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The Radical-Zariski Topology on the Radical Spectrum of Modules

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Abstract. For a module M over a commutative ring R with identity, let $\mathrm{RSpec}(M)$ denote the collection of all submodules L of M such that $\sqrt{(L:M)}$ is a prime ideal of R and is equal to $(\mathrm{rad}\, L : M)$. In this article, we topologies $\mathrm{RSpec}(M)$ with a topology which enjoys analogs of many of the properties of the Zariski topology on the prime spectrum $\mathrm{Spec}(M)$ (as a subspace topology). We investigate this topological space from the point of view of spectral spaces by establishing interrelations between $\mathrm{RSpec}(M)$ and $\mathrm{Spec}(R/\mathrm{Ann}(M))$.

1. Introduction

Throughout all rings are commutative with identity and all modules are unitary. For a submodule N of an R-module M, $(N:M)$ is the ideal $\{r \in R \mid rM \subseteq N\}$ of R. As usual $(0:M)$ is the annihilator of M and is denoted by $\mathrm{Ann}(M)$. A proper submodule N of M is called *prime* (resp. *primary*) if for any $r \in R$ and any $m \in M$, $rm \in N$ implies that either $m \in N$ or $r \in (N:M)$ (resp. $r \in \sqrt{(N:M)}$) (see e.g. [4], [7] and [12]). The set of all prime submodules of an R-module M is denoted by $\mathrm{Spec}(M)$. *The radical* of a submodule N of M, denoted by $\mathrm{rad}\, N$, is the intersection of all elements of $\mathrm{Spec}(M)$ containing N or, in case there are no such elements, $\mathrm{rad}\, N$ is M. A submodule N of M is called *radical* if $\mathrm{rad}\, N = N$. For an ideal I of a ring R, we assume throughout that $\sqrt{I}$ denotes the radical of I. For any R-module M, by $N \prec M$ we mean that N is a submodule of M such that $(\mathrm{rad}\, N : M) = \sqrt{(N:M)}$. If $N \prec M$ and $\sqrt{(N:M)} = p$ is a prime ideal of R, we write $N \prec_p M$. It is evident that for any prime submodule N of M, $N \prec_p M$. If M is a module over a one-dimensional domain R and N is a primary submodule of M, then by [13, Theorem 1.3], $\mathrm{rad}\, N$ is a prime submodule of M and hence $N \prec_p M$. An R-module M is called a *primeful* module, if either $M = 0$ or $M \neq 0$ and the *natural map* $\psi : \mathrm{Spec}(M) \to \mathrm{Spec}(R/\mathrm{Ann}(M))$ defined by $\psi(N) = (N:M)/\mathrm{Ann}(M)$ is surjective.

In [9, Proposition 5.3], it has been shown that if M/N is a primeful R-module, then $N \prec M$. But the converse is not true in general. For example, let $M = \prod_{p\in\Omega} \mathbb{Z}/p\mathbb{Z}$ and $N = \bigoplus_{p\in\Omega} \mathbb{Z}/p\mathbb{Z}$, where Ω is the set of prime integers. Then M is a primeful $\mathbb{Z}$-module while N and M/N are not. Moreover N is a 0-prime submodule of M which implies that $N \prec_0 M$ (see [9, Example 1]). However, if M is a non-zero primeful R-module, then for every prime ideal p of R containing $\mathrm{Ann}(M)$, M/pM is a primeful R-module and in particular $pM \prec_p M$ (see [9, Proposition 4.5 and Proposition 5.3]).

Let M be an R-module. The *radical spectrum* of M, denoted $\mathrm{RSpec}(M)$, is the set $\{L \mid L \prec_p M\}$. It is clear that $\mathrm{Spec}(M) \subseteq \mathrm{RSpec}(M)$, and $\mathrm{RSpec}(M) = \varnothing$ if and only if $\mathrm{Spec}(M) = \varnothing$.We remark that in [11] there are found necessary and sufficient conditions for a module M such that $\mathrm{Spec}(M) = \varnothing$.

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Recall that *the prime spectrum* of a ring R, denoted by Spec(R), consists of all prime ideals of R and is non-empty. For any ideal I of R, let $V(I) = \{p \in \mathrm{Spec}(R) : I \subseteq p\}$. It is well-known that the sets $V(I)$, where I is an ideal of R, satisfy the axioms for the closed sets of a topology on Spec(R), called the *Zariski topology* (see, for example, [3, p. 98]). In the literature, there are many different generalizations of the Zariski topology from rings to modules (see e.g. [1], [8] and [11]). The Zariski topology on Spec(M), denoted τ, is one of them which has the sets $V(N) = \{P \in \mathrm{Spec}(M) \mid (P : M) \supseteq (N : M)\}$ as closed sets (see [8]). Here, we topologize RSpec(M) with a topology which is called the *radical-Zariski topology*, denoted by $\mathcal{T}$ and described by taking the set $\{\mathcal{V}(N) \mid N \text{ is a submodule of } M\}$ as the family of closed sets in which $\mathcal{V}(N) = \{L \prec_p M \mid p \supseteq \sqrt{(N : M)}\}$. The topological space $(\mathrm{RSpec}(M), \mathcal{T})$ has $(\mathrm{Spec}(M), \tau)$ as a subspace with the usual subspace topology. The *radical natural map* $\varphi : \mathrm{RSpec}(M) \to \mathrm{Spec}(\bar{R})$, defined by $\varphi(L) = \sqrt{(L : M)}/\,\mathrm{Ann}(M)$, that plays a remarkable role in the study of radical-Zariski topology is a continuous map (Theorem 4.3). In particular, if φ is surjective, then φ is bijective if and only if φ is a homeomorphism (Corollary 4.4). It is shown that if φ is surjective, then

(1) $(\mathrm{RSpec}(M), \mathcal{T})$ is quasi-compact and has a basis of quasi-compact open subsets (Theorem 4.11);

(2) The quasi-compact open subsets of $(\mathrm{RSpec}(M), \mathcal{T})$ are closed under finite intersection and form an open base (Theorem 4.13);

(3) Every irreducible closed subset of $(\mathrm{RSpec}(M), \mathcal{T})$ has a generic point, if R can be embedded in a zero-dimensional ring (Corollary 5.8).

Finally, according to Hochster's characterization, it is shown that if R is embedded in a zero-dimensional ring and φ is surjective, then $(\mathrm{RSpec}(M), \mathcal{T})$ is a spectral space if and only if $(\mathrm{RSpec}(M), \mathcal{T})$ is a T_0-space if and only if φ is injective (Theorem 5.11).

2. Some Properties of $\prec$ and $\prec_p$

In this section, we give some basic results, particularly the interplay between $\prec$ (resp. $\prec_p$) and some usual operations which are needed in next sections.

Let I be a radical ideal of R and M be a finitely generated R-module. By [7, p. 65, Proposition 8] $(IM : M) = M$ if and only if $\mathrm{Ann}(M) \subseteq I$. This assertion holds in primeful modules as a class wider than finitely generated modules [9, Proposition 3.1]. It follows that, if M is a primeful module and N a submodule of M, then $(\sqrt{(N : M)}M : M) = \sqrt{(N : M)}$. This may be compared with the following lemma.

Lemma 2.1. *Let M be an R-module and $N \prec M$. Then $(\sqrt{(N : M)}M : M) = \sqrt{(N : M)}$.*

Proof. Since $N \prec M$, we have $(\sqrt{(N : M)}M : M) = ((\mathrm{rad}\, N : M)M : M) \subseteq (\mathrm{rad}\, N : M) = \sqrt{(N : M)}$. The reverse containment is clear. □

Lemma 2.2. *Let $f : M \to M'$ be an epimorphism of R-modules. Then*

(1) If $L' \prec M'$, then $f^{-1}(L') \prec M$.

(2) If $L \prec M$ and $\mathrm{Ker} f \subseteq L$, then $f(L) \prec M'$.

Moreover, the analogous statements also hold if we replace " $\prec$" by " $\prec_p$ " in the above.

Proof. (1) Let $r \in (\mathrm{rad}\, f^{-1}(L') : M)$. Then $rM \subseteq \mathrm{rad}\, f^{-1}(L')$ and hence, by using [10, Corollary 1.3.], $rM' \subseteq f(\mathrm{rad}\, f^{-1}(L')) = f(f^{-1}(\mathrm{rad}\, L')) = \mathrm{rad}\, L'$. Thus $r \in (\mathrm{rad}\, L' : M') = \sqrt{(L' : M')}$. Therefore there exists a positive integer n such that $r^n M' \subseteq L'$, which implies that $r^n M \subseteq f^{-1}(L')$. This means that $r \in \sqrt{(f^{-1}(L') : M)}$ and hence we have $\sqrt{(f^{-1}(L') : M)} = (\mathrm{rad}\, f^{-1}(L') : M)$. (2) Let $r \in (\mathrm{rad}\, f(L) : M')$. Again, by [10, Corollary 1.3.], we have $f(rM) = rM' \subseteq f(\mathrm{rad}\, L)$. Now, by assumption $rM \subseteq \mathrm{rad}\, L$ and then $r \in \sqrt{(L : M)}$. This follows that $r \in \sqrt{(f(L) : M')}$ and hence we are done.

