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Weak Demicompactness Involving Measures of Weak Noncompactness and Invariance of the Essential Spectrum

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Abstract. In this paper, we show that an unbounded weakly S_0-demicompact linear operator T, introduced in [18], acting on a Banach space, can be characterized by some measures of weak noncompactness. Moreover, our results are illustrated to discuss the relationship with Fredholm and upper semi-Fredholm operators as well as the stability of the essential spectrum of T.

1. Introduction

Let X and Y be two Banach spaces. The set of all closed densely defined (resp. bounded) linear operators acting from X into Y is denoted by $\mathcal{C}(X,Y)$ (resp. $\mathcal{L}(X,Y)$). We denote by $\mathcal{K}(X,Y)$ the subset of compact operators of $\mathcal{L}(X,Y)$. For $T \in \mathcal{C}(X,Y)$, we use the following notations: $\alpha(T)$ is the dimension of the kernel $\mathcal{N}(T)$ and $\beta(T)$ is the codimension of the range $\mathcal{R}(T)$ in Y. The next sets of upper semi-Fredholm, lower semi-Fredholm, Fredholm and semi-Fredholm operators from X into Y are, respectively, defined by:

$$\Phi_+(X,Y) = \{T \in \mathcal{C}(X,Y) \text{ such that } \alpha(T) < \infty \text{ and } \mathcal{R}(T) \text{ closed in } Y\},$$
$$\Phi_-(X,Y) = \{T \in \mathcal{C}(X,Y) \text{ such that } \beta(T) < \infty \text{ and } \mathcal{R}(T) \text{ closed in } Y\},$$
$$\Phi(X,Y) := \Phi_-(X,Y) \cap \Phi_+(X,Y),$$

and

$$\Phi_\pm(X,Y) := \Phi_-(X,Y) \cup \Phi_+(X,Y).$$

The set of bounded upper (resp. lower) semi-Fredholm operators from X into Y is defined by

$$\Phi_+^b(X,Y) = \Phi_+(X,Y) \cap \mathcal{L}(X,Y) \quad (\text{resp. } \Phi_-^b(X,Y) = \Phi_-(X,Y) \cap \mathcal{L}(X,Y)).$$

We denote by $\Phi^b(X,Y) = \Phi(X,Y) \cap \mathcal{L}(X,Y)$ the set of bounded Fredholm operators from X into Y. The index of an operator $T \in \Phi_\pm(X,Y)$ is defined by $ind(T) := \alpha(T) - \beta(T)$. A complex number λ is in $\Phi_{+T}, \Phi_{-T}, \Phi_{\pm T}$ or Φ_T if $\lambda - T$ is in $\Phi_+(X,Y)$, $\Phi_-(X,Y)$, $\Phi_\pm(X,Y)$ or $\Phi(X,Y)$, respectively. If $X = Y$, then $\mathcal{L}(X,Y)$, $\mathcal{C}(X,Y)$,

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$\mathcal{K}(X,Y)$, $\Phi(X,Y)$, $\Phi_+(X,Y)$, $\Phi_-(X,Y)$ and $\Phi_\pm(X,Y)$ are replaced by $\mathcal{L}(X)$, $C(X)$, $\mathcal{K}(X)$, $\Phi(X)$, $\Phi_+(X)$, $\Phi_-(X)$ and $\Phi_\pm(X)$, respectively. If $T \in C(X)$, we denote by $\rho(T)$ the resolvent set of T and by $\sigma(T)$ the spectrum of T. Let $T \in C(X)$. For $x \in D(T)$, the graph norm $\|.\|_T$ of x is defined by $\|x\|_T = \|x\| + \|Tx\|$. It follows from the closedness of T that $X_T := (D(T), \|.\|_T)$ is a Banach space. Clearly, for every $x \in D(T)$ we have $\|Tx\| \leq \|x\|_T$, so that $T \in \mathcal{L}(X_T, X)$. A linear operator B is said to be T-defined if $D(T) \subseteq D(B)$. If the restriction of B to $\mathcal{D}(T)$ is bounded from X_T into X, we say that B is T-bounded.

Definition 1.1. *Let X and Y be two Banach spaces and let $F \in \mathcal{L}(X,Y)$.*
(*a*) *The operator F is a Fredholm perturbation if $T + F \in \Phi(X,Y)$ whenever $T \in \Phi(X,Y)$.*
(*b*) *The operator F is an upper semi-Fredholm perturbation if $T + F \in \Phi_+(X,Y)$ whenever $T \in \Phi_+(X,Y)$.*
(*c*) *The operator F is a lower semi-Fredholm perturbation if $T + F \in \Phi_-(X,Y)$ whenever $T \in \Phi_-(X,Y)$.*

Now, we define the S-resolvent set of T by

$$\rho_S(T) := \{\lambda \in \mathbb{C} \text{ such that } \lambda S - T \text{ has a bounded inverse}\}.$$

The S-spectrum of T is defined by

$$\sigma_S(T) := \mathbb{C} \backslash \rho_S(T).$$

A complex number λ is in $\Phi_{+S,T}$, $\Phi_{-S,T}$ or $\Phi_{S,T}$ if $\lambda S - T$ is in $\Phi_+(X)$, $\Phi_-(X)$ or $\Phi(X)$ respectively. Note that the concept of S-essential spectrum is introduced in [9] as a generalization of the usual notion of Wolf essential spectrum (see [31]). In the end of this section, we recall that there are many ways to define the essential spectrum of an operator $T \in C(X)$ (see for example [12, 26]). In this work, we are concerned with the S-essential spectrum, the S-approximate essential spectrum (see [24, 25]) and the S-essential defect spectrum of T (see [27]), defined respectively by

$$\sigma_{e,S}(T) := \bigcap_{K \in \mathcal{K}(X)} \sigma_S(T+K),$$

$$\sigma_{eap,S}(T) := \bigcap_{K \in \mathcal{K}(X)} \sigma_{ap,S}(T+K),$$

and

$$\sigma_{e\delta,S}(T) := \bigcap_{K \in \mathcal{K}(X)} \sigma_{\delta,S}(T+K),$$

where

$$\sigma_{ap,S}(T) := \{\lambda \in \mathbb{C} \text{ such that } \inf_{\|x\|=1, x \in D(T)} \|(\lambda S - T)x\| = 0\},$$

and

$$\sigma_{\delta,S}(T) := \{\lambda \in \mathbb{C} \text{ such that } \lambda S - T \text{ is not onto}\}.$$

Now, we give some well-known properties.

Proposition 1.2. [1, 28] Let X be a Banach space. Then,
(*i*) The sets $\Phi_+^b(X)$, $\Phi_-^b(X)$ and $\Phi^b(X)$ are open.
(*ii*) The index is constant on every component of each of the sets: $\Phi_+^b(X)$, $\Phi_-^b(X)$ and $\Phi^b(X)$.
(*iii*) If $S, T \in \Phi_+^b(X)$ (resp. $S, T \in \Phi_-^b(X)$), then $ST \in \Phi_+^b(X)$ (resp. $ST \in \Phi_-^b(X)$) and $ind(ST) = ind(S) + ind(T)$.

If $x \in X$ and $r > 0$, then $B(x, r)$ will denote the closed ball of X with center at x and radius r. We denote by B_X the closed unit ball of X. The family of all nonempty and bounded subsets of X will be denoted by $\mathcal{M}_X$ and $\mathcal{M}_X^w$ its subfamily consisting of all relatively weakly compact sets. Moreover, let $conv(A)$ denotes the convex hull of a set $A \subset X$. An operator $T \in \mathcal{L}(X,Y)$ is said to be weakly compact if $T(M)$ is relatively weakly compact for every $M \in \mathcal{M}_X$. The family of weakly compact operators from X into Y is denoted by $\mathcal{W}(X,Y)$. If $X = Y$, we denote $\mathcal{W}(X)$ instead of $\mathcal{W}(X,X)$. The set $\mathcal{W}(X)$ is a closed two-sided ideal of $\mathcal{L}(X)$ containing $\mathcal{K}(X)$ (see [8, 10]). The operator T is said to be a Dunford-Pettis (for short property DP operator) if it maps weakly compact sets into compact sets. In particular, if T is a DP operator, then $x_n \xrightarrow{w} 0$ implies $\lim \|Tx_n\| = 0$ (see [5]). The Banach space X is said to have the Dunford-Pettis property (see [7]) if for each Banach space Y, every weakly compact operator $T : X \longrightarrow Y$ takes weakly compact sets in X into norm compact sets of Y. Note that, if X is a Banach space with the Dunford-Pettis property, then every $T \in \mathcal{W}(X,Y)$ is a DP operator.

Definition 1.3. *Let X be a Banach space and $T \in \mathcal{L}(X)$. We say that T has a right weak-Fredholm inverse if there exists $T_r \in \mathcal{L}(X)$ such that $I - TT_r \in \mathcal{W}(X)$. Let*

$$\Phi^{r,w}(X) = \Big\{T \in \mathcal{L}(X) : \quad T \text{ has a right weak-Fredholm inverse}\Big\}.$$

We denote by $\mathcal{G}^{r,w}$ the set of right weak-Fredholm inverse of T.

In 1966, W. V. Petryshyn [22, 23] has developed the concept of demicompactness for nonlinear operator. Several applications of this concept were provided, especially on fixed point theory. Moreover, the demicompactness concept was used to provide several results on Fredholm theory (see [2, 23]). The class of demicompact operators acting on a Banach space contains the class of compact operator. Hence, the class of demicompact operators play an important role when studying perturbations of Fredholm operators. Recently, W. Chaker, A. Jeribi and B. Krichen [14] have utilized demicompact operators in order to investigate the essential spectra of closed linear operators. In 2014, B. Krichen [17], introduced the relative demicompactness class with respect to a given closed linear operator as a generalization of the demicompactness notion. This definition asserts that if X is a Banach space, $T : D(T) \subset X \longrightarrow X$, and $S_0 : D(S_0) \subset X \longrightarrow X$ are two linear operators with $D(T) \subset D(S_0)$, then T is said to be S_0-demicompact (or relative demicompact with respect to S_0), if every bounded sequence $(x_n)_n$ in $D(T)$ such that $(S_0x_n - Tx_n)_n$ converges in X, have a convergent subsequence. Recently, B. Krichen and D. O'Regan developed in [18, 19] some Fredholm and perturbation results involving the class of weakly demicompact linear operators. Moreover, they studied the relationship between this class and measures of weak noncompactness of linear operator with respect to an axiomatic one.

