subseteq \mathcal{B}_{K_i}$. Lemma 3.6 assures that

$$\mathcal{B}_{Z_i} = \mathcal{B}_{(K_i)_2(e_i)} = \left\{x_i \in \mathcal{B}_{K_i} : \|x_i - m_i\| \leq 1 \text{ for all } m_i \in M_i = \{e_i\}^{\perp}_{min} \cap K_i\right\}.$$

By construction,

$$\|x - (a_i)_{i\in F}\| \leq \|x - (x_i)_{i\in F}\| + \|(x_i)_{i\in F} - (a_i)_{i\in F}\| < \varepsilon.$$

Finally, for each $i \in F$, Z_i is finite dimensional or reflexive, therefore $\bigoplus_{i\in F}^{\ell_\infty} Z_i$ is a JBW*-triple, and thus its closed unit ball satisfies the strong Mankiewicz property by [29, Corollary 1.2]. □

Remark 5.3. *It is worth to note that every compact C*-algebra of the form $A = \bigoplus_{i\in I}^{c_0} K(H_i)$, where the H_i's are complex Hilbert spaces (cf. [1, Theorem 8.2]). Obviously, A is a weakly compact JB*-triple, and hence every convex body $K \subset A$ satisfies the strong Mankiewicz property.*

We have already developed enough tools to address the question whether every weakly compact JB*-triple satisfies the Mazur–Ulam property.

Theorem 5.4. *Every weakly compact JB*-triple E satisfies the Mazur–Ulam property, that is, for every real Banach space Y, every surjective isometry from $S(E)$ onto $S(Y)$ admits a (unique) extension to a surjective real linear isometry from E onto Y.*

Proof. We begin with an observation. For each non-zero tripotent $u \in E$, we consider the proper face $F_u = u + \mathcal{B}_{E_0(u)}$. Lemma 3.1 assures that $\Delta(F_u)$ is an intersection face of $S(Y)$, in particular a convex subset. Let us observe that $E_0(u)$ is a weakly compact JB*-triple. Corollary 5.2 implies that $\mathcal{B}_{E_0(u)}$ satisfies the strong Mankiewicz property. We consider the next commutative diagram

$$\begin{array}{ccc} F_u = u + \mathcal{B}_{E_0(u)} & \xrightarrow{\ \Delta\ } & \Delta\left(u + \mathcal{B}_{E_0(u)}\right) \\ \tau_{-u}\big\downarrow & & \big\uparrow\tau_{\Delta(u)} \\ \mathcal{B}_{E_0(u)} & \overset{\Delta_u}{\dashrightarrow} & \Delta\left(u + \mathcal{B}_{E_0(u)}\right) - \Delta(u) \end{array}$$

Since $E_0(u)$ satisfies the strong Mankiewicz property and Δ_u is a surjective isometry from $\mathcal{B}_{E_0(u)}$ onto a convex set, we can guarantee the existence of a linear isometry $T_u : E_0(u) \to Y$ satisfying

$$\Delta(u + x) = T_u(x) + \Delta(u), \text{ for all } x \in \mathcal{B}_{E_0(u)}, \tag{8}$$

(cf. [35]). In particular $\Delta|_{F_u}$ is affine.

We claim that

$$\Delta(w) = T_u(w), \text{ for every non-zero tripotents } u, w \in E \text{ with } w \in E_0(u). \tag{9}$$

Namely, for each non-zero tripotent $w \in E_0(u)$ ($u \perp w$) the element $u \pm w$ is a tripotent in F_u. Therefore

$$\Delta(u) = \Delta\left(\frac{1}{2}(u+w) + \frac{1}{2}(u-w)\right) = \frac{1}{2}\Delta(u+w) + \frac{1}{2}\Delta(u-w).$$

By Lemma 3.4 (see also [29, Corollary 2.4]) we have $-\Delta(u-w) = \Delta(-u+w)$, and by similar arguments to those given above, but now with respect to the face F_w, we have

$$\Delta(w) = \Delta\left(\frac{1}{2}(u+w) + \frac{1}{2}(-u+w)\right) = \frac{1}{2}\Delta(u+w) + \frac{1}{2}\Delta(-u+w)$$

$$= \frac{1}{2}\Delta(u+w) - \frac{1}{2}\Delta(u-w).$$

It then follows that

$$\Delta(u) + \Delta(w) = \Delta(u+w). \tag{10}$$

The claim in (9) is a straight consequence of (10) and (8).

We continue with our argument. If E is an elementary JB*-triple the result follows from Theorem 4.1. We can therefore assume that E decomposes as the orthogonal sum of two non-zero weakly compact JB*-triples A and B. Let us pick two non-zero tripotents $u_1 \in A$ and $u_2 \in B$ and the corresponding linear isometries $T_{u_j} : E_0(u_j) \to Y$ given by (8).

Let us observe that $A \subseteq E_0(u_2)$, $B \subseteq E_0(u_1)$ and $E = A \oplus^{\ell_\infty} B$. Therefore the mapping $T : E \to Y$, $T(a+b) = T_{u_2}(a) + T_{u_1}(b)$ is a well-defined linear operator. We shall next show that

$$T(x) = \Delta(x), \text{ for all } x \in S(E). \tag{11}$$

Let us fix $x \in S(E)$. By Remark 4.6 in [6] there exists a possible finite at most countable family $\{e_n\}$ of mutually orthogonal minimal tripotents in E and $(\lambda_n) \subseteq \mathbb{R}^+$ such that $\lambda_1 = 1$ and $x = \sum_{n\geq 1} \lambda_n e_n$. Each e_n lies in A or in B, and hence it follows from the definition of T that $T(e_n) = T_{u_2}(e_n)$ if $e_n \in A$ and $T(e_n) = T_{u_1}(e_n)$ if $e_n \in B$. We deduce from (9) that in any case we have $\Delta(e_n) = T(e_n)$ for all $n \geq 1$. Now we regard x as an element in the face F_{e_1} and we apply the properties of T_{e_1} and (9) to deduce that

$$\Delta(x) = \Delta\left(e_1 + \sum_{n\geq 2} \lambda_n e_n\right) = \Delta(e_1) + \sum_{n\geq 2} \lambda_n T_{e_1}(e_n) = \Delta(e_1) + \sum_{n\geq 2} \lambda_n \Delta(e_n)$$

$$= T(e_1) + \sum_{n\geq 2} \lambda_n T(e_n) = T\left(e_1 + \sum_{n\geq 2} \lambda_n e_n\right) = T(x),$$

witnessing the validity of (11).

Finally, since T is linear we derive from (11) and the hypothesis on Δ that T is a surjective real linear isometry. $\square$

We have previously mentioned that compact C*-algebras are examples of weakly compact JB*-triples. The next corollary perhaps deserves its own place.

Corollary 5.5. *Suppose* $A = \bigoplus_{i \in I}^{c_0} K(H_i)$ *is a compact* C*-*algebra, where each* H_i *is a complex Hilbert space. Then every surjective isometry from the unit sphere of* A *onto the unit sphere of any other real Banach space* Y *admits a (unique) extension to a surjective real linear isometry from* A *onto* Y.

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