For the "moreover" statement, the proof of (1) shows that $\sqrt{(f^{-1}(L') : M)} = \sqrt{(L' : M')} = (\mathrm{rad}\, f^{-1}(L') : M)$ and also the proof of (2) shows that $(\mathrm{rad}\, f(L) : M') = \sqrt{(L : M)} = (f(L) : M')$. Hence, if $\sqrt{(L' : M')}$ and $\sqrt{(L : M)}$ are prime ideals of R, then so are $\sqrt{(f^{-1}(L') : M)}$ and $(\mathrm{rad}\, f(L) : M')$. □

Corollary 2.3. *Let M be an R-module and L, N be submodules of M such that $N \subseteq L$. Then $L \prec M$ (resp., $L \prec_p M$) if and only if $L/N \prec M/N$ (resp., $L/N \prec_p M/N$).*

Proof. It suffices to consider the natural surjection $\pi : M \to M/N$ defined by $\pi(m) = m+N$ and apply Lemma 2.10. □

It is well-known that the radical and intersection of a finite family of ideals commute with each other. However, this is not true for infinite families in general. For example, if we consider the family $\{2^n\mathbb{Z}\}_{n\geq 1}$ of ideals of $\mathbb{Z}$, then $\sqrt{\bigcap_{n\geq 1} 2^n\mathbb{Z}} = 0 \subsetneq \bigcap_{n\geq 1} \sqrt{2^n\mathbb{Z}} = 2\mathbb{Z}$. The following lemma characterizes rings in which the commutativity holds will be used in assertions.

Lemma 2.4. *Let R be a ring. Then R is embedded in a zero-dimensional ring if and only if $\sqrt{\bigcap_{\lambda\in\Lambda} I_\lambda} = \bigcap_{\lambda\in\Lambda} \sqrt{I_\lambda}$, for every family $\{I_\lambda\}_{\lambda\in\Lambda}$ of ideals of R.*

Proof. By [2, Theorem 1.3 and Theorem 2.4] □

In the rest of paper, if R is embedded in a zero-dimentional ring, then we say simply that R is an $EZ-ring$.

Lemma 2.5. *Let R be an EZ-ring, M be an R-module and let $\{L_i \prec_{p_i} M : i \in I\}$ be a non-empty chain of submodules of M. Then $\bigcap_{i\in I} L_i \prec_p M$ where $p = \bigcap_{i\in I} p_i$.*

Proof. By using Lemma 2.12, we have

$$\sqrt{(\bigcap_{i\in I} L_i : M)} = \bigcap_{i\in I} \sqrt{(L_i : M)} = \bigcap_{i\in I} (\mathrm{rad}\, L_i : M) = (\bigcap_{i\in I} \mathrm{rad}\, L_i : M) \supseteq (\mathrm{rad}(\bigcap_{i\in I} L_i) : M) \supseteq \sqrt{(\bigcap_{i\in I} L_i : M)}$$

Thus $\sqrt{(\bigcap_{i\in I} L_i : M)} = (\mathrm{rad}(\bigcap_{i\in I} L_i) : M) = p$, that is $\bigcap_{i\in I} L_i \prec_p M$. □

Let M be an R-module and N be a proper submodule of M. Let $E_M(N) = \{rx : r \in R$ and $x \in M$ such that $r^n x \in N$ for some $n \in \mathbb{N}\}$. The *envelop submodule* of N in M is defined to be a submodule of M generated by $E_M(N)$. Following [10], the submodule N is said to *satisfy the radical formula* if $\mathrm{rad}\, N = E_M(N)$. Also M is said to *satisfy the radical formula* if every submodule of M satisfies the radical formula.

Proposition 2.6. *Let M be an R-module which satisfies the radical formula. Then the following are equivalent:*

(1) $L_1 \cap L_2 \prec M$, for all finitely generated submodules L_1 and L_2 of M whit $L_1 \prec M$, $L_2 \prec M$.

(2) $L_1 \cap L_2 \prec M$, for all submodules L_1 and L_2 of M whit $L_1 \prec M$, $L_2 \prec M$.

Moreover, the analogous statements also hold if we replace " $\prec$" by " $\prec_p$ " in the above.

Proof. (1) ⇒ (2) Let L_1 and L_2 be two submodules of M whit $L_1 \prec M$ and $L_2 \prec M$. We have $\sqrt{L_1 \cap L_2 : M} = \sqrt{L_1 : M} \cap \sqrt{L_2 : M} = (\mathrm{rad}\, L_1 : M) \cap (\mathrm{rad}\, L_2 : M) = (\mathrm{rad}\, L_1 \cap \mathrm{rad}\, L_2 : M)$. We show that $\mathrm{rad}\, L_1 \cap \mathrm{rad}\, L_2 = \mathrm{rad}(L_1 \cap L_2)$. Clearly $\mathrm{rad}(L_1 \cap L_2) \subseteq \mathrm{rad}\, L_1 \cap \mathrm{rad}\, L_2$. Let $m \in \mathrm{rad}\, L_1 \cap \mathrm{rad}\, L_2$. Since M satisfies the radical formula, $m \in RE_M(L_1) \cap RE_M(L_2)$. Hence $m = \sum_{i=1}^{s} r_i x_i$ for some $r_i \in R$ and $x_i \in M (1 \leqslant i \leqslant s)$ where $x_i = a_i u_i$ and $a_i^{n_i} u_i \in L_1$, for some $a_i \in R$, $u_i \in M$ and positive integers n_i $(1 \leqslant i \leqslant s)$. Also $m = \sum_{j=1}^{t} s_j y_j$ for some $s_j \in R$ and $y_j \in M(1 \leqslant j \leqslant t)$ where $y_j = b_j v_j$ and $b_j^{m_j} v_j \in L_2$, for some $b_j \in R$, $v_j \in M$ positive integers m_j $(1 \leqslant j \leqslant t)$. Now let $L_1' = Ra_1^{n_1}u_1 + Ra_2^{n_2}u_2 + ... + Ra_s^{n_s}u_s \subseteq L_1$ and $L_2' = Rb_1^{m_1}v_1 + Rb_2^{m_2}v_2 + ... + Rb_t^{m_t}v_t \subseteq L_2$. Thus $m \in RE(L_1') \cap RE(L_2') = \mathrm{rad}\, L_1' \cap \mathrm{rad}\, L_2'$, and , by (1), $m \in \mathrm{rad}(L_1' \cap L_2')$. Now since $\mathrm{rad}(L_1' \cap L_2') \subseteq \mathrm{rad}(L_1 \cap L_2)$, we have $m \in \mathrm{rad}(L_1 \cap L_2)$. Hence $\mathrm{rad}\, L_1 \cap \mathrm{rad}\, L_2 \subseteq \mathrm{rad}(L_1 \cap L_2)$. It follows that $\sqrt{L_1 \cap L_2 : M} = (\mathrm{rad}(L_1 \cap L_2) : M)$, that is $L_1 \cap L_2 \prec M$. (2) ⇒ (1). Clear.

The "moreover" statement is clear. □

Proposition 2.7. *Let M be a finitely generated R-module and $L \prec_p M$. Then $L_p \prec_{p_p} M_p$.*

Proof. Let $L \prec_p M$ and $\sqrt{(L:M)} = (\operatorname{rad} L : M) = p$. First we show that $(\operatorname{rad} L)_P$ is a proper submodule of M. Let $\{m_1, ..., m_n\}$ be a set of generators of M and suppose on the contrary that $(\operatorname{rad} L)_P = M_P$. Then there exists $s \in R \setminus p$ such that $sm_i \in \operatorname{rad} L$ for $i = 1, ..., n$. This implies that $sM \subseteq \operatorname{rad} L$ and hence $s \in p$ a contradiction. Thus $(\operatorname{rad} L)_p \neq M_p$ and so $p_p \subseteq ((\operatorname{rad} L)_p : M_p) \neq R_p$ which follows that $p_p = ((\operatorname{rad} L)_p : M_p)$. Hence, by [14, Lemma 1.7], $(\operatorname{rad} L)_p$ is a prime submodule of M_p containing L_p. Thus $p_p = (\sqrt{(L:M)})_p = \sqrt{(L:M)_p} = \sqrt{(L_p : M_p)} = (\operatorname{rad} L_p : M_p) \subseteq ((\operatorname{rad} L)_p : M_p) = p_p$. Therefore $p_p = \sqrt{(L_p : M_p)} = (\operatorname{rad} L_p : M_p)$, i.e., $L_p \prec_{p_p} M_p$. $\square$

Proposition 2.8. *Let M be an R-module and $L \prec M$(resp. $L \prec_p M$). Then $\operatorname{rad} L \prec M$(resp. $\operatorname{rad} L \prec_p M$).*

Proof. We have $(\operatorname{rad} L : M) = \sqrt{(L:M)} \subseteq \sqrt{(\operatorname{rad} L : M)} \subseteq (\operatorname{rad}(\operatorname{rad} L) : M) = (\operatorname{rad} L : M)$. Thus $\sqrt{(\operatorname{rad} L : M)} = (\operatorname{rad}(\operatorname{rad} L) : M)$. Now the proof is clear. $\square$

Let I be a radical ideal of R and M be a finitely generated R-module. By [7, p. 65, Proposition 8] $(IM : M) = M$ if and only if $\operatorname{Ann}(M) \subseteq I$. This assertion holds in primeful modules as a class wider than finitely generated modules [9, Proposition 3.1]. It follows that, if M is a primeful module and N a submodule of M, then $(\sqrt{(N:M)}M : M) = \sqrt{(N:M)}$. This may be compared with the following lemma.