In this paper, we show that an unbounded weakly S_0-demicompact linear operator T acting on a Banach space, can be characterized by some measures of weak noncompactness. The obtained results are used to discuss the relationship with Fredholm and upper semi-Fredholm operators as well as the invariance of the essential spectrum of T. Finally, let us mention that results obtained in the paper generalize a few ones contained in the papers [14, 17, 18], for example.

2. Measure of Weak Noncompactness

First, we recall the axiomatic approach in defining of measures of weak noncompactness [4]. Let $(X, \|\cdot\|)$ be an infinite dimensional complex Banach space. We denote by $\mathcal{M}_X^w$ the subfamily of $\mathcal{M}_X$ consisting of all relatively weakly compact sets, and $\overline{M}^w$ denote the weak closure of a set $M \subset X$.

Definition 2.1. A function $\mu : \mathcal{M}_X \longrightarrow \mathbb{R}_+$ is said to be a measure of weak noncompactness if, for all $A, B \in \mathcal{M}_X$, it satisfies the following conditions:
(i) $\mu(A) = 0 \Longleftrightarrow A \in \mathcal{M}_X^w$.
(ii) $A \subset B \Longrightarrow \mu(A) \leq \mu(B)$.
(iii) $\mu(\overline{conv}(A)) = \mu(A)$.
(iv) $\mu(A \cup B) = \max\{\mu(A), \mu(B)\}$.
(v) $\mu(A + B) \leq \mu(A) + \mu(B)$.
(vi) $\mu(\lambda A) = |\lambda|\mu(A), \lambda \in \mathbb{C}$.

Let us recall that each measure of weak noncompactness satisfies the Cantor intersection condition (see [4]) i.e., If $(A_n) \subseteq \mathcal{M}_X^w$ such that $A_n = \overline{A}_n$ and $A_{n+1} \subset A_n$ for $n = 1, 2, ...$ and if $\lim\limits_{n\to+\infty} \mu(A_n) = 0$, then $A_\infty = \bigcap_{n=1}^{+\infty} A_n \neq \emptyset$.
By using the relation

$$M \subset \overline{M}^w \subset \overline{conv}(M),$$

we infer that the measure μ satisfies

$$\mu(M) = \mu(\overline{M}^w),$$

for all $M \in \mathcal{M}_X$.
A measure of weak noncompactnes is said to be regular if it satisfies the condition:

$$\mu(M) = 0 \Longleftrightarrow M \in \mathcal{M}_X^w.$$

An important example of a regular measure of weak noncompactness is the De Blasi measure (see [6]), defined as follows:

$$w(M) = \inf\{r > 0, \text{ there exists } W \in \mathcal{M}_X^w \text{ with } M \subseteq W + rB_X\},$$

for each $M \in \mathcal{M}_X$. This function has several useful properties (see [6]). For example, $w(B_X) = 0$ whenever X is reflexive and $w(B_X) = 1$ otherwise.

In the sequel, all Banach spaces considered are supposed to be non reflexive.

Remark 2.2. We notice that the definition of a measure of weak noncompactness μ on a Banach space X can be extended to all subsets by imposing the following condition: $\mu(D) = +\infty$ whenever the subset D is unbounded. Obviously, the function μ defined on every set of X is a weak measure of noncompactness.

Now, we provide a definition which gives an axiomatic approach to the notion of measure of weak noncompactness of operators.

Definition 2.3. Let X and Y be two complex Banach spaces, and let μ be a measure of weak noncompactness in Y. We define the function

$$\begin{array}{rccl} \Psi_\mu : & \mathcal{L}(X,Y) & \longrightarrow & [0,+\infty[\\ & T & \longmapsto & \Psi_\mu(T) = \mu(T(B_X)). \end{array}$$

Ψ_μ is called a measure of weak noncompactness of operators associated to μ.

In view of this definition, we get easily the following properties of the function Ψ_μ.

Proposition 2.4. Let X and Y be two complex Banach spaces, μ be a measure of weak noncompactness in Y and let Ψ_μ a measure of weak noncompactness of operators associated to μ. For all $S, T \in \mathcal{L}(X,Y)$, we have
(*i*) $\Psi_\mu(T) = 0 \Longleftrightarrow T \in \mathcal{W}(X)$.
(*ii*) $\Psi_\mu(T) \leq \Psi_\mu(T) + \Psi_\mu(S)$.
(*iii*) $\Psi_\mu(\lambda S) = |\lambda|\Psi_\mu(S), \lambda \in \mathbb{C}$.
(*iv*) $\Psi_\mu(S + K) = \Psi_\mu(S)$, for all $K \in \mathcal{W}(X)$.

Definition 2.5. Let X and Y be two complex Banach spaces, μ be a measure of weak noncompactness in Y and let Ψ_μ be a measure of weak noncompactness of operators associated to μ. Ψ_μ is said to be algebraic semi-multiplicative if it satisfies the condition:

$$\Psi_\mu(ST) \leq \Psi_\mu(S)\Psi_\mu(T), \text{ for all } S,\ T \in \mathcal{L}(X,Y).$$

Let us recall an important example of measure of weak noncompactness of operators.

Definition 2.6. Let X and Y be two complex Banach spaces and w be the De Blasi measure of weak noncompactness in the space Y. We define the function

$$\begin{array}{rccc}\Theta_w: & \mathcal{L}(X,Y) & \longrightarrow & [0,+\infty[\\ & T & \longmapsto & \Theta_w(T)=w(T(B_X)).\end{array}$$

Θ_w is a measure of weak noncompactness of operators associated to w.

Now, we give some properties of Θ_w.

Proposition 2.7. Let X and Y be two complex Banach spaces and $T, S \in \mathcal{L}(X,Y)$. Then,
(i) $w(T(D)) \leq \Theta_w(T)w(D)$ for every $D \in \mathcal{M}_X$.
(ii) $\Theta_w(T) \leq \|T\|$.
(iii) If $X = Y$ then, $\Theta_w(ST) \leq \Theta_w(S)\Theta_w(T)$.
(iv) $\Theta_w(S+T) \leq \Theta_w(S) + \Theta_w(T)$.
(v) If $X = Y$, then $\Theta_w(T^n) \leq (\Theta_w(T))^n$ for every $n \in \mathbb{N}$.
(vi) Let $C \geq 0$ such that for every $x \in X$, $\|Tx\| \leq C\|x\|$. Then,

$$w(T(D)) \leq Cw(D).$$

(vii) Let $C \geq 0$ such that for every $x \in X$, $\|x\| \leq C\|Tx\|$. Then,

$$w(D) \leq Cw(T(D)),$$

and the result holds even if $T \in \mathcal{C}(X)$.
($viii$) If $\Theta_w(T) = 0$, then $T \in \mathcal{W}(X,Y)$.

Proof. (i) Let $D \in \mathcal{M}_X$ and $r > w(D)$. Then, there exists $W \in \mathcal{M}_X^w$ such that $D \subseteq W + rB_X$. Hence,

$$T(D) \subseteq T(W) + rT(B_X).$$

Accordingly,

$$w(T(D)) \leq rw(T(B_X)).$$

Taking into account the fact that $w(T(B_X)) = \Theta_w(T)$ and letting $r \to w(D)$ we deduce that

$$w(T(D)) \leq \Theta_w(T)w(D).$$

For the proof of (ii), it suffices to see that $T(B_X) \subseteq \|T\|B_X$. Then,

$$\begin{array}{rcl}\Theta_w(T) & = & w(T(B_X)) \\ & \leq & \|T\|.\end{array}$$

Now, let $S, T \in \mathcal{L}(X,Y)$, then

$$\begin{array}{rcl}\Theta_w(S+T) & = & w\Big((S+T)(B_X)\Big) \\ & \leq & w\Big(S(B_X)\Big) + w\Big(T(B_X)\Big) \\ & = & \Theta_w(S) + \Theta_w(T).\end{array}$$

This proves (iii).
To prove (iv), let $S, T \in \mathcal{L}(X,Y)$, then

$$\begin{array}{rcl}\Theta_w(ST) & = & w(ST(B_X)) \\ & \leq & \Theta_w(S)w(T(B_X)) \\ & = & \Theta_w(S)\Theta_w(T).\end{array}$$

We prove easily (v) by induction. For the proof of (vi), take $D \in \mathcal{M}_X$ and $r > w(D)$ such that $D \subseteq W + rB_X$. Then,

$$T(D) \subseteq T(W) + rT(B_X).$$

Since we have $\|Tx\| \leq C\|x\|$ for every $x \in X$, we infer that

$$T(B_X) \subseteq CB_X.$$

Hence,

$$T(D) \subseteq T(W) + rCB_X.$$

Accordingly,

$$w(T(D)) \leq rC.$$

Letting $r \to w(D)$ we deduce that

$$w(T(D)) \leq Cw(D).$$

Now, let us prove (vii). Suppose that $T \in \mathcal{C}(X)$ satisfies the inequality $\|x\| \leq C\|Tx\|$, for some positive scalar C. It yields that T has a bounded inverse $\widetilde{T}^{-1} : T(X) \longrightarrow X$. Let D be a bounded set of $D(T)$. If $w(T(D)) = +\infty$, then we have clearly $w(D) \leq w(T(D))$. Suppose that $w(T(D)) < +\infty$, then $T(D)$ is bounded. Furthermore, we have $D \subset \widetilde{T}^{-1}T(D)$. One can readily show that $\|\widetilde{T}^{-1}\| \leq C$. This fact combined with property (vi) achieves the proof of (vii).
Now, suppose that $\Theta_w(T) = 0$. Let $D \in \mathcal{M}_X$, then there exists $r > 0$ such that $D \subseteq rB_X$. Thus, $w\Big(T(D)\Big) \leq r\Theta_w(T))$. This implies that $w\Big(T(D)\Big) = 0$ which means that $T(D) \subseteq \mathcal{M}_Y^w$. This achieves the proof of $(viii)$. □

3. Relatively weakly demicompact operators and measures of weak noncompactness

We denote $\to$ for the strong convergence and $\rightharpoonup$ for the weak convergence. Recall the following definition introduced in [18].

Definition 3.1. Let X be a Banach space and let $A : D(A) \subset X \longrightarrow X, S_0 : D(S_0) \subset X \longrightarrow X$ be two linear operators with $D(A) \subset D(S_0)$. The operator A is said to be weakly S_0-demicompact (or weakly relative demicompact with respect to S_0), if for every bounded sequence $(x_n)_n \subset D(A)$ such that $S_0x_n - Ax_n \rightharpoonup x \in X$, for some $x \in X$, then there exists a weakly convergent subsequence of $(x_n)_n$.

Given a Banach space X and $S_0 \in \mathcal{L}(X)$, we define the following sets:

$$\mathcal{WDC}(S_0)(X) := \{T \in \mathcal{C}(X), \text{ such that } T \text{ is weakly } S_0\text{-demicompact }\},$$

and

$$\mathcal{WDC}^b(S_0)(X) := \mathcal{WDC}(S_0)(X) \cap \mathcal{L}(X).$$

Note that if we put $S_0 = I$, then we recover the usual definition of weakly demicompact operator. Furthermore, the spaces $\mathcal{WDC}(S_0)(X)$ and $\mathcal{WDC}^b(S_0)(X)$ will be replaced, respectively, by $\mathcal{WDC}(X)$ and $\mathcal{WDC}^b(X)$.
Next, we introduce special classes of weakly demicompact operators. Before that, let us recall some definitions.

Definition 3.2. [13, 16, 21] Let X be a Banach space and $T \in \mathcal{L}(X)$. T is said to be weakly quasi-compact if there exists a positive integer m and $W \in \mathcal{W}(X)$ such that $\|T^m - W\| < 1$.

We note $\mathcal{WQP}(X)$ for the set of weakly quasi-compact operators acting on a Banach space X.

Remark 3.3. Let X be a Banach space. Then, one can check that $\mathcal{K}(X) \subseteq \mathcal{W}(X) \subseteq \mathcal{WQP}(X)$.

Proposition 3.4. Let X be a Banach space. Then,

$$\mathcal{WQ}P(X) \subset \mathcal{WDC}^b(X).$$

Proof. Let $T \in \mathcal{WQ}P(X)$. Then, there exists a positive integer m and $W \in \mathcal{W}(X)$ such that $\|T^m - W\| < 1$. By using Newmann series, we readily see that $I - T^m + W$ is boundedly invertible. Moreover, we have $\left(I - T^m + W\right)^{-1} = \sum\limits_{k=0}^{+\infty}(T^m - W)^k$. Now, take a bounded sequence $(x_n)_n$ such that

$$x_n - Tx_n \rightharpoonup x \in X.$$

Put $z = x + Tx + ...T^{m-1}x \in X$. Then,

$$x_n - T^m x_n \rightharpoonup z.$$

Since W is weakly demicompact, there exists a subsequence $(x_{\varphi(n)})_n$ and $y \in X$ such that

$$Wx_{\varphi(n)} \underset{\to+\infty}{\longrightarrow} y.$$

Thus,

$$x_{\varphi(n)} - T^m x_{\varphi(n)} + Wx_{\varphi(n)} \rightharpoonup z + y.$$

Accordingly,

$$x_{\varphi(n)} \underset{\to+\infty}{\longrightarrow} \left(I - T^m + W\right)^{-1}(z + y).$$

Hence, $T \in \mathcal{WDC}^b(X)$. This completes the proof. □

Proposition 3.5. *Let X be a Banach space and let T and S_0 be two bounded operators on X. Then, we have*

$$T \text{ is weakly } S_0- \text{ demicompact} \Longrightarrow \mathcal{G}^{r,w}(I - S_0 + T) \subseteq \mathcal{WDC}^b(X).$$

Proof. Let $A \in \mathcal{G}^{r,w}(I - S_0 + T)$, then there exists $W \in \mathcal{W}(X)$ such that

$$(I - S_0 + T)A = I + W.$$

Now, let $(x_n)_n$ be a bounded sequence of X such that

$$x_n - Ax_n \rightharpoonup x \in X.$$

Then,

$$x_n - (I + W - TA + S_0A)x_n \rightharpoonup x.$$

It follows that

$$-Wx_n + (T - S_0)Ax_n \rightharpoonup x.$$

Since W is weakly compact, then there exists a subsequence $(x_{\varphi(n)})$ such that

$$Wx_{\varphi(n)} \rightharpoonup a \in X.$$

Therefore,

$$(T - S_0)Ax_{\varphi(n)} \rightharpoonup a + x.$$

By observing that

$$(T - S_0)x_{\varphi(n)} = (T - S_0)(x_{\varphi(n)} - Ax_{\varphi(n)}) + (T - S_0)Ax_{\varphi(n)},$$

for all $n \in \mathbb{N}$, we deduce that $\left((T - S_0)(x_{\varphi(n)}\right)_n$ converges. Since T is weakly S_0-demicompact, we conclude that $(x_n)_n$ has a weakly convergent subsequence, then $A \in \mathcal{WDC}^b(X)$. This achieves the proof. □

Proposition 3.6. *Let Z be a Banach space, $T \in \mathcal{L}(Z)$ and $\mathbb{T}$ be the unit circle of the complex plane.*
(i) *If T^m is weakly demicompact for some positive integer m, then T is weakly demicompact.*
(ii) *If T is weakly demicompact and $\sigma(T) \cap \mathbb{T} = \emptyset$, then T^m is weakly demicompact for all $m \in \mathbb{N}\backslash\{0\}$.*

Proof. Let $(x_n)_n$ be a bounded sequence in Z such that

$$x_n - Tx_n \rightharpoonup x \in Z.$$

Then,

$$x_n - T^m x_n \rightharpoonup x + Tx + ... + T^{m-1}x.$$

Since T^m is weakly demicompact, $(x_n)_n$ has a weakly convergent subsequence. This achieves the proof of (i).
Now, we prove (ii). Let $m \in \mathbb{N}\backslash\{0\}$ and suppose that T is weakly demicompact. We may suppose that $m \geq 2$. Let $(x_n)_n$ be a bounded sequence in Z such that

$$x_n - T^m x_n \rightharpoonup x \in Z.$$

Equivalently,

$$Q(T)(I - T)x_n \rightharpoonup x.$$

where $Q(X) = 1 + X + ... + X^{m-1} \in \mathbb{C}[X]$. According to the spectral mapping theorem, $\sigma\big(Q(T)\big) = Q\big(\sigma(T)\big)$. Now, note that if $\lambda \in \mathbb{C}$ satisfies $Q(\lambda) = 0$, then $\lambda^m = 1$ so that $\lambda \in \mathbb{T}$. Hence, $0 \notin \sigma\big(Q(T)\big)$.
This implies

$$x_n - Tx_n \rightharpoonup Q(T)^{-1}x.$$

Since T is weakly demicompact, $(x_n)_n$ has a convergent subsequence. We conclude that T^m is weakly demicompact. $\square$

Definition 3.7. Let X, Y be two Banach spaces. An operator $T : D(T) \subseteq X \longrightarrow Y$ is said to be weakly closed, if for every sequence $(x_n)_n$ in $D(T)$ such that $x_n \rightharpoonup x \in X$ and $Tx_n \rightharpoonup y \in Y$, we have $x \in D(T)$ and $y = Tx$.

Remark 3.8. Let X and Y be two Banach spaces and $T \in \mathcal{L}(X, Y)$. Then, T is weakly closed.

The following key lemma will be useful for some proofs.

Lemma 3.9. [30] An operator A is in $\Phi_+(X)$ with $ind(A) \leq 0$, if and only if, there exists two operators A_0 and K such that A_0 is in $\Phi_+(X)$ and one to one, and K is a finite rank operator such that $A = A_0 + K$.

Now, we are in position to state the following result.

Theorem 3.10. Let X be a Banach space and $T, S_0 \in \mathcal{C}(X)$ with $D(T) \subset D(S_0)$. Suppose that
(i) $T(D(T)) \subseteq D(T)$ and $S_0(D(T)) \subseteq D(T)$.
(ii) $S_0 - T$ is weakly closed.
(iii) There exists a complex polynomial P such that $P(1) = 1$ and $P(I - S_0 + T + K)$ is a DP operator for all $K \in \mathcal{K}(X)$.
Then,

$$T \text{ is weakly } S_0\text{-demicompact, if and only if, } S_0 - T \in \Phi_+(X).$$

Proof. First, note that, since $S_0 - T$ is weakly closed then $S_0 - T$ is closed. By applying Theorem 1 in [30], it suffices to prove that, for any compact operator $K \in \mathcal{K}(X)$, $\alpha(S_0 - T - K) < \infty$. To do so, it suffices to establish that, for every $K \in \mathcal{K}(X)$, the set $B_X \cap \mathcal{N}(S_0 - T - K)$ is compact. Let $K \in \mathcal{K}(X)$ and take a sequence $(x_n)_n \subseteq B_X \cap \mathcal{N}(S_0 - T - K)$. Then, for every $n \in \mathbb{N}$,

$$(S_0 - T - K)x_n = 0. \tag{1}$$

Since K is compact, there exists a subsequence of $(x_n)_n$, still denoted $(x_n)_n$, such that

$$Kx_n \longrightarrow x \in X. \tag{2}$$

Hence,

$$(S_0 - T)x_n \longrightarrow x \in X.$$

Then,

$$(S_0 - T)x_n \rightharpoonup x \in X.$$

Taking into account the fact that T is S_0-weakly demicompact, we deduce that $(x_n)_n$ has a subsequence $(x_{\varphi(n)})_n$ such that

$$x_{\varphi(n)} \rightharpoonup a \in X. \tag{3}$$

Since $S_0 - T$ is weakly closed, we get

$$a \in D(T) \text{ and } (S_0 - T)a = x. \tag{4}$$

Furthermore, we have

$$\|a\| \leq \liminf \|x_{\varphi(n)}\| = 1. \tag{5}$$

Keeping in mind equations (2) and (3) and using the fact that K is compact we get

$$Ka = x. \tag{6}$$

Thus, by using equations (4), (5) and (6) we deduce that

$$a \in B_X \cap \mathcal{N}(S_0 - T - K).$$

By using equation (1), we obtain for every $n \in \mathbb{N}$,

$$P(I - S_0 + T + K)x_n = x_n.$$

Indeed, let $P = \sum_{k=0}^{N} a_k X^k$, thus

$$\begin{aligned} P(I - S_0 + T + K)x_n &= \sum_{k=0}^{N} a_k (I - S_0 + T + K)^k x_n \\ &= \sum_{k=0}^{N} a_k x_n \\ &= P(1)x_n \\ &= x_n. \end{aligned}$$

Similarly, since

$$(S_0 - T - K)a = 0$$

we deduce that

$$P(I - S_0 + T + K)a = a.$$

Combinig equation (3) and the fact that $P(I - S_0 + T + K)$ is DP, we infer that

$$x_{\varphi(n)} \longrightarrow a.$$

Accordingly, $B_X \cap \mathcal{N}(S_0 - T - K)$ is a compact set. This achieves the proof. Now, we suppose that $S_0 - T \in \Phi_+(X)$. There are two cases.