Lemma 2.9. *Let M be an R-module and $N \prec M$. Then $(\sqrt{(N:M)}M : M) = \sqrt{(N:M)}$.*

Proof. Since $N \prec M$, we have $(\sqrt{(N:M)}M : M) = ((\operatorname{rad} N : M)M : M) \subseteq (\operatorname{rad} N : M) = \sqrt{(N:M)}$. The reverse containment is clear. $\square$

Lemma 2.10. *Let $f : M \to M'$ be an epimorphism of R-modules. Then*

(1) If $L' \prec M'$, then $f^{-1}(L') \prec M$.

(2) If $L \prec M$ and $Kerf \subseteq L$, then $f(L) \prec M'$.

Moreover, the analogous statements also hold if we replace " $\prec$" by " $\prec_p$ " in the above.

Proof. (1) Let $r \in (\operatorname{rad} f^{-1}(L') : M)$. Then $rM \subseteq \operatorname{rad} f^{-1}(L')$ and hence, by using [10, Corollary 1.3.], $rM' \subseteq f(\operatorname{rad} f^{-1}(L')) = f(f^{-1}(\operatorname{rad} L')) = \operatorname{rad} L'$. Thus $r \in (\operatorname{rad} L' : M') = \sqrt{(L' : M')}$. Therefore there exists a positive integer n such that $r^n M' \subseteq L'$, which implies that $r^n M \subseteq f^{-1}(L')$. This means that $r \in \sqrt{(f^{-1}(L') : M)}$ and hence we have $\sqrt{(f^{-1}(L') : M)} = (\operatorname{rad} f^{-1}(L') : M)$. (2) Let $r \in (\operatorname{rad} f(L) : M')$. Again, by [10, Corollary 1.3.], we have $f(rM) = rM' \subseteq f(\operatorname{rad} L)$. Now, by assumption $rM \subseteq \operatorname{rad} L$ and then $r \in \sqrt{(L:M)}$. This follows that $r \in \sqrt{(f(L) : M')}$ and hence we are done.

For the "moreover" statement, the proof of (1) shows that $\sqrt{(f^{-1}(L') : M)} = \sqrt{(L' : M')} = (\operatorname{rad} f^{-1}(L') : M)$ and also the proof of (2) shows that $(\operatorname{rad} f(L) : M') = \sqrt{(L:M)} = (f(L) : M')$. Hence, if $\sqrt{(L' : M')}$ and $\sqrt{(L:M)}$ are prime ideals of R, then so are $\sqrt{(f^{-1}(L') : M)}$ and $(\operatorname{rad} f(L) : M')$. $\square$

Corollary 2.11. *Let M be an R-module and L, N be submodules of M such that $N \subseteq L$. Then $L \prec M$ (resp., $L \prec_p M$) if and only if $L/N \prec M/N$ (resp., $L/N \prec_p M/N$).*

Proof. It suffices to consider the natural surjection $\pi : M \to M/N$ defined by $\pi(m) = m + N$ and apply Lemma 2.10. $\square$

It is well-known that the radical and intersection of a finite family of ideals commute with each other. However, this is not true for infinite families in general. For example, if we consider the family $\{2^n\mathbb{Z}\}_{n\geq 1}$ of ideals of $\mathbb{Z}$, then $\sqrt{\bigcap_{n\geq 1} 2^n\mathbb{Z}} = 0 \subsetneq \bigcap_{n\geq 1} \sqrt{2^n\mathbb{Z}} = 2\mathbb{Z}$.

The following lemma characterizes rings in which the commutativity holds will be used in assertions.

Lemma 2.12. *Let R be a ring. Then R is embedded in a zero-dimensional ring if and only if* $\sqrt{\bigcap_{\lambda\in\Lambda} I_\lambda} = \bigcap_{\lambda\in\Lambda}\sqrt{I_\lambda}$, *for every family* $\{I_\lambda\}_{\lambda\in\Lambda}$ *of ideals of R.*

Proof. By [2, Theorem 1.3 and Theorem 2.4] □

In the rest of paper, if R is embedded in a zero-dimentional ring, then we say simply that R is an $EZ-ring$.

Lemma 2.13. *Let R be an EZ-ring, M be an R-module and let* $\{L_i \prec_{p_i} M : i \in I\}$ *be a non-empty chain of submodules of M. Then* $\bigcap_{i\in I} L_i \prec_p M$ *where* $p = \bigcap_{i\in I} p_i$.

Proof. By using Lemma 2.12, we have

$$\sqrt{(\bigcap_{i\in I} L_i : M)} = \bigcap_{i\in I}\sqrt{(L_i : M)} = \bigcap_{i\in I}(\operatorname{rad} L_i : M) = (\bigcap_{i\in I}\operatorname{rad} L_i : M) \supseteq (\operatorname{rad}(\bigcap_{i\in I} L_i) : M) \supseteq \sqrt{(\bigcap_{i\in I} L_i : M)}.$$

Thus $\sqrt{(\bigcap_{i\in I} L_i : M)} = (\operatorname{rad}(\bigcap_{i\in I} L_i) : M) = p$, that is $\bigcap_{i\in I} L_i \prec_p M$. □

Let M be an R-module and N be a proper submodule of M. Let $E_M(N) = \{rx : r \in R$ and $x \in M$ such that $r^n x \in N$ for some $n \in \mathbb{N}\}$. The *envelop submodule* of N in M is defined to be a submodule of M generated by $E_M(N)$. Following [10], the submodule N is said to *satisfy the radical formula* if $\operatorname{rad} N = E_M(N)$. Also M is said to *satisfy the radical formula* if every submodule of M satisfies the radical formula.

Proposition 2.14. *Let M be an R-module which satisfies the radical formula. Then the following are equivalent:*

(1) $L_1 \cap L_2 \prec M$, *for all finitely generated submodules* L_1 *and* L_2 *of M whit* $L_1 \prec M$, $L_2 \prec M$.

(2) $L_1 \cap L_2 \prec M$, *for all submodules* L_1 *and* L_2 *of M whit* $L_1 \prec M$, $L_2 \prec M$.

Moreover, the analogous statements also hold if we replace " $\prec$*" by "* $\prec_p$ *" in the above.*

Proof. (1) ⇒ (2) Let L_1 and L_2 be two submodules of M whit $L_1 \prec M$ and $L_2 \prec M$. We have $\sqrt{L_1 \cap L_2 : M} = \sqrt{L_1 : M} \cap \sqrt{L_2 : M} = (\operatorname{rad} L_1 : M) \cap (\operatorname{rad} L_2 : M) = (\operatorname{rad} L_1 \cap \operatorname{rad} L_2 : M)$. We show that $\operatorname{rad} L_1 \cap \operatorname{rad} L_2 = \operatorname{rad}(L_1 \cap L_2)$. Clearly $\operatorname{rad}(L_1 \cap L_2) \subseteq \operatorname{rad} L_1 \cap \operatorname{rad} L_2$. Let $m \in \operatorname{rad} L_1 \cap \operatorname{rad} L_2$. Since M satisfies the radical formula, $m \in RE_M(L_1) \cap RE_M(L_2)$. Hence $m = \sum_{i=1}^{s} r_i x_i$ for some $r_i \in R$ and $x_i \in M (1 \leqslant i \leqslant s)$ where $x_i = a_i u_i$ and $a_i^{n_i} u_i \in L_1$, for some $a_i \in R$, $u_i \in M$ and positive integers n_i $(1 \leqslant i \leqslant s)$. Also $m = \sum_{j=1}^{t} s_j y_j$ for some $s_j \in R$ and $y_j \in M (1 \leqslant j \leqslant t)$ where $y_j = b_j v_j$ and $b_j^{m_j} v_j \in L_2$, for some $b_j \in R$, $v_j \in M$ positive integers m_j $(1 \leqslant j \leqslant t)$. Now let $L'_1 = Ra_1^{n_1}u_1 + Ra_2^{n_2}u_2 + ... + Ra_s^{n_s}u_s \subseteq L_1$ and $L'_2 = Rb_1^{m_1}v_1 + Rb_2^{m_2}v_2 + ... + Rb_t^{m_t}v_t \subseteq L_2$. Thus $m \in RE(L'_1) \cap RE(L'_2) = \operatorname{rad} L'_1 \cap \operatorname{rad} L'_2$, and , by (1), $m \in \operatorname{rad}(L'_1 \cap L'_2)$. Now since $\operatorname{rad}(L'_1 \cap L'_2) \subseteq \operatorname{rad}(L_1 \cap L_2)$, we have $m \in \operatorname{rad}(L_1 \cap L_2)$. Hence $\operatorname{rad} L_1 \cap \operatorname{rad} L_2 \subseteq \operatorname{rad}(L_1 \cap L_2)$. It follows that $\sqrt{L_1 \cap L_2 : M} = (\operatorname{rad}(L_1 \cap L_2) : M)$, that is $L_1 \cap L_2 \prec M$. (2) ⇒ (1). Clear.