First case: If $ind(S_0 - T) > 0$, then $S_0 - T \in \Phi(X)$. By using Theorem 7.2 in [28], there exists $A \in \mathcal{L}(X)$ and $K \in \mathcal{K}(X)$ such that

$$A(S_0 - T) = I + K.$$

Let $(x_n)_n$ be a bounded sequence of $D(T)$ such that:

$$(S_0 - T)x_n \rightharpoonup x \in X.$$

Then,

$$A(S_0 - T)x_n \rightharpoonup Ax.$$

Hence, $(x_n + Kx_n)_n$ converges weakly to Ax. Since K is compact, then $(Kx_n)_n$ has a convergent, and then a weakly convergent, subsequence. It follows that $(x_n)_n$ admits a weakly convergent subsequence.

Second case: If $ind(S_0 - T) \leq 0$, then, in view of Lemma 3.9, there exists a bounded below operator A_0 and $K \in \mathcal{K}(X)$ such that

$$S_0 - T = A_0 + K.$$

Let $(x_n)_n$ be a bounded sequence in $D(T)$ such that:

$$(S_0 - T)x_n \rightharpoonup x \in X.$$

Then, $((A_0 + K)x_n)_n$ converges weakly on X. Since K is compact, then $(Kx_n)_n$ has a convergent subsequence $(Kx_{\varphi(n)})_n$. Consequently, $(A_0 x_{\varphi(n)})_n$ is a weakly convergent sequence. Since A_0 is bounded below, then there exist a positive constant C such that

$$\|x\| \leq C\|A_0 x\|,$$

for all $x \in D(T)$.
Let $D = \{x_n; n \in \mathbb{N}\}$, then by using Proposition 2.7 we get

$$w(D) \leq Cw(A_0(D)).$$

Thus, $w(D) = 0$ and therefore, $(x_{\varphi(n)})_n$ has a weakly convergent subsequence. This achieves the proof. □
If we consider bounded operators, we have the following result.

Proposition 3.11. Let X be a Banach space and $T \in \mathcal{L}(X)$. Assume that there exists an entire function $f : \mathbb{C} \longrightarrow \mathbb{C}$ such that $f(T)$ is a DP operator and $f(1) = 1$. Then,

$$T \text{ is weakly-demicompact, if and only if, } I - T \in \Phi_+(X).$$

Proof. If $I - T \in \Phi_+(X)$, then we prove that T is weakly demicompact identically to the proof of Theorem 3.10 with $S_0 = I$.
Now, suppose that T is weakly S_0-demicompact. According to Theorem 1 in [30], it suffices to prove that $\alpha(S_0 - T - K) < \infty$ for any compact operator $K \in \mathcal{K}(X)$. For that purpose, let $K \in \mathcal{K}(X)$ and take a sequence $(x_n)_n \subseteq B_X \cap \mathcal{N}(I - T - K)$. Then, for every $n \in \mathbb{N}$, we have

$$(I - T - K)x_n = 0.$$

Since K is compact, there exists a subsequence of $(x_n)_n$, still denoted $(x_n)_n$, such that

$$Kx_n \longrightarrow x \in X.$$

Hence,

$$(I - T)x_n \longrightarrow x \in X.$$

Therefore,

$$(I-T)x_n \rightharpoonup x \in X.$$

Taking into account the fact that T is weakly demicompact, we deduce that $(x_n)_n$ has a subsequence $(x_{\varphi(n)})_n$ such that

$$x_{\varphi(n)} \rightharpoonup a \in X.$$

It comes that

$$(I-T)a = x.$$

Furthermore, we have $\|a\| \leq \liminf \|x_{\varphi(n)}\| = 1$. Using the fact that K is compact, we get $Ka = x$. We deduce that

$$a \in B_X \cap \mathcal{N}(I-T-K).$$

Since $(T+K)x_n = x_n$ for every $n \in \mathbb{N}$ and $f(1) = 1$, we readily get

$$f(T+K)a = a \qquad \text{and} \qquad f(T+K)x_n = x_n, \quad \forall n \in \mathbb{N}. \tag{7}$$

Now, let $f(z) = \sum\limits_{n=0}^{+\infty} a_n z^n$, $z \in \mathbb{C}$. Then,

$$f(T+K) - f(T) = \sum_{n=1}^{+\infty} a_n \Big[(T+K)^n - T^n\Big]. \tag{8}$$

Since $\mathcal{K}(X)$ is an ideal, then $(T+K)^n - T^n \in \mathcal{K}(X)$ for all $n \in \mathbb{N}\backslash\{0\}$. Taking into account that $\mathcal{K}(X)$ is closed, we deduce from equation 8 that $f(T+K) - f(T) \in \mathcal{K}(X)$. This implies that $f(T+K)$ is a DP operator. Recall that $x_{\varphi(n)} \rightharpoonup a$, then

$$f(T+K)x_n \longrightarrow f(T+K)a.$$

By using equation 7, we infer that

$$x_n \longrightarrow a.$$

We conclude that $B_X \cap \mathcal{N}(S_0 - T - K)$ is a compact set. This achieves the proof. □

Now, we give a characterization of relative weakly demicompact operators by means of the De Blasi measure of weak noncompactness.

Theorem 3.12. Let X be a Banach space, T and S_0 be two operators acting on X such that $D(T) \subset D(S_0)$. Assume that:
(*i*) $T(D(T)) \subseteq D(T)$ and $S_0(D(T)) \subseteq D(T)$.
(*ii*) $S_0 - T$ is weakly closed.
(*iii*) There exists a complex polynomial P satisfying $P(1) = 1$ such that $P(I - S_0 + T + K)$ is DP for all $K \in \mathcal{K}(X)$.
Then, T is weakly S_0-demicompact, if and only if, there exists a positive constant C such that for all bounded sets $D \subseteq D(T)$,

$$w(D) \leq C w((S_0 - T)(D)).$$

Proof. (*i*) $\Longrightarrow$ (*ii*) Suppose first that T is weakly S_0-demicompact. Then, by using Theorem 3.10, $S_0 - T \in \Phi_+(X)$. Now, if $ind(S_0 - T) > 0$, then by using Theorem 7.2 in [28], there exists a bounded operator A and a compact operator K such that:

$$A(S_0 - T) = I + K.$$

Let D be a bounded set of X. Then,

$$\begin{aligned} w(D) &\leq w((A(S_0-T)(D)) \\ &\leq \|A\| w((S_0-T)(D)). \end{aligned}$$

In the case where $ind(S_0 - T) \leq 0$, then, by using Lemma 3.9, there exists a compact operator K and a bounded below operator A_0 such that:

$$S_0 - T = K + A_0.$$

Since A_0 is bounded below, there exists a positive constant C such that:

$$\|x\| \leq C\|A_0 x\|,$$

for all $x \in D(T)$.
Hence, by applying Proposition 2.7, we get

$$w(D) \leq Cw((S_0 - T)(D)),$$

for any bounded set $D \subset D(T)$. Now, choose $C' = \max(\|A\|, C)$, then for any bounded subset D of $D(T)$ we have

$$w(D) \leq C'(w(S_0 - T)(D)).$$

$(ii) \Longrightarrow (i)$ Suppose that there exists a positive constant C such that for every bounded set D of X,

$$w(D) \leq Cw((S_0 - T)(D)).$$

Let $(x_n)_n$ be a bounded sequence of $D(T)$ such that

$$(S_0 - T)x_n \rightharpoonup x \in X.$$

Choose $D = \{x_n;\ n \in \mathbb{N}\}$. It is clear that D is a bounded set of $D(T)$ such that $w((S_0 - T)(D)) = 0$. Hence, $w(D) = 0$ so that $(x_n)_n$ has a weakly convergent subsequence. We conclude that T is weakly S_0-demicompact. □

Corollary 3.13. Let X be a Banach space and $T \in \mathcal{L}(X)$. Assume that there exists an entire function $f : \mathbb{C} \longrightarrow \mathbb{C}$ such that $f(T)$ is a DP operator and $f(1) = 1$.
Then, T is weakly demicompact, if and only if, there exists a positive constant C such that for all bounded sets $D \subseteq X$,

$$w(D) \leq Cw((I - T)(D)).$$

Proof. Suppose that T is weakly demicompact. By Proposition 3.11, $I - T \in \Phi_+^b(X)$. By considering both cases $ind(I - T) > 0$ and $ind(I - T) \leq 0$ and using a similar proof to the one of Theorem 3.12 with $S_0 = I$, we show the requested inequality. Now, let $C > 0$ such that $w(D) \leq Cw((I - T)(D))$ for all $D \in \mathcal{M}_X$. A similar proof to the one of Theorem 3.12 with $S_0 = I$ proves that T is weakly demicompact. □

Now, we derive a characterization of upper semi-Fredholm operators by means of De Blasi measure of weak noncompactness.

Proposition 3.14. Let X be a Banach space and $T \in \mathcal{L}(X)$. Assume that there exists an entire function $f : \mathbb{C} \longrightarrow \mathbb{C}$ such that $f(T)$ is a DP operator and $f(1) = 1$. Then,

$$T \in \Phi_+^b(X) \Longleftrightarrow \forall D \in \mathcal{M}_X, \quad w(T(D)) = 0 \Longrightarrow w(D) = 0.$$

.

Proof. Let $S_0 = I$ and suppose that $T \in \Phi_+^b(X)$. Then, Proposition 3.11 shows that $I - T$ is weakly demicompact. Corollary 3.13 ensures the existence of a positive constant C such that for all bounded sets $D \subseteq D(T)$, we have $w(D) \leq Cw(T(D))$. Hence, if $w(T(D)) = 0$ then $w(D) = 0$ for all $D \in \mathcal{M}_X$. Conversely, assume that $w(T(D)) = 0 \Longrightarrow w(D) = 0$ whenever $D \in \mathcal{M}_X$. According to Proposition 3.11, to prove that $T \in \Phi_+(X)$, it suffices to prove that $I - T$ is weakly demicompact. For this purpose, let $(x_n)_n \subseteq X$ be a bounded sequence such that $Tx_n \rightharpoonup x$, for some $x \in X$. Put $D = \{x_n, n \in \mathbb{N}\}$. Hence, $w(T(D)) = 0$ and so $w(D) = 0$. Accordingly, $I - T$ is weakly demicompact. This ends the proof. □

Now, we give a characterization of weakly demicompact bounded projections. First, let us recall the following lemma.

Lemma 3.15. *[20, 28]* Let X be a Banach space and $K \in \mathcal{K}(X)$. Then, $I \pm K \in \Phi^b(X)$ and $ind(I \pm K) = 0$.

Proposition 3.16. *Let X be a Banach space and P be a bounded projection on X. Assume that P is a DP operator. Then, the following assertions are equivalents*

(i) $P \in \mathcal{WDC}^b(X)$.
(ii) $P \in \mathcal{K}(X)$.
(iii) $I \pm P \in \Phi^b(X)$ and $ind(I \pm P) = 0$.
(iv) $\pm P$ is demicompact.

Proof. $(i) \Rightarrow (ii)$ Let P be a bounded DP projection on X. If P is weakly demicompact then, By using Theorem 3.10, we deduce that $I - P \in \Phi_+^b(X)$. Consequently, $\mathcal{R}(P) = \mathcal{N}(I - P)$ is finite dimensional. This proves that P is a finite rank operator which implies that $P \in \mathcal{K}(X)$.
$(i) \Rightarrow (ii)$ Suppose that $P \in \mathcal{K}(X)$, then $\pm P \in \mathcal{K}(X)$. By using Lemma 3.15, we get the desired result.
$(iii) \Rightarrow (iv)$ Since $I \pm P$ is a bounded Fredholm operator the, according to Theorem 3.1 in [14], $\pm P$ is demicompact.
$(iv) \Rightarrow (iii)$ According to Theorem 2.1 in [14], we have $I \pm P \in \Phi_+^b(X)$. By Proposition 1.2, we get

$$\begin{aligned} ind(I-P) &= ind(I-P^2) \\ &= ind(I-P) + ind(I+P). \end{aligned}$$

Consequently, $ind(I + P) = 0$. Therefore, $I + P \in \Phi^b(X)$. Now, using the fact that $(I - P)^2 = I - P$, we get $ind(I - P) = 0$ and thus $I - P \in \Phi^b(X)$.
$(iii) \Rightarrow (i)$ Since $I-P \in \Phi^b(X)$, then $\mathcal{R}(I-P) = \mathcal{N}(I-P)$ is finite dimensional. Hence, P is a finite rank operator. this proves that P is a DP operator. In view of Theorem 3.10, we deduce that P is weakly demicompact. □

Proposition 3.17. Let X be a Banach space, $\mathbb{D}$ be the unit disk of the complex plane and $T \in \mathcal{L}(X)$. Assume that $\Theta_w(T^m) < 1$ for some positive integer m. Then, the following assertions holds,

1. $\lambda T \in \mathcal{WDC}^b(X)$ for all $\lambda \in \mathbb{D}$.
2. If, moreover, T^k is a DP operator for some $k \in \mathbb{N}\backslash\{0\}$, then
   (a) $I \pm T \in \Phi^b(X)$ and $ind(I \pm T) = 0$.
   (b) For all $p \in \mathbb{N}\backslash\{0\}$, $I + T^p \in \Phi^b(X)$ and $ind(I + T^p) = 0$.

Proof. For the proof of assertion (1), let $\lambda \in \mathbb{D}$ and $(x_n)_n$ be a bounded sequence such that

$$x_n - \lambda T x_n \rightharpoonup x \in X.$$

Obviously, there exists a bounded operator $S \in \mathcal{L}(X)$ such that

$$I - \lambda^m T^m = S(I - \lambda T).$$

Hence,

$$x_n - \lambda^m T^m x_n \rightharpoonup S(x).$$

Using the properties of the De Blasi measure of weak noncompactness, we get

$$\begin{aligned} w\{x_n\} &\leq w\{\lambda^m T^m x_n\} + w\{x_n - \lambda^m T^m x_n\} \\ &\leq |\lambda|^m \Theta_w(T^m) w\{x_n\} \\ &\leq \Theta_w(T^m) w\{x_n\}. \end{aligned}$$

Thus, $w\{x_n\} = 0$. This shows that λT is weakly demicompact.
Now we prove the second assertion. Let us choose arbitrarily $\lambda \in \mathbb{D}$. Obviously, we have $\Theta_w((\lambda T)^m)) < 1$.

Since $(\lambda T)^k$ is DP, then by applying Theorem 3.10, we deduce that $I - \lambda T \in \Phi_+^b(X)$. Taking into account Proposition 1.2, we obtain

$$\begin{array}{rcl} ind(I-T) & = & ind(I-\lambda T) \\ & = & ind(I+T) \\ & = & ind(I) \\ & = & 0. \end{array}$$

Hence , $I \pm T \in \Phi^b(X)$. This proves 2-a.
To prove 2-b, let $p \in \mathbb{N}\backslash\{0\}$. We have

$$I + T^p = \prod_{k=0}^{p-1} \Big(I - \lambda_k T \Big)$$

where

$$\lambda_k = \exp\Big(- i \frac{2k+1}{p} \pi \Big), \quad \forall k \in [\![0, p-1]\!].$$

Combining the result in the previous assertion and Theorem 3.10, we see that $I - \lambda_k T \in \Phi_+^b(X)$. By using Proposition 1.2, we get

$$ind(I - \lambda_k T) = ind(I - \lambda T)$$

for all $\lambda \in \mathbb{D}$ and $k \in [\![0, p-1]\!]$. In particular,

$$ind(I - \lambda_k T) = ind(I) = 0$$

for all $k \in [\![0, p-1]\!]$. Applying Proposition 1.2 again, we get

$$\begin{array}{rcl} ind(I+T^p) & = & \displaystyle\sum_{k=0}^{p-1} ind(I - \lambda_k T) \\ & = & 0. \end{array}$$

This implies that $I + T^p \in \Phi^b(X)$ and completes the proof. □

Now, we give another characterization of weakly demicompact operators by means of weak essential norm.

Definition 3.18. *[29] Let X and Y be two Banach spaces. For $S \in \mathcal{L}(X,Y)$, we call*

$$\|S\|_w = \inf\Big\{ \|S - W\| : \quad W \in \mathcal{W}(X,Y) \Big\},$$

the weak essential norm of S.

Remark 3.19. *Let X and Y be two Banach spaces and $S \in \mathcal{L}(X,Y)$. Then, $\|S\|_w$ is also called the quotient norm of the operator S in the Banach space $\mathcal{L}(X,Y)/\mathcal{W}(X,Y)$.*

Let us recall some properties of the weak essential norm through the following proposition.

Proposition 3.20. *[29]* Let X and Y be two Banach spaces and $S \in \mathcal{L}(X,Y)$. Then,
(*i*) $\|S\|_w = d\Big(S, \mathcal{W}(X,Y)\Big)$, where $d\Big(S, \mathcal{W}(X,Y)\Big)$ is the distance between S and $\mathcal{W}(X,Y)$).
(*ii*) $\|S\|_w = 0$, if and only if, $S \in \mathcal{W}(X,Y)$.
(*iii*) $\| \ \|_w$ is a semi-norm on $\mathcal{L}(X,Y)$.
(*iv*) $\|S^*\|_w \leq \|S\|_w$, where S^* is the adjoint of S.

Proposition 3.21. *Let T be a bounded operator acting on a Banach space X. Suppose that $\|T^n\|_w < 1$ for some integer $n \geq 1$. Then, T is weakly demicompact.*

Proof. Suppose $\|T^n\|_w < 1$ or some integer $n \geq 1$. Hence, there exists $W \in \mathcal{W}(X)$ such that $\|T^n - W\| < 1$. It follows that $T \in \mathcal{WQP}(X)$. By using Proposition 3.4, we see that $T \in \mathcal{WDC}^b(X)$. □

Theorem 3.22. Let X be a Banach space and $T \in \mathcal{L}(X)$. Suppose that $\|T\|_w < 1$. Then, we have the following assertions

(i) $T \in \mathcal{WDC}^b(X)$.

(ii) $T^* \in \mathcal{WDC}^b(X^*)$.

(iii) If T is DP, then for every $\varepsilon \in \{-1, 1\}$, $I - \varepsilon T \in \Phi^b(X)$ and $ind(I - \varepsilon T) = 0$.

Proof. For the proof of (i), take a bounded sequence $(x_n)_n$ of X such that

$$y_n := x_n - Tx_n \rightharpoonup x \in X.$$

Since $\|T\|_w < 1$, there exists $W_0 \in \mathcal{W}(X)$ such that

$$\begin{aligned} \|T\|_w &\leq \|T - W_0\| \\ &= 1. \end{aligned}$$

Obviously,

$$\{x_n\} \subseteq \{y_n\} + \{W_0 x_n\} + \{(T - W_0)x_n\}.$$

Hence,

$$\begin{aligned} w(\{x_n\}) &\leq w(\{y_n\}) + w(\{W_0 x_n\}) + w(\{(T - W_0)x_n\}) \\ &\leq w(\{(T - W_0)x_n\}) \\ &\leq \|T - W_0\| w(\{x_n\}). \end{aligned}$$

Thus,

$$\Big(1 - \|T - W_0\|\Big) w(\{x_n\}) \leq 0.$$

Then, $w(\{x_n\}) = 0$. This shows that (x_n) has a weakly convergent subsequence and consequently $T \in \mathcal{WDC}^b(X)$. In view of Proposition 3.20, we infer that $\|T^*\|_w < 1$. Now, applying the result of the assertion (i) we complete the proof of (ii).

To prove (iii), let λ be an arbitrarily scalar in $[-1, 1]$. Since $\|T^*\|_w < 1$, then $\|\lambda T^*\|_w < 1$. Notice that λT is also a DP operator. Then, by using Theorem 3.10, we get

$$I - \lambda T \in \Phi^b_+(X), \quad \forall \lambda \in [-1, 1].$$

On the other hand, from Proposition 1.2, the index is constant on any component of $\Phi^b_+(X)$. It follows that, for $\lambda \in [-1, 1]$,

$$\begin{aligned} ind(I - T) &= ind(I - \lambda T) \\ &= ind(I + T) \\ &= ind(I) \\ &= 0. \end{aligned}$$

Thus, $I - \varepsilon T \in \Phi^b(X)$ for $\varepsilon \in \{-1, 1\}$. □

Now, we introduce a natural quantity measuring the deviation of an operator from weak compactness. Then, we will establish some results related to weakly demicompact operators.

Definition 3.23. *[11] Let X, Y be two Banach spaces and $T \in \mathcal{L}(X, Y)$. Set*

$$\beta_w(T) = \inf\Big\{\varepsilon > 0 \Big| \quad \exists Z, \exists U \in \mathcal{W}(X, Z) : \|Tx\| \leq \|Ux\| + \varepsilon\|x\| \, \textit{for all } x \in X\Big\},$$

where Z is a Banach space.