The "moreover" statement is clear. □

Proposition 2.15. *Let M be a finitely generated R-module and* $L \prec_p M$. *Then* $L_p \prec_{p_p} M_p$.

Proof. Let $L \prec_p M$ and $\sqrt{(L : M)} = (\operatorname{rad} L : M) = p$. First we show that $(\operatorname{rad} L)_P$ is a proper submodule of M. Let $\{m_1, ..., m_n\}$ be a set of generators of M and suppose on the contrary that $(\operatorname{rad} L)_P = M_P$. Then there exists $s \in R \setminus p$ such that $sm_i \in \operatorname{rad} L$ for $i = 1, ..., n$. This implies that $sM \subseteq \operatorname{rad} L$ and hence $s \in p$ a contradiction. Thus $(\operatorname{rad} L)_p \neq M_p$ and so $p_p \subseteq ((\operatorname{rad} L)_p : M_p) \neq R_p$ which follows that $p_p = ((\operatorname{rad} L)_p : M_p)$. Hence, by [14, Lemma 1.7], $(\operatorname{rad} L)_p$ is a prime submodule of M_p containing L_p. Thus $p_p = (\sqrt{(L : M)})_p = \sqrt{(L : M)_p} = \sqrt{(L_p : M_p)} = (\operatorname{rad} L_p : M_p) \subseteq ((\operatorname{rad} L)_p : M_p) = p_p$. Therefore $p_p = \sqrt{(L_p : M_p)} = (\operatorname{rad} L_p : M_p)$, i.e., $L_p \prec_{p_p} M_p$. □

Proposition 2.16. *Let M be an R-module and* $L \prec M$*(resp.* $L \prec_p M$*). Then* $\operatorname{rad} L \prec M$*(resp.* $\operatorname{rad} L \prec_p M$*).*

Proof. We have $(\operatorname{rad} L : M) = \sqrt{(L : M)} \subseteq \sqrt{(\operatorname{rad} L : M)} \subseteq (\operatorname{rad}(\operatorname{rad} L) : M) = (\operatorname{rad} L : M)$. Thus $\sqrt{(\operatorname{rad} L : M)} = (\operatorname{rad}(\operatorname{rad} L) : M)$. Now the proof is clear. □

3. RSpec(M) and Topologies on it

In this section, first we determine RSpec(M) and compare it with Spec(M) for some R-modules M and then we turn our attention to the topologies on RSpec(M).

Example 3.1. *(1) Let V be a vector space over a field F. Then* $\mathrm{RSpec}(V) = \mathrm{Spec}(V)$ = *the set of all proper subspaces of V and $\mathcal{V}(W) = V(W)$ for every subspace W of V.*

(2) Let M be the $\mathbb{Z}$-module $\mathbb{Q}$, the set of rational numbers. It is clear that for every proper submodule N of $\mathbb{Q}$, $(N : \mathbb{Q}) = 0$ and 0 is the unique prime submodule of $\mathbb{Q}$. Thus we have $\mathrm{Spec}(\mathbb{Q}) = \{0\} = \mathrm{RSpec}(\mathbb{Q})$.

(3) Let $R = \mathbb{Z}$ and $M = \mathbb{Z}_{p^n}$ for some prime integer p and positive integer n. Any submodule of M is of the form $p^rM, 1 \leq r \leq n$ and we have $\sqrt{(p^rM : M)} = (\mathrm{rad}(p^rM) : M) = (pM : M) = p$. Thus, clearly, $\mathrm{Spec}(M) = \{p\mathbb{Z}_{p^n}\}$ *and* $\mathrm{RSpec}(M)$ = *the set of all proper submodules of M.*

(4) Let $X = \mathrm{Spec}(M)$. An R-module M is said to be X-injective if either $X = \emptyset$ or $X \neq \emptyset$ and the natural map $\psi : \mathrm{Spec}(M) \to \mathrm{Spec}(R/\mathrm{Ann}(M))$ is injective. Let M be the $\mathbb{Z}$-module $\mathbb{Q} \oplus \mathbb{Z}_{(p^\infty)}$. Since $\mathbb{Z}_{(p^\infty)}$ is a torsion X-injective module, by [1, Corollary 3.8(b)], M is X-injective and hence by [1, Proposition 3.7(b)] $\mathrm{Spec}(M) = \{0 \oplus \mathbb{Z}_{(p^\infty)}\}$. *Now, if $L \in rs(M)$, then $\mathrm{rad}\, L \neq M$ and so $L \subseteq 0 \oplus \mathbb{Z}_{(p^\infty)}$. Therefore, we can conclude that* $\mathrm{RSpec}(M) = \{0 \oplus N : N$ *is a submodule of* $\mathbb{Z}_{(p^\infty)}\}$ *which follows that* $\mathrm{Spec}(M) \subsetneq \mathrm{RSpec}(M)$.

For an R-module M and submodule N of M, let $V(N) = \{P \in \mathrm{Spec}(M) : (P : M) \supseteq (N : M)\}$ and $V^*(N) = \{P \in \mathrm{Spec}(M) : P \supseteq N\}$. Also, let $\zeta(M) = \{V(N) : N$ is a submodule of $M\}$, $\zeta^*(M) = \{V^*(N) : N$ is a submodule of $M\}$ and $\zeta'(M) = \{V^*(IN) : I$ is an ideal of $R\}$. In [8], Lu has introduced topologies on Spec(M) induced, respectively, by these three sets. In fact, $\zeta(M)$ and $\zeta'(M)$ always induce topologies on Spec(M) while $\zeta^*(M)$ induces a topology on Spec(M) if and only if $\zeta^*(M)$ is closed under finite union. Following [11], a module M is called a *top module* if $\zeta^*(M)$ induces a topology on Spec(M).

For a submodule N of M we consider two different types of varieties $\mathcal{V}^*(N)$ and $\mathcal{V}(N)$ on RSpec(M). We define $\mathcal{V}^*(N) = \{L \in \mathrm{RSpec}(M) : \mathrm{rad}\, L \supseteq N\}$ and $\mathcal{V}(N) = \{L \in \mathrm{RSpec}(M) : \sqrt{L : M} \supseteq \sqrt{N : M}\}$. Now, we consider the following sets and next the topologies on RSpec(M) induced by these sets: $\delta^*(M) = \{\mathcal{V}^*(N) \mid N$ is a submodule of $M\}$, $\delta'(M) = \{\mathcal{V}^*(IM) \mid I$ is an ideal of $R\}$ and $\delta(M) = \{\mathcal{V}(N) \mid N$ is a submodule of $M\}$.

By Lemma 3.2 below, it is clear that there exists a topology, $\mathcal{T}^*$ say, on RSpec(M) having $\delta^*(M)$ as the collection of all closed sets if and only if $\delta^*(M)$ is closed under finite union. When this is the case, we call the topology $\mathcal{T}^*$ the *radical quasi Zariski topology* on RSpec(M). An R-module M is called a *radical top module* if $\delta^*(M)$ induces the topology $\mathcal{T}^*$. On the other hand, Lemma 3.3 below, shows that for any module M there always exists a topology, $\mathcal{T}$ say, on RSpec(M) having $\delta(M)$ as the family of all closed sets. We call this topology the *radical-Zariski topology* on RSpec(M).