Some basic properties of the quantity β_w will be provided in the following proposition.

Proposition 3.24. *[3, 29]* Let X, Y be two Banach spaces and $T \in \mathcal{L}(X, Y)$. Then,
(i) β_w is a seminorm on $\in \mathcal{L}(X, Y)$.
(ii) $\beta_w(T) = 0$ if, and only if, $T \in \mathcal{W}(X, Y)$.
(iii) $\beta_w(ST) \leq \beta_w(S)\beta_w(T)$ for all bounded operator S such that ST is defined.
(iv) $\beta_w(T) \leq \|T\|_w$.

Now, we state the following result.

Theorem 3.25. Let X be a Banach space and $T \in \mathcal{L}(X)$. Assume that $\beta_w(T) < 1$. Then, for every $k \in \mathbb{N}\backslash\{0\}$ and for every $\epsilon \in \{-1, 1\}$, we have
(i) ϵT^k is weakly demicompact.
(ii) If T is also a DP operator, then $I - T^k \in \Phi^b(X)$ and $ind(I - T^k) = 0$.

Proof. First, let us prove (i) in the case $k = 1$. Take a bounded sequence $(x_n)_n$ on X such that

$$x_n - Tx_n \rightharpoonup x \in X.$$

Since the operator T satisfies the condition $\beta_w(T) < 1$, then there exists $\varepsilon \in]0, 1[$, a Banach space Z and a weakly compact operator $U \in \mathcal{W}(X, Z)$ such that

$$\|Tx\| \leq \|Ux\| + \varepsilon\|x\|,$$

for all $x \in X$. Now, we have

$$\begin{aligned} \|x\| &\leq \|Tx\| + \|x - Tx\| \\ &\leq \|Ux\| + \|x - Tx\| + \varepsilon\|x\|, \end{aligned}$$

for all $x \in X$. Hence, we obtain for all $x \in X$,

$$\|x\| \leq \frac{1}{1-\varepsilon}\Big(\|Ux\| + \|x - Tx\|\Big).$$

This yields

$$\begin{aligned} w(D) &\leq \frac{1}{1-\varepsilon}\Big(w\Big(U(D)\Big)\| + w\Big((I-T)(D)\Big)\Big) \\ &\leq \frac{1}{1-\varepsilon} w\Big((I-T)(D)\Big), \end{aligned}$$

for all $D \in \mathcal{M}_X$. Now, choose $D = \{x_n : n \in \mathbb{N}\}$. Then, we get $w(D) = 0$. Hence, $(x_n)_n$ has a weakly convergent subsequence. Consequently, T is a weakly demicompact operator. Taking into account Proposition 3.24 (iii), we prove by a simple induction that for every or every $k \in \mathbb{N}\backslash\{0\}$ and for every $\epsilon \in \{-1, 1\}$,

$$\begin{aligned} \beta_w(\epsilon T^k) &= \beta_w(T^k) \\ &\leq \Big(\beta_w(T)\Big)^k \\ &< 1. \end{aligned}$$

Hence, ϵT^k is weakly demicompact.
To prove (ii), let k be a positive integer and take arbitrarily $\lambda \in [-1, 1]$. Then,

$$\begin{aligned} \beta_w\Big(\lambda T)^k\Big) &= |\lambda|^k \beta_w(T^k) \\ &\leq \Big(\beta_w(T)\Big)^k \\ &< 1. \end{aligned}$$

It follows that $(\lambda T)^k$ is weakly demicompact for all $\lambda \in [-1,1]$. Since $(\lambda T)^k$ is also a DP operator for all $\lambda \in [-1,1]$, then by using Theorem 3.10, we deduce that $I-(\lambda T)^k \in \Phi_+^b(X)$. Taking into account the fact that $[-1,1]$ is connected, we conclude by using Proposition 1.2 that

$$\begin{aligned} ind(I-T^k) &= ind\Big(I-(\lambda T)^k\Big) \\ &= ind(I) \\ &= 0. \end{aligned}$$

Hence, $I-T^k \in \Phi^b(X)$. □

Now, we give some consequences of Theorem 3.25.

Corollary 3.26. *Let X be a Banach space and $T \in \mathcal{L}(X)$. Assume that $\lim\limits_{n\to+\infty}\Big(\beta(T^n)\Big)^{\frac{1}{n}} = 0$. Then,*

(i) T is weakly demicompact.

(ii) If T is also a DP operator, then for every $\epsilon \in \{-1,1\}$, $I-\epsilon T \in \Phi^b(X)$ and we have $ind(I-\epsilon T) = 0$.

Proof. Since $\lim\limits_{n\to+\infty}\Big(\beta(T^n)\Big)^{\frac{1}{n}} = 0$, there exists a positive integer n_0 such that $\Big(\beta(T^{n_0})\Big)^{\frac{1}{n_0}} < 1$. Thus, $\beta(T^{n_0}) < 1$. By applying Theorem 3.25, we deduce that T^{n_0} is weakly demicompact and then T is weakly demicompact. For the proof of (ii), take $\lambda \in [-1,1]$. then, λT is a DP operator. Moreover,

$$\begin{aligned} \lim_{n\to+\infty}\Big[\beta\Big((\lambda T)^n\Big)\Big]^{\frac{1}{n}} &= |\lambda| \lim_{n\to+\infty}\Big(\beta(T^n)\Big)^{\frac{1}{n}} \\ &= 0. \end{aligned}$$

Hence, by using assertion (i), we deduce that λT is weakly demicompact for all $\lambda \in [-1,1]$. Now, by applying Theorem 3.10, we infer that $I-\lambda T \in \Phi_+^b(X)$. The rest of the proof is similar to that of Theorem 3.25 (ii). □

Corollary 3.27. *Let X be a Banach space and $(S,T) \in \mathcal{L}(X)\times\mathcal{L}(X)$. Assume that $\beta_w(S) < 1$ and $\beta_w(T) < 1$. The, the set*

$$\mathcal{H}(S,T) = \Big\{S^{k_1}T^{k_2}...S^{k_n} | \quad n \in \mathbb{N}\backslash\{0\}, (k_1,...,k_n) \in \mathbb{N}^n\backslash\{(0,...,0\}\Big\}$$

has the following properties

(i) $\mathcal{H}(S,T) \subseteq \mathcal{WDC}^b(X)$.

(ii) If S and T are DP, then every $H \in \mathcal{H}(S,T)$ satisfy $ind(I-H) = 0$.

Proof. First, it is easy to show, by a simple induction, that $\beta_w(T^k) \leq \Big(\beta(T)\Big)^k$ for every $k \in \mathbb{N}\backslash\{0\}$. Now, let us take an element $H \in \mathcal{H}(S,T)$. Then, there exists $n \in \mathbb{N}\backslash\{0\}$ and $(k_1,...,k_n) \in \mathbb{N}^n\backslash\{(0,...,0\}$ such that

$$H = S^{k_1}T^{k_2}...S^{k_n}.$$

Hence,

$$\begin{aligned} \beta_w(H) &= \beta_w\Big(S^{k_1}T^{k_2}...S^{k_n}\Big) \\ &\leq \beta_w(S^{k_1})\beta_w(T^{k_2})...\beta_w(S^n) \\ &\leq \Big(\beta_w(S)\Big)^{k_1}\Big(\beta_w(T)\Big)^{k_2}...\Big(\beta_w(S)\Big)^{k_n} \\ &< 1. \end{aligned}$$

By using Theorem 3.25, we infer that $H \in \mathcal{WDC}^b(X)$. For the proof of (ii), we suppose that S and T are DP operators. Then, it is clear that every element of $\mathcal{H}(S,T)$ is a DP operator. In view of Theorem 3.25, we see that every element of $\mathcal{H}(S,T)$ has a null index. □

Corollary 3.28. *Let X be a complex Banach space, and $S \in \mathcal{L}(X)$ be a DP operator, $P(X) = \sum_{k=1}^{n} a_k X^k$ be a complex polynomial satisfying $P(0) = 0$ and let $|P(X)| = \sum_{k=1}^{n} |a_k| X^k$. Let $Q(X) = \frac{P(X)}{|P|(1)}$. Assume that $\beta_w(S) < 1$, then*

(i) $Q(S)$ is weakly demicompact.
(ii) $I - Q(S) \in \Phi^b(X)$ and $ind\Big(I - Q(S)\Big) = 0$.

Proof. First, it is obvious that $Q(S)$ is a DP operator. In view of Proposition 3.24, we can show by a simple induction that $\beta_w(S^k) \leq \Big(\beta(S)\Big)^k$ for every $k \in \mathbb{N}\backslash\{0\}$. Hence,

$$\begin{aligned} \beta_w\Big(Q(S)\Big) &= \frac{1}{|P|(1)}\beta_w\Big(P(S)\Big) \\ &\leq \frac{1}{\sum_{k=1}^{n}|a_k|}\sum_{k=1}^{n}|a_k|\beta_w(S^k) \\ &\leq \frac{1}{\sum_{k=1}^{n}|a_k|}\sum_{k=1}^{n}|a_k|\Big(\beta_w(S)\Big)^k \\ &< \frac{1}{\sum_{k=1}^{n}|a_k|}\sum_{k=1}^{n}|a_k| \\ &= 1. \end{aligned}$$

Finally, invoking Theorem 3.25, we achieve the proof. □

4. characterization and invariance of the essential spectrum

In this section, we give some characterizations of the Schechter essential spectrum of an operator acting on a Banach space. In what follows, we consider a Banach space X and an operator $T \in \mathcal{C}(X)$. Let J be a linear operator. If $D(T) \subset D(J) \subset$, then J will be called T-defined. The restriction of J to $D(T)$ will be denoted by $\widehat{J}$. Moreover, if $J \in \mathcal{L}(X_T, X)$, we say that J is T-bounded (see [15] for more details). Furthermore, we have the obvious relations

$$\begin{cases} \alpha(\widehat{T}) = \alpha(T), \beta(\widehat{T}) = \beta(T), \mathcal{R}(\widehat{T}) = \mathcal{R}(T), \\ \alpha(\widehat{T} + \widehat{J}) = \alpha(T + J), \beta(\widehat{T} + \widehat{J}) = \beta(T + J), \\ \mathcal{R}(\widehat{T} + \widehat{J}) = \mathcal{R}(T + J). \end{cases} \quad (3.9)$$

Hence, $T \in \Phi(X)$ (resp. $\Phi_+(X)$), if and only if, $\widehat{T} \in \Phi(X_T, X)$ (resp. $\Phi_+(X_T, X)$).