Lemma 3.2. *Using the above notation, the following statements hold:*

(1) $\mathcal{V}^*(0) = \mathrm{RSpec}(M)$ *and* $\mathcal{V}^*(M) = \emptyset$,

(2) $\bigcap_{i \in I} \mathcal{V}^*(N_i) = \mathcal{V}^*(\sum_{i \in I} N_i)$ *for any index set I and submodules $N_i (i \in I)$ of M,*

(3) $\mathcal{V}^*(N_1) \cup \mathcal{V}^*(N_2) \subseteq \mathcal{V}^*(N_1 \cap N_2)$ *for any submodules N_1 and N_2 of M.*

Proof. It is clear. □

Lemma 3.3. *Using the above notation, the following statements hold:*

(1) $\mathcal{V}(N) = \mathrm{RSpec}(M)$ *and* $\mathcal{V}(M) = \emptyset$,

(2) $\bigcap_{i \in I} \mathcal{V}(N_i) = \mathcal{V}(\sum_{i \in I} (N_i : M)M)$ *for any index set I and submodules $N_i (i \in I)$ of M,*

(3) $\mathcal{V}(N_1) \cup \mathcal{V}(N_2) = \mathcal{V}(N_1 \cap N_2)$ *for any submodules N_1 and N_2 of M.*

Proof. (1) It is clear. (2) Let $L \in \bigcap_{i\in I}\mathcal{V}(N_i)$. Then $\sqrt{(L:M)} \supseteq \sqrt{(N_i:M)} \supseteq (N_i:M)$ for all $i \in I$ and hence $\sqrt{(L:M)} \supseteq \sum_{i\in I}(N_i:M)$. This implies that $\sqrt{(L:M)} = (\sqrt{(L:M)}M:M) \supseteq (\sum_{i\in I}(N_i:M)M:M)$. Thus $L \in \mathcal{V}(\sum_{i\in I}(N_i:M)M)$ and we have $\bigcap_{i\in I}\mathcal{V}(N_i) \subseteq \mathcal{V}(\sum_{i\in I}(N_i:M)M)$. The reverse inclusion is clear as we have $((N:M)M:M) = (N:M)$ for any submodule N of M. (3) It is clear. □

The following lemma shows that Spec(M) together with the Zariski topology (resp. quasi Zariski topology) is a topological subspace of RSpec(M) together with the radical-Zariski topology (resp. radical quasi Zariski topology).

Lemma 3.4. *Let M be an R-module and N a submodule of M. Then*

(1) $\mathcal{V}^*(N) \cap \mathrm{Spec}(M) = V^*(N)$.

(2) $\mathcal{V}(N) \cap \mathrm{Spec}(M) = V(N)$.

Proof. It is clear. □

Recall that an R-module M is said to be *multiplication* if for every submodule N of M there exists an ideal I of R such that $N = IM$ [5]. Any multiplication module over a ring R is a radical top module as the following lemma, item (3), shows.

Lemma 3.5. *Let M be an R-module and I, J two ideals of R. Then*

(1) $\mathcal{V}(IJM) = \mathcal{V}((I\cap J)M) = \mathcal{V}(IM \cap JM)$

(2) $\mathcal{V}(IM) = \mathcal{V}(\sqrt{I}M) = \mathcal{V}^*(IM) = \mathcal{V}^*(\sqrt{I}M)$.

(3) $\mathcal{V}^*(IM) \cup \mathcal{V}^*(JM) = \mathcal{V}^*(IJM)$.

(4) $\mathcal{V}(IM) = \bigcap_{a_i\in I}\mathcal{V}(Ra_iM)$

Proof. (1) Since $IJM \subseteq (I\cap J)M \subseteq IM \cap JM$, we have $\mathcal{V}(IM\cap JM) \subseteq \mathcal{V}((I\cap J)M) \subseteq \mathcal{V}(IJM)$. Let $L \in \mathcal{V}(IJM)$. Then $(\mathrm{rad}\, L : M) = \sqrt{(L:M)} \supseteq \sqrt{(IJM:M)} \supseteq IJ$. Since $\sqrt{(L:M)}$ is prime, without loss of generality, we may assume that $\sqrt{(L:M)} \supseteq I$. Thus, by Lemma 2.9, $\sqrt{(L:M)} \supseteq (IM:M)$ and hence $\sqrt{(L:M)} \supseteq (IM:M) \cap (JM:M) = (IM\cap JM:M)$. Therefore $L \in \mathcal{V}(IM\cap JM)$ and we have the desired equalities. (2) Firstly, we show that $\mathcal{V}(IM) = \mathcal{V}(\sqrt{I}M)$. Clearly $\mathcal{V}(\sqrt{I}M) \subseteq \mathcal{V}(IM)$. Now, let $L \in \mathcal{V}(IM)$. Then $\sqrt{(L:M)} \supseteq \sqrt{(IM:M)} \supseteq \sqrt{I}$ and hence $\sqrt{(L:M)}M \supseteq \sqrt{I}M$. By using Lemma 2.9, we have $\sqrt{(L:M)} = (\sqrt{(L:M)}M:M) \supseteq (\sqrt{I}M:M)$. Thus $\sqrt{(L:M)} \supseteq \sqrt{(\sqrt{I}M:M)}$ which means that $L \in \mathcal{V}(\sqrt{I}M)$ and we have the desired equality. Now, by considering the following obvious inclusions, we have the final result: $\mathcal{V}^*(\sqrt{I}M) \subseteq \mathcal{V}^*(IM) \subseteq \mathcal{V}(IM) \subseteq \mathcal{V}^*(\sqrt{I}M)$. (3) By (1) and (2) above and Lemma 3.3 we have, $\mathcal{V}^*(IM) \cup \mathcal{V}^*(JM) = \mathcal{V}(IM) \cup \mathcal{V}(JM) = \mathcal{V}(IM\cap JM) = \mathcal{V}(IJM) = \mathcal{V}^*(IJM)$. (4) By using Lemma 3.3 (2), $\mathcal{V}(IM) = \mathcal{V}(\sum_{a_i\in I}Ra_iM) = \mathcal{V}(\sum_{a_i\in I}(Ra_i:M)M) = \bigcap_{a_i\in I}\mathcal{V}(Ra_iM)$. □

Theorem 3.6. *Let M be an R-module. Consider the following statements:*

(1) M is a multiplication module.

(2) M is a radical top module.

(3) M is a top module.

Then (1) ⇒ (2) ⇒ (3). *Moreover, if M is finitely generated then* (3) ⇒ (1).

Proof. (1) ⇒ (2) It is clear by Lemma 3.2 and Lemma 3.5 (3). (2) ⇒ (3) Since $\mathrm{RSpec}(M)$ is a topological space, Lemma 3.4 (1) shows that $\mathrm{Spec}(M)$ is a subspace of $\mathrm{RSpec}(M)$ and hence M is a top module. (3) ⇒ (1) It is clear by [11, Theorem 3.5]. □

Lemma 3.7. *Let M be an R-module with submodules N, N_1 and N_2. Then*

(1) If $(N_1 : M) = (N_2 : M)$, then $\mathcal{V}(N_1) = \mathcal{V}(N_2)$. The converse is true if both N_1 and N_2 are prime submodules of M.

(2) If $\sqrt{(N_1 : M)} = \sqrt{(N_2 : M)}$, then $\mathcal{V}(N_1) = \mathcal{V}(N_2)$. The converse is true if both N_1 and N_2 are primary submodules of M.

(3) $\mathcal{V}(N) = \mathcal{V}((N : M)M) = \mathcal{V}(\sqrt{(N : M)}M) = \mathcal{V}^*((N : M)M) = \mathcal{V}^*(\sqrt{(N : M)}M)$.

(4) $\mathcal{V}(N) = \bigcup_{J \in \mathcal{V}(N:M)} \mathrm{RSpec}_J(M)$, where $\mathrm{RSpec}_J(M) = \{L \in \mathrm{RSpec}(M) : \sqrt{(L : M)} = \sqrt{J}\}$ and $\mathcal{V}((N : M))$ is the set of all ideals J of R such that $\sqrt{J}$ is a prime ideal containing $\sqrt{(N : M)}$.

(5) If M is multiplication, then $\mathcal{V}(N) = \mathcal{V}(\mathrm{rad}\, N) = \mathcal{V}^*(N)$.

Proof. (1), (2) and (4) are clear.

(3) Since $(N : M) = ((N : M)M : M)$, $\sqrt{(N : M)} = \sqrt{((N : M)M : M)}$, and so $\mathcal{V}(N) = \mathcal{V}((N : M)M)$. Now, apply Lemma 3.5 (2).