Note that, if $T \in \mathcal{C}(X)$, S is a nonzero bounded operator defined on X, K is a T-bounded operator and $\lambda \in \rho_S(T + K)$, then by using Lemma 2.1 in [26], $K(\lambda S - T - K)^{-1}$ is a closed operator defined on X and therefore is bounded by the closed graph theorem.

We consider the following sets

$\mathcal{A}^w_{S,T}(X) := \{K \in \mathcal{L}(X_T, X) : \forall \lambda \in \rho_S(T + K),\ \Upsilon_r(\lambda)$ is a DP operator and $\|\Upsilon_r(\lambda)\|_w < 1\}$,

$\mathcal{Z}^{w}_{S,T}(X) := \{K \in \mathcal{L}(X_T, X) : \forall \lambda \in \rho_S(T + K), \Upsilon_r(\lambda) \in \Gamma^{w}(X)\}$,

$\mathcal{Q}^{w}_{S,T}(X) := \{K \in \mathcal{L}(X_T, X) : \forall \lambda \in \rho_S(T + K), \Upsilon_l(\lambda) \in \Gamma^{w}(X)\}$,

where

$$\Upsilon_r(\lambda) = K(\lambda S - T - K)^{-1}, \;\; \Upsilon_l(\lambda) = (\lambda S - T - K)^{-1}K,$$

and

$$\Gamma^{w}(X) := \{T \in \mathcal{L}(X) : T \text{ is a DP operator and } \beta_w(T) < 1\}.$$

Now, we are in position to state the following result.

Theorem 4.1. Let X be a Banach space and S be a nonzero bounded operator on X. For each $T \in \mathcal{C}(X)$, we have

(*i*) $\sigma_{e,S}(T) = \bigcap_{K \in \mathcal{A}^{w}_{S,T}(X)} \sigma_S(T + K)$.

(*ii*) $\sigma_{e,S}(T) = \bigcap_{K \in \mathcal{Z}^{w}_{S,T}(X)} \sigma_S(T + K)$.

(*iii*) $\sigma_{e,S}(T) = \bigcap_{K \in \mathcal{Q}^{w}_{S,T}(X)} \sigma_S(T + K)$.

Proof. For the proof of (*i*), it is obvious that $\mathcal{K}(X) \subseteq \mathcal{A}^{w}_{S,T}(X)$. Hence,

$$\bigcap_{K \in \mathcal{A}^{w}_{S,T}(X)} \sigma_S(T + K) \subseteq \sigma_{e,S}(T).$$

Now, take λ such that $\lambda \notin \bigcap_{K \in \mathcal{A}^{w}_{S,T}(X)} \sigma_S(T + K)$, then there exists $K \in \mathcal{A}^{w}_{S,T}(X)$ such that $\lambda \in \rho_S(T + K)$. Thus, $\lambda S - T - K$ has a bounded inverse, $\Upsilon_r(\lambda)$ is a DP operator and $\|\Upsilon_r(\lambda)\|_w < 1$. By using Theorem 3.22, we get

$$I + \Upsilon_r(\lambda) \in \Phi^{b}(X) \text{ and } ind[I + \Upsilon_r(\lambda)] = 0.$$

Furthermore, we have

$$\lambda S - T = [I + \Upsilon_r(\lambda)][\lambda S - T - K].$$

Then, applying Atkinson's theorem (see [20], Theorem 12, p.159), we obtain

$$\lambda S - T \in \Phi^{b}(X) \text{ and } ind(\lambda S - T) = 0.$$

Taking into account Theorem 7.27 in [28], we infer that $\lambda \notin \sigma_{e,S}(T)$. Hence, we deduce that $\sigma_{e,S}(T) \subseteq \bigcap_{K \in \mathcal{A}^{w}_{S,T}(X)} \sigma_S(T + K)$. This achieves the proof of (*i*).

Now, we prove the assertion (*ii*). Obviously, we have $\mathcal{K}(X) \subseteq \mathcal{Z}^{w}_{S,T}(X)$. Then,

$$\bigcap_{K \in \mathcal{Z}^{w}_{S,T}(X)} \sigma_S(T + K) \subseteq \sigma_{e,S}(T).$$

Now, take λ such that $\lambda \notin \bigcap_{K \in \mathcal{Z}^{w}_{S,T}(X)} \sigma_S(T+K)$. Then, there exists $K \in \mathcal{Z}^{w}_{S,T}(X)$ such that $\lambda \in \rho_S(T+K)$. It follows that $\lambda S - T - K$ has a bounded inverse, $\Upsilon_r(\lambda)$ is a DP operator and $\beta_w(\Upsilon_r(\lambda)) < 1$. Hence, $\Upsilon_r(\lambda) \in \mathcal{L}(X)$. In view of Theorem 3.25, we infer that

$$I + \Upsilon_r(\lambda) \in \Phi^{b}(X) \text{ and } ind[I + \Upsilon_r(\lambda)] = 0.$$

Moreover, we have

$$\lambda S - T = [I + \Upsilon_r(\lambda)][\lambda S - T - K].$$

We deduce, as in the proof of (i), that $\lambda \notin \sigma_{e,S}(T)$. Accordingly,

$$\sigma_{e,S}(T) \subseteq \bigcap_{K \in \mathcal{Z}^{w}_{S,T}(X)} \sigma_S(T+K).$$

Consequently,

$$\bigcap_{K \in \mathcal{Z}^{w}_{S,T}(X)} \sigma_S(T+K) = \sigma_{e,S}(T).$$

The proof of the assertion (iii) is similar to (ii). □

Corollary 4.2. Let X be a Banach space, $T \in \mathcal{C}(X)$ and let S be a nonzero bounded operator defined on X. Let $\mathcal{V}^{w}_{S,T}(X)$ be any one of the sets $\mathcal{H}^{w}_{S,T}(X)$, $\mathcal{Z}^{w}_{S,T}(X)$ and $\mathcal{Q}^{w}_{S,T}(X)$. Then, the following statements hold

(i) $\sigma_{e,S}(T) = \bigcap_{K \in \mathcal{U}^{w}_{S,T}(X)} \sigma_S(T+K)$, where $\mathcal{U}^{w}_{S,T}(X)$ is a subset of $\mathcal{V}^{w}_{S,T}(X)$ satisfying

$$\mathcal{K}(X) \subset \mathcal{U}^{w}_{S,T}(X) \subset \mathcal{V}^{w}_{S,T}(X).$$

(ii) Let $\Omega^{w}_{S,T}(X)$ be a subset of $\mathcal{C}(X)$ included in $\mathcal{V}^{w}_{S,T}(X)$ and containing the subspace $\mathcal{K}(X)$. If for all $K, K' \in \Omega^{w}_{S,T}(X)$, we have $K \pm K' \in \Omega^{w}_{S,T}(X)$, then the equality

$$\sigma_{e,S}(T) = \sigma_{e,S}(T+K),$$

holds for every $K \in \Omega_{S,T}(X)$.

Proof. (i) Since $\mathcal{K}(X) \subset \mathcal{U}^{w}_{S,T}(X) \subset \mathcal{V}^{w}_{S,T}(X)$, it follows that

$$\bigcap_{K \in \mathcal{V}^{w}_{S,T}(X)} \sigma_S(T+K) \subseteq \bigcap_{K \in \mathcal{U}^{w}_{S,T}(X)} \sigma_S(T+K) \subseteq \bigcap_{K \in \mathcal{K}(X)} \sigma_S(T+K).$$

According to Theorem 4.1, we infer that $\sigma_{e,S}(T) = \bigcap_{K \in \mathcal{V}^{w}_{S,T}(X)} \sigma_S(T+K)$. This completes the proof of the assertion (i). Now, we prove (ii). By assumption, we have $\mathcal{K}(X) \subset \Omega_{S,T}(X) \subset \mathcal{V}^{w}_{S,T}(X)$. By Applying ($i$), we get

$$\sigma_{e,S}(T) = \bigcap_{K \in \Omega^{w}_{S,T}(X)} \sigma_S(T+K).$$

Now, fix arbitrarily $K \in \Omega^{w}_{S,T}(X)$. Taking into account the fact that for all $K, K' \in \Omega^{w}_{S,T}(X)$, we have $K \pm K' \in \Omega_{S,T}(X)$. We can readily prove that the application $\Psi : K' \longmapsto K + K'$ is a bijection from $\Omega^{w}_{S,T}(X)$ to itself. Hence,

$$\begin{aligned} \sigma_{e,S}(T+K) &= \bigcap_{K' \in \Omega^{w}_{S,T}(X)} \sigma_S(T+K'+K) \\ &= \bigcap_{K' \in \Omega^{w}_{S,T}(X)} \sigma_S(T+K') \\ &= \sigma_{e,S}(T). \end{aligned}$$

□

Now, we give some characterizations of the approximate essential spectrum of an operator.

Theorem 4.3. Let X be a Banach space, $T \in \mathcal{C}(X)$ and S be a nonzero bounded operator defined on X. Let $\mathcal{V}^w_{S,T}(X)$ be any one of the sets $\mathcal{A}^w_{S,T}(X)$, $\mathcal{Z}^w_{S,T}(X)$ and $\mathcal{Q}^w_{S,T}(X)$. Then, the following statements hold

(i) $\sigma_{eap,S}(T) = \bigcap_{K\in\mathcal{V}^w_{S,T}(X)} \sigma_{ap,S}(T+K)$.

(ii) $\sigma_{eap,S}(T) = \bigcap_{K\in\mathcal{U}^w_{S,T}(X)} \sigma_{ap,S}(T+K)$, where $\mathcal{U}^w_{S,T}(X)$ is a subset of $\mathcal{V}^w_{S,T}(X)$ satisfying

$$\mathcal{K}(X) \subset \mathcal{U}^w_{S,T}(X) \subset \mathcal{V}^w_{S,T}(X).$$

(iii) Let $\Omega^w_{S,T}(X)$ be a subset of $\mathcal{C}(X)$ included in $\mathcal{V}^w_{S,T}(X)$ and containing the subspace $\mathcal{K}(X)$. If for all $K, K' \in \Omega^w_{S,T}(X)$, we have $K \pm K' \in \Omega^w_{S,T}(X)$, then the equality

$$\sigma_{eap,S}(T) = \sigma_{eap,S}(T+K),$$

holds for every $K \in \Omega^w_{S,T}(X)$.