(5) Clearly $\mathcal{V}^*(N) \subseteq \mathcal{V}(\mathrm{rad}\, N) \subseteq \mathcal{V}(N)$. Now, let $L \in \mathcal{V}(N)$. So $(\mathrm{rad}\, L : M) = \sqrt{(L : M)} \supseteq \sqrt{(N : M)} \supseteq (N : M)$. Hence $N = (N : M)M \subseteq \mathrm{rad}\, L$. That is $L \in \mathcal{V}^*(N)$. □

Corollary 3.8. *Let M be an R-module. Then $\delta(M) = \delta'(M) \subseteq \delta^*(M)$.*

Proof. Clearly $\delta'(M) \subseteq \delta^*(M)$. Let N be a submodule of M. Then, by Lemma 3.7, $\mathcal{V}(N) = \mathcal{V}^*((N : M)M)$. Thus $\delta(M) \subseteq \delta'(M)$. Also, by Lemma 3.5 (2), we have $\mathcal{V}(IM) = \mathcal{V}^*(IM)$ for any ideal I of R, which means that $\delta'(M) \subseteq \delta(M)$. □

4. Relating RSpec(*M*), Spec(*M*) and Spec(*R*/ Ann(*M*))

Throughout the rest of this paper, we assume that $\mathrm{RSpec}(M)$ is non-empty, unless stated otherwise, and is equipped with the radical-Zariski topology for every R-module M under consideration. We will use $\mathcal{X}$, X and $X^{\bar{R}}$ to represent $\mathrm{RSpec}(M)$, $\mathrm{Spec}(M)$ and $\mathrm{Spec}(\bar{R})$ respectively, where $\bar{R} = R/\mathrm{Ann}(M)$. Let M be an R-module. In [8] the natural map $\psi : X \to X^{\bar{R}}$ defined by $\psi(P) = \overline{(P : M)}$ has been introduced which is continuous automatically. In [9], various condition for a module M have been given under which ψ is surjective. If ψ is surjective, then ψ is both closed and open [8, Theorem 3.6]. Consequently ψ is bijective if and only if ψ is homeomorphic [8, Corollary 3.7]. It also may be found conditions for a module M for which ψ is injective [1].

We define the mapping $\varphi : \mathcal{X} \to X^{\bar{R}}$ by $\varphi(L) = \overline{\sqrt{(L : M)}}$ and we call φ the *radical natural map* of $\mathcal{X}$.

Proposition 4.1. *Let M be an R-module, L_1 and L_2 be elements of $\mathcal{X}$. Then the following statements are equivalent.*

(1) The radical natural map φ is injective.

(2) If $\mathcal{V}(L_1) = \mathcal{V}(L_2)$, then $L_1 = L_2$.

(3) $|\mathrm{RSpec}_p(M)| \leq 1$, for any $p \in \mathrm{Spec}(R)$.

Proof. (1) ⇒ (2). Since $\mathcal{V}(L_1) = \mathcal{V}(L_2)$, we have $L_1 \in \mathcal{V}(L_2)$ and $L_2 \in \mathcal{V}(L_1)$. Thus $\sqrt{L_1 : M} = \sqrt{L_2 : M}$. This implies that $\varphi(L_1) = \varphi(L_2)$ and the injectivity of φ gives the result. (2) ⇒ (3). Let $p \in \mathrm{Spec}(R)$ and $L, L' \in \mathrm{RSpec}_p(M)$. Thus $\sqrt{(L : M)} = \sqrt{(L' : M)} = p$ and hence $\mathcal{V}(L) = \mathcal{V}(L')$. Now (2) gives that $L = L'$. (3) ⇒ (1). Let $\varphi(L) = \varphi(L')$ which implies that $L, L' \in \mathrm{RSpec}_p(M)$ where $p = \sqrt{(L : M)} = \sqrt{(L' : M)}$. Now by (3), $L = L'$, that is φ is injective. □

Proposition 4.2. *Let M be an R-module. Then the radical natural map φ is surjective if and only if for every prime ideal $p \in X^{\bar{R}}$, $pM \prec_p M$.*

Proof. (⇒). Let $p \in X^{\bar{R}}$. Since φ is surjective, there exists $L \in \mathcal{X}$ such that $p = \sqrt{(L : M)} = (\mathrm{rad}\, L : M)$. Thus we have $pM \subseteq \mathrm{rad}\, L \subset M$ and hence $p \subseteq (pM : M) \subseteq (\mathrm{rad}\, pM : M) \subseteq (\mathrm{rad}\, L : M) = p$. Therefore $p = \sqrt{(pM : M)} = (\mathrm{rad}\, pM : M)$, that is $pM \prec_p M$. (⇐). The converse is obvious. □

Theorem 4.3. *Let M be an R-module. Then the radical natural map $\varphi : \mathcal{X} \to X^{\bar{R}}$ is continuous. Moreover, if φ is surjective, then it is both closed and open.*

Proof. Let I be any ideal of R containing $\mathrm{Ann}(M)$. We will show that $\varphi^{-1}(V^{\bar{R}}(\bar{I})) = \mathcal{V}(IM)$. Let $L \in \varphi^{-1}(V^{\bar{R}}(\bar{I}))$. Then $\varphi(L) = \overline{\sqrt{(L : M)}} \in V^{\bar{R}}(\bar{I})$. So we have $\sqrt{(L : M)} \supseteq I$ and hence by Lemma 2.9, $\sqrt{(L : M)} = (\sqrt{(L : M)}M : M) \supseteq (IM : M)$. Thus $L \in \mathcal{V}(IM)$. The reverse inclusion is clear. Therefore φ is continuous. For the "moreover" statement, let N be a submodule of M. By using Proposition 3.7(3) and the first part, $\varphi(\mathcal{V}(N)) = \varphi(\mathcal{V}((N : M)M)) = \varphi \circ \varphi^{-1}(V(\overline{(N : M)})) = V(\overline{(N : M)})$ and

$$\begin{aligned}\varphi(\mathcal{X} - \mathcal{V}(N)) &= \varphi(\mathcal{X} - \mathcal{V}((N : M)M)) = \varphi(\varphi^{-1}(X^{\bar{R}}) - \varphi^{-1}(V(\overline{(N : M)}))) \\ &= \varphi \circ \varphi^{-1}(X^{\bar{R}} - V(\overline{(N : M)})) = X^{\bar{R}} - V(\overline{(N : M)}).\end{aligned}$$

which show that φ is closed and open, respectively. □

Corollary 4.4. *Let M be an R-module and the corresponding radical natural map $\varphi : \mathcal{X} \to X^{\bar{R}}$ is surjective. Then φ is bijective if and only if φ is homeomorphic.*

A topological space is *connected* if and only if it contains no non-empty proper subset which is both open and closed.

Corollary 4.5. *Let M be an R-module with the surjective radical natural map $\varphi : \mathcal{X} \to X^{\bar{R}}$. Then $\mathcal{X}$ is connected if and only if $X^{\bar{R}}$ is connected.*

Proof. (⇒) Let $\mathcal{X}$ be a connected space. By Theorem 4.3, φ is continuous and hence, this together with surjectivity of φ implies that $X^{\bar{R}}$ is connected. (⇐) Let $X^{\bar{R}}$ be a connected space and assume the contrary. Then there exists a non-empty proper subset Y in $\mathcal{X}$ which is both open and closed. Therefore, by Theorem 4.3, $\varphi(Y)$ is open and closed. Now, it suffices to show that $\varphi(Y)$ is a proper subset of $X^{\bar{R}}$, which implies that $X^{\bar{R}}$ is not connected, a contradiction. Let $\varphi(Y) = X^{\bar{R}}$. Since Y is open, there exists a submodule N of M such that $Y = \mathcal{X} - \mathcal{V}(N)$ and hence by Proposition 4.3, $X^{\bar{R}} = \varphi(Y) = \varphi(\mathcal{X} - \mathcal{V}(N)) = X^{\bar{R}} - V(\overline{(N : M)})$. Therefore, $V(\overline{(N : M)}) = \emptyset$ and hence $\overline{(N : M)} = \bar{R}$. It follows that $N = M$ and so $Y = \mathcal{X} - \mathcal{V}(N) = \mathcal{X} - \emptyset = \mathcal{X}$ which is impossible. Thus $\varphi(Y)$ is a proper subset of $X^{\bar{R}}$ and we are done. □

Corollary 4.6. *Let M be an R-module with the surjective radical natural map $\varphi : \mathcal{X} \to X^{\bar{R}}$. Then both $\mathcal{X}$ and $X^{\bar{R}}$ are connected if R is a local ring or* $\mathrm{Ann}(M)$ *is a prime ideal.*

Proof. If R is a local ring or $\mathrm{Ann}(M)$ is a prime ideal of R, then the only idempotent elements of $\bar{R}$ are $\bar{0}$ and $\bar{1}$. By [3, p.104, Corollary 2 to proposition 15], this is equivalent to the connectedness of $X^{\bar{R}}$. Now by Corollary 4.5, we have the result. □

Corollary 4.7. *Let M be an R-module and ψ be surjective. Then Then $\mathcal{X}$ is connected if and only if X is connected.*

Proof. ψ is surjective and hence by [8, Corollary 3.8], X is connected if and only if $X^{\bar{R}}$ is connected. Also, it is clear that $\varphi|_{\text{Spec}(M)} = \psi$ and hence φ is surjective too. Thus, by Corollary 4.5, $\mathcal{X}$ is connected if and only if $X^{\bar{R}}$ is connected. Hence we have the result. □

Proposition 4.8. *Let M and M' be R-modules, $\mathcal{X} = \text{RSpec}(M)$, $\mathcal{X}' = \text{RSpec}(M')$ and let $f : M \to M'$ be an R-module epimorphism. Then the mapping $\vartheta : \mathcal{X}' \to \mathcal{X}$ defined by $\vartheta(L') = f^{-1}(L')$ is continuous.*

Proof. By Lemma 2.10, ϑ is well-defined. Now let N be any submodule of M and consider the closed set $\mathcal{V}(N)$ in $\mathcal{X}$. By Lemma 3.7 (3), $\mathcal{V}(N) = \mathcal{V}^*(\sqrt{(N : M)}M)$. Hence we have:

$$\begin{aligned} L' \in \vartheta^{-1}(\mathcal{V}(N)) &\Leftrightarrow \vartheta(L') = f^{-1}(L') \in \mathcal{V}(N) = \mathcal{V}^*(\sqrt{(N : M)}M) \\ &\Leftrightarrow f^{-1}(L') \supseteq \sqrt{(N : M)}M \Leftrightarrow L' \supseteq \sqrt{(N : M)}M' \\ &\Leftrightarrow L' \in \mathcal{V}^*(\sqrt{(N : M)}M'). \end{aligned}$$

Thus $\vartheta^{-1}(\mathcal{V}(N)) = \mathcal{V}(\sqrt{(N : M)}M')$ and we are done. □

For each $r \in R$, set $D_r = X^R - V(Rr)$. It is well-known that $\{D_r | r \in R\}$ form a base for the Zariski topology on X in which D_r is quasi-compact [8]. For each $r \in R$, we define $B_r = \mathcal{X} - \mathcal{V}(RrM)$. So, every B_r is an open set of $\mathcal{X}$, $B_0 = \mathcal{X}$ and $B_1 = \emptyset$. We show that $\{B_r | r \in R\}$ is a base for $\mathcal{X}$ and $r \in R$, B_r is quasi-compact provided that the corresponding radical natural map $\varphi : \mathcal{X} \to X^{\bar{R}}$ is surjective.

Proposition 4.9. *Let M be an R-module and $\varphi : \mathcal{X} \to X^{\bar{R}}$ be the corresponding radical natural map and $r, s \in R$. Then*

(1) $\varphi^{-1}(D_{\bar{r}}) = B_r$.

(2) $\varphi(B_r) \subseteq D_{\bar{r}}$. *Moreover, the equality holds if φ is surjective.*

(3) $B_{rs} = B_r \cap B_s$.

Proof. (1) $\varphi^{-1}(D_{\bar{r}}) = \varphi^{-1}(X^R - V(\bar{R}\bar{r})) = \varphi^{-1}(X^R) - \varphi^{-1}(V(\bar{R}\bar{r})) = \mathcal{X} - \mathcal{V}(RrM) = B_r$. (2) is clear by (1). (3) By using (1), we have,

$$B_{rs} = \varphi^{-1}(D_{\bar{r}\bar{s}}) = \varphi^{-1}(D_{\bar{r}} \cap D_{\bar{s}}) = \varphi^{-1}(D_{\bar{r}}) \cap \varphi^{-1}(D_{\bar{s}}) = B_r \cap B_s.$$

□

Proposition 4.10. *Let M be an R-module and $B = \{B_r : r \in R\}$. Then B forms a base for the radical-Zariski topology on $\mathcal{X}$.*

Proof. The case $\mathcal{X} = \emptyset$ is trivial. Let $\mathcal{X} \neq \emptyset$ and U be any open set in $\mathcal{X}$. Then, by Corollary 3.8, $U = \mathcal{X} - \mathcal{V}(IM)$ for some ideal I of R. Now by Lemma 3.5 (4), we have:

$$U = \mathcal{X} - \mathcal{V}(IM) = \mathcal{X} - \mathcal{V}(\sum_{a \in I} RaM) = \mathcal{X} - \bigcap_{a \in I} \mathcal{V}(RaM) = \bigcup_{a \in I} (\mathcal{X} - \mathcal{V}(RaM)) = \bigcup_{a \in I} B_a.$$

This proves that B is a base for the radical-Zariski topology. □

Theorem 4.11. *Let M be an R-module and the corresponding radical natural map $\varphi : \mathcal{X} \to X^{\bar{R}}$ be surjective. Then for each $r \in R$, B_r is quasi-compact. In particular, $\mathcal{X}$ is quasi-compact.*

Proof. Without loss of generality, we may assume that $B_r \subseteq \bigcup_{\lambda \in \Lambda} B_{r_\lambda}$ is an open cover of B_r. Then

$$D_{\bar{r}} = \varphi(B_r) \subseteq \varphi(\bigcup_{\lambda \in \Lambda} B_{r_\lambda}) = \bigcup_{\lambda \in \Lambda} \varphi(B_{r_\lambda}) = \bigcup_{\lambda \in \Lambda} D_{\bar{r}_\lambda}.$$

Now, since $D_{\bar{r}}$ is quasi-compact, there exists a finite subset Λ' of Λ such that $D_{\bar{r}} \subseteq \bigcup_{\lambda' \in \Lambda'} D_{\bar{r}_{\lambda'}}$. Therefore, by Proposition 4.9(1),

$$B_r = \varphi^{-1}(D_{\bar{r}}) \subseteq \varphi^{-1}(\bigcup_{\lambda' \in \Lambda'} D_{\bar{r}_{\lambda'}}) = \bigcup_{\lambda' \in \Lambda'} \varphi^{-1}(D_{\bar{r}_{\lambda'}}) = \bigcup_{\lambda' \in \Lambda'} B_{\bar{r}_{\lambda'}}.$$

Thus B_r is quasi-compact. The "in particular" part is clear. □

As we mention in the proof of corollary 4.7, surjectivity of ψ implies the surjectivity of φ and hence by using [8, Proposition 3.3 and Proposition 3.5], we have the following:

Corollary 4.12. *Let M be an R-module. Then $\mathcal{X}$ is quasi-compact in each of the following cases:*

(1) M is a nonzero finitely generated R-module.

(2) M is a nonzero faithfully flat R-module.

(4) $pM_p \neq M_p$ for every prime ideal p of R with $p \supseteq \mathrm{Ann}(M)$.

Theorem 4.13. *Let M be an R-module and the corresponding radical natural map $\varphi : \mathcal{X} \to X^{\bar{R}}$ be surjective. Then the quasi-compact open sets of $\mathcal{X}$ are closed under finite intersection and form an open base.*

Proof. Let $U = U_1 \cap U_2$ where U_1 and U_2 are two quasi-compact open sets of $\mathcal{X}$. By Proposition 4.9 and Theorem 4.11, it is easily seen that $U = \bigcup_{i=1}^{n} B_{r_i}$ for some $r_i \in R$ and so U is quasi-compact by Theorem 4.11. □

5. Irreducible closed subsets and spectral space

Let M be an R-module and let RSpec(M) be endowed with the radical-Zariski topology. For each subset Y of RSpec(M), we will denote the closure of Y in RSpec(M) by $\overline{Y}$, and intersection of all elements of Y by $\Im(Y)$ (note that if $Y = \emptyset$, then $\Im(Y) = M$).

A topological space W is said to be *irreducible* if for any decomposition $W = W_1 \cup W_2$ with closed subsets W_1 and W_2 of W, we have $W_1 = W$ or $W_2 = W$. A subset W_0 of W is irreducible if it is irreducible as a subspace of W. An irreducible component of a topological space W is a maximal irreducible subset of W.

Proposition 5.1. *Let R be an EZ-ring, M be an R-module and $Y \subseteq \mathcal{X}$. Then*

(1) $\mathcal{V}(\Im(Y)) = \overline{Y}$.

(2) Y is closed if and only if $\mathcal{V}(\Im(Y)) = Y$.

Proof. (1) For any $L \in Y$, we have $L \supseteq \Im(Y)$ and hence $\sqrt{L : M} \supseteq \sqrt{\Im(Y) : M}$. Thus $L \in \mathcal{V}(\Im(Y))$. It follows that $Y \subseteq \mathcal{V}(\Im(Y))$. We show that $\mathcal{V}(\Im(Y))$ is the smallest closed set in $\mathcal{X}$ containing Y and hence $\mathcal{V}(\Im(Y)) = \overline{Y}$. Let N be a submodule of M such that $Y \subseteq \mathcal{V}(N)$ and let $L \in \mathcal{V}(\Im(Y))$. Then, by using Lemma 2.12, we have

$$\sqrt{(L : M)} \supseteq \sqrt{(\mathcal{V}(\Im(Y)) : M)} \supseteq \sqrt{(N : M)}.$$

Hence $\mathcal{V}(\Im(Y)) \subseteq \mathcal{V}(N)$ and we are done. (2) is clear by (1). □

By using the first part of the above proposition, we have the following corollary:

Corollary 5.2. *Let R be an EZ-ring and M an R-module and Y be the singleton set $\{L\}$, for some $L \in \mathcal{X}$. Then*

(1) $\overline{Y} = \mathcal{V}(L)$.