Proof. (i) Suppose for example that $\mathcal{V}^w_{S,T}(X) = \mathcal{A}^w_{S,T}(X)$. Since $\mathcal{K}(X) \subseteq \mathcal{A}^w_{S,T}(X)$, then

$$\bigcap_{K\in\mathcal{A}^w_{S,T}(X)} \sigma_{ap,S}(T+K) \subseteq \sigma_{eap,S}(T).$$

Now, take λ such that $\lambda \notin \bigcap_{K\in\mathcal{A}^w_{S,T}(X)} \sigma_{ap,S}(T+K)$. Then, there exists $K \in \mathcal{A}^w_{S,T}(X)$ such that

$$\inf_{\|x\|=1, x\in D(T)} \|(\lambda S - T - K)x\| > 0.$$

Hence, $\alpha(\lambda S - T - K) = 0$ so that $ind(\lambda S - T - K) \leq 0$. By using Theorem IV.1.6 in [10], we infer that $\lambda S - T - K \in \Phi_+(X)$. Since $\lambda S - T - K$ is closed, then $\lambda S - \widehat{T} - K \in \mathcal{L}(X_T, X)$. Now, we have

$$\lambda S - \widehat{T} = [I + K(\lambda S - T - K)^{-1}][\lambda S - \widehat{T} - K].$$

Combining Theorem 3.22 and Atkinson's theorem, we infer that

$$\lambda S - \widehat{T} \in \Phi_+(X_T, X) \text{ and } ind(\lambda S - \widehat{T}) \leq 0.$$

Taking into account (3.9), we get

$$\lambda S - T \in \Phi_+(X) \text{ and } ind(\lambda S - T) \leq 0.$$

Then we easily shows that $\lambda \notin \sigma_{eap,S}(T)$, this proves the assertion (i). The cases $\mathcal{V}^w_{S,T}(X) = \mathcal{Z}^w_{S,T}(X)$ and $\mathcal{V}^w_{S,T}(X) = \mathcal{Q}^w_{S,T}(X)$ can be treated similarily. For the proof of (ii), assume that $\mathcal{K}(X) \subset \mathcal{U}^w_{S,T}(X) \subset \mathcal{V}^w_{S,T}(X)$. Then,

$$\bigcap_{K\in\mathcal{V}^w_{S,T}(X)} \sigma_{ap,S}(T+K) \subset \bigcap_{K\in\mathcal{U}^w_{S,T}(X)} \sigma_{ap,S}(T+K) \subset \bigcap_{K\in(X)} \sigma_{ap,S}(T+K).$$

In view of assertion (i), we infer that

$$\sigma_{eap,S}(T) = \bigcap_{K\in\mathcal{V}^w_{S,T}(X)} \sigma_{ap,S}(T+K).$$

The proof of (iii) is similar to the one of of Corollary 4.2. □

The next proposition will provide a characterization of the essential defect spectrum of an operator.

Proposition 4.4. Let X be a Banach space, $T \in \mathcal{C}(X)$ and let S be a nonzero bounded operator defined on X. Let $\mathcal{V}^w_{S,T}(X)$ be any one of the sets $\mathcal{A}^w_{S,T}(X)$, $\mathcal{Z}^w_{S,T}(X)$ and $\mathcal{Q}^w_{S,T}(X)$. Then, the following statements hold

(i) $\sigma_{e\delta,S}(T) = \bigcap_{K\in\mathcal{V}^w_{S,T}(X)} \sigma_{\delta,S}(T+K)$.

(ii) $\sigma_{e\delta,S}(T) = \bigcap_{K\in\mathcal{U}^w_{S,T}(X)} \sigma_{\delta,S}(T+K)$, where $\mathcal{U}^w_{S,T}(X)$ is a subset of $\mathcal{V}^w_{S,T}(X)$ satisfying

$$\mathcal{K}(X) \subset \mathcal{U}^w_{S,T}(X) \subset \mathcal{V}^w_{S,T}(X).$$

(iii) Let $\Omega^w_{S,T}(X)$ be a subset of $\mathcal{C}(X)$ included in $\mathcal{V}^w_{S,T}(X)$ and containing the subspace $\mathcal{K}(X)$. If for all $K, K' \in \Omega^w_{S,T}(X)$, we have $K \pm K' \in \Omega^w_{S,T}(X)$, then the equality

$$\sigma_{e\delta,S}(T) = \sigma_{e\delta,S}(T+K),$$

holds for every $K \in \Omega_{S,T}(X)$.

Proof. (i) Assume that $\mathcal{V}^w_{S,T}(X) = \mathcal{A}^w_{S,T}(X)$. Since $\mathcal{K}(X) \subseteq \mathcal{A}^w_{S,T}(X)$, then

$$\bigcap_{K\in\mathcal{A}^w_{S,T}(X)} \sigma_{\delta,S}(T+K) \subseteq \sigma_{e\delta,S}(T).$$

Now, take λ such that $\lambda \notin \bigcap_{K\in\mathcal{A}^w_{S,T}(X)} \sigma_{\delta,S}(T+K)$. Then, there exists $K \in \mathcal{A}^w_{S,T}(X)$ such that $\lambda S - T - K$ is onto. Thus, $\lambda S - T - K \in \Phi_-(X)$ and $ind(\lambda S - T - K) = \alpha(\lambda S - T - K) \geq 0$. Reasoning as in the proof of Theorem 4.3, we get $\lambda S - T \in \Phi_-(X)$ and $ind(\lambda S - T) \geq 0$. Hence we obtain $\lambda \notin \sigma_{e\delta,S}(T)$. This completes the proof of (i). We prove in the same way the cases $\mathcal{V}^w_{S,T}(X) = \mathcal{Z}^w_{S,T}(X)$ and $\mathcal{V}^w_{S,T}(X) = \mathcal{Q}^w_{S,T}(X)$. The same reasoning used in the proof of Theorem 4.3 can be applied to achieve the proof of assertions (ii) and (iii). □

References

[1] P. Aiena, Semi Fredholm Operators, Perturbation Theory and Localized Svep, IVIC (2007).
[2] W. Y. Akashi, On the Perturbation Theory for Fredholm Operators, J. Maths 21, 603–612 (1983).
[3] K. Astala, On measures of noncompactness and ideal variations in Banach spaces, Ann. Acad. Sci. Fenn. Ser. A. I. Math. Dissertationes 29, 1–42 (1980).
[4] J. Banas, J. Rivero, On measures of weak noncompactness. Ann. Mat. Pura Appl. 151, 213–224 (1988).
[5] J. B. Conway, A Course in Functiounal Analysis, Springer. (1990)
[6] F.S. De Blasi, On a property of the unit sphere in a Banach space. Bull. Math. Soc. Sci. Math. R.S. Roumanie (N.S) 21: 259–262 (1977).
[7] N. Dunford, J.T. Schwartz, Linear operations on summable functions. Trans. Amer. Math. Soc. 47, 323–392 (1940).
[8] N. Dunford, J.T. Schwartz, Linear Operators, Part I. General Theory, Interscience, New York, (1958).
[9] M. Faierman, R. Mennicken, M. Moller, A boundary eigenvalue problem for a system of partial differential operators occuring in magnetohydrodynamics. Math Nachr, 173, 141–167 (1995).
[10] S. Goldberg, Unbounded Linear Operators, Theory and Applications, McGraw-Hill Book Co, New York, (1966).
[11] M. Gonzalez, Operational Quantities Derived From The Norm And Measures Of Noncompactness, Extracta Mathematica 5, 56–58 (1990).
[12] K. Gustafson, J. Weidmann, On the Essential Spectrum, J. Math. Anal. Appl. 25, 121–127 (1969).
[13] H. Hennion, L. Hervé, Limit Theorems for Markoc chains and Stochastic Properties of Dynamical Systems by Quasi-Compactness. Springer-Verlag Berlin Heidelberg (2001).
[14] A. Jeribi, B. Krichen, M. Salhi, Characterization of relatively demicompact operators by means of measures of noncompactness. J. Korean Math. Soc. 119 (2018)
[15] T. Kato, Perturbation Theory for Linear Operators. New York, Springer-Velag, (1980)
[16] U. Krengel, Ergodic Theorems, Walter de Gruyter, Berlin, New York (1985).
[17] B. Krichen, Relative essential spectra involving relative demicompact unbounded linear operators, Acta Math. Sci. Ser. B Engl. Ed.2, 546–556 (2014).
[18] B. Krichen, D. O'Regan, On the Class of Relatively Weakly Demicompact Nonlinear Operators, Fixed Point Theory, 19, 625–630 (2018).
[19] B. Krichen, D. O'Regan, Weakly demicompact linear operators and axiomatic measures of weak noncompactness. Mathematica Slovaca 69 (6), 1403–1412 (2019).

[20] V. Muller, Spectral Theory of Linear Operator and Spectral Systems in Banach Algebras. Oper. Theo. Adva. Appl. 139 (2003).
[21] J. Neveu, Mathematical Foundations of the Calculus of Probability, Holden-Day Inc., (1965).
[22] W. V. Petryshyn, Construction of Fixed points Of Demicompact Mappings in Hilbert Spaces, Journal of Functional Analysis And Applications 14, 276–284 (1966).
[23] W. V. Petryshyn, Remarks on Condensing and k-set Contractive Mappings, Journal of Functional Analysis And Applications 39, 717–741 (1972).
[24] V. Rakočević, On one subset of M. Schechter's essential spectrum, Math. Vesnik 33, 389–391 (1981).
[25] V. Rakočević, On the Essential Approximate Spectrum II, Math. Vesnik 36, 89–97 (1984).
[26] M. Schechter, On the Essential Spectrum of an Arbitrary Operator, J. Math. Anal. Appl. 13, 205–215 (1966).
[27] C. Schmoeger, The Spectral Mapping Theorem for the Essential Approximate Point Spectrum, Colloq. Math. 2, 167–176 (1997).
[28] M. Schechter, Principles of Functional Analysis, Amer. Maths. Soc 36. (2002)
[29] H. O. Tylli, Duality of the Weak Essential Norm, Proceedings of the American Mathematical Society 129, 1437–1443 (2000).
[30] V. Williams, Closed Fredholm and Semi-Fredholm Operators, Essential Spectra and Perturbations, Journal of Functional Analysis 20,1–25 (1975).
[31] F. Wolf, On the invariance of the essential spectrum under a change of boundary conditions of partial differential boundary operators. Nederl Akad Wetensch Proc Ser A 62 Indag Math, 21, 142–147 (1959).