(2) $L' \in \overline{Y}$ if and only if $\sqrt{(L' : M)} \supseteq \sqrt{(L : M)}$ if and only if $\mathcal{V}(L') \subseteq \mathcal{V}(L)$.

(3) Y is closed if and only if

(a) $p = \sqrt{L : M}$ *is a maximal element in the set* $\{q = \sqrt{(L' : M)} : L' \in \mathcal{X}\}$.

(b) $\text{RSpec}_p(M) = Y$, *where* $p = \sqrt{(L : M)}$.

(4) $\mathcal{V}(L)$ *is an irreducible closed subset of* $\mathcal{X}$.

Proof. (1) is clear by Proposition 5.1 (1). (2) is clear. (3) (⇒) Since Y is closed, by (1), we have $Y = \overline{Y} = \mathcal{V}(L)$. Let $L' \in \mathcal{X}$ and $q = \sqrt{(L' : M)} \supseteq \sqrt{(L : M)} = p$. Then $L' \in \mathcal{V}(L) = Y = \{L\}$ and hence $L' = L$ which implies that $q = p$. This proves (*a*). Now, let $L' \in \text{RSpec}_p(M)$. Then $\sqrt{(L' : M)} = p = \sqrt{(L : M)}$ and hence $L' \in \mathcal{V}(L) = \{L\}$. Thus $L' = L$, so we get (*b*). (⇐) Let $L' \in \mathcal{V}(L)$. By (*a*), $\sqrt{(L' : M)} = P$ and by (*b*), $L' = L$. Thus $\overline{Y} = \mathcal{V}(L) \subseteq Y$. So $\overline{Y} = Y$ and hence Y is closed. (4) It is easily seen that the closure of a singleton subset of a topological space is irreducible; so by (1), we have the result. □

Lemma 5.3. *Let R be a ring and* $Y \subseteq X^R$. *Then Y is irreducible if and only if* $\Im(Y)$ *is a prime ideal of R.*

Proof. [3, p. 102, Proposition 14]. □

Theorem 5.4. *Let R be an EZ-ring, M be an R-module and* $Y \subseteq \mathcal{X} = \text{RSpec}(M)$. *Then*

(1) If $\Im(Y) \prec_p M$, *then Y is irreducible.*

(2) If R is a zero-dimensional ring and Y is irreducible, then the subset $S = \{\sqrt{(L : M)} : L \in Y\}$ *of* X^R *is irreducible.*

Proof. (1) Since $\Im(Y) \prec_p M$, by Corollary 5.2 (4), $\mathcal{V}(\Im(Y))$ is irreducible. Thus by Proposition 5.1 (1), $\overline{Y}$ is irreducible and hence Y is irreducible. (2) Let Y be irreducible. Then $\varphi(Y)$ is irreducible. Now, by Lemma 5.3, $\Im(\varphi(Y)) = \overline{\Im(S)}$ is a prime ideal of $\bar{R}$. Therefore, by Lemma 2.12, $\Im(S) = \sqrt{\Im(Y) : M}$ is a prime ideal of R and hence by Lemma 5.3, S is irreducible. □

Corollary 5.5. *Let R be an EZ-ring, M be an R-module and* $p \in X^R$ *such that* $\text{RSpec}_p(M) \neq \emptyset$. *Then,* $\text{RSpec}_p(M)$ *is an irreducible subset of* $\mathcal{X}$. *Moreover, if p is a maximal ideal of R, then* $\text{RSpec}_p(M)$ *is a closed subset of* $\mathcal{X}$.

Proof. Similar to the proof of Lemma 2.13, we can see that $\Im(\text{RSpec}_p(M)) \prec_p M$. Hence by Theorem 5.4(1), $\text{RSpec}_p(M)$ is irreducible in $\mathcal{X}$. Now, let p be a maximal ideal of R. It is easily seen that $\text{RSpec}_p(M) = \mathcal{V}(pM)$, i.e., $\text{RSpec}_p(M)$ is closed. □

Proposition 5.6. *Let R be an EZ-ring, M be an R-module and Y be a subset of* $\text{RSpec}(M)$ *such that* $\sqrt{\Im(Y) : M} = p$ *is a prime ideal of R and let* $\text{RSpec}_p(M) \neq \emptyset$. *Then Y is irreducible.*

Proof. Let $L \in \text{RSpec}_p(M)$. Then $p = \sqrt{L : M} = \sqrt{\Im(Y) : M}$. By using Lemma 3.7(2) and Proposition 5.1(1), we have $\mathcal{V}(L) = \mathcal{V}(\Im(Y)) = \overline{Y}$. Now, by Proposition 5.1(4), $\overline{Y}$ and hence Y is irreducible. □

Theorem 5.7. *Let R be an EZ-ring, M be an R-module and* $Y \subseteq \mathcal{X}$. *Let the radical natural map* $\varphi : \mathcal{X} \to X^{\bar{R}}$ *be surjective. Then Y is an irreducible closed subset of* $\mathcal{X}$ *if and only if* $Y = \mathcal{V}(L)$ *for some* $L \in \mathcal{X}$.

Proof. (⇒) Let Y be an irreducible closed subset of $\mathcal{X}$. Then $Y = \mathcal{V}(N)$ for some submodule N of M. By Theorem 5.4 and Lemma 5.3, $\sqrt{(\Im(\mathcal{V}(N)) : M)} = \sqrt{\Im(Y) : M} = \bigcap_{L_i \in Y} \sqrt{(L_i : M)} = p$ for some prime ideal p of R. Since φ is surjective, there exists a submodule $L \in \mathcal{X}$ such that $\sqrt{(L : M)} = p$. Now, by Lemma 3.7(2), we have $\mathcal{V}(\Im(\mathcal{V}(N)) = \mathcal{V}(L)$ and so by Proposition 5.1(2), $\mathcal{V}(N) = \mathcal{V}(L)$.

(⇐) Let $Y = \mathcal{V}(L)$ for some $L \in \mathcal{X}$. Then by Proposition 5.2(4), Y is irreducible. □

Let W_0 be a subset of a topological space W. An element $w \in W_0$ is called a *generic point* of W_0 if $W_0 = \overline{\{w\}}$.

Corollary 5.8. *Let R be an EZ-ring, M be an R-module and the radical natural map $\varphi : \mathcal{X} \to X^{\bar{R}}$ be surjective. Then every irreducible closed subset of $\mathcal{X}$ has a generic point.*

Proof. Let Y be an irreducible closed subset of $\mathcal{X}$. Then by Theorem 5.7, $Y = \mathcal{V}(L)$ for some $L \in \mathcal{X}$. Thus L is a generic point of Y. □

Corollary 5.9. *Let M be a module over an EZ-ring R and the corresponding radical natural map $\varphi : \mathcal{X} \to X^{\bar{R}}$ be surjective. Let A and B denote the set of all irreducible closed sets and irreducible components of $\mathcal{X}$ respectively, and $Min(R)$ denotes the set of all minimal prime ideals of R. Then,*

(1) $f : \mathcal{X} \to A$ defined by $f(L) = \mathcal{V}(L)$ is a surjection.

(2) $g : B \to Min(\bar{R})$ defined by $g(\mathcal{V}(L)) = \overline{\sqrt{L : M}}$ is a bijection.

Proof. (1) Let $Y \in A$. Then by Theorem 5.7, there exists $L \in \mathcal{X}$ such that $Y = \mathcal{V}(L)$; so f is surjective. (2) Let Y be an irreducible component of $\mathcal{X}$. Then by Theorem 5.7, Y is a maximal element of $\{\mathcal{V}(L) : L \in \mathcal{X}\}$. Thus $Y = \mathcal{V}(L)$ for some $L \in \mathcal{X}$ and $\sqrt{L : M}$ is a minimal prime ideal of R containing $\mathrm{Ann}(M)$. Hence g is well-defined as well as being bijection. □

A topological space W is called a T_0-*space* (or a *Kolmogorov space*) if for every pair of distinct points $w_1, w_2 \in W$ there exists an open subset of W containing exactly one of these points. Equivalently, for every pair of distinct points $w_1, w_2 \in W$ there exists a closed subset of W containing exactly one of these points. It is easily verified that, a topological space W is a T_0-space if and only if the closures of distinct points are distinct.

Example 5.10. *Let V be a vector space over a field F and $dim(V) \geq 2$. Let u and v be two distinct elements of V. By* Example 3.1(1)*, it is clear that there does not exist any closed subset of $\mathcal{X}$ containing exactly one of these points. Thus $\mathcal{X}$ is not a T_0-space. It is obvious that, $\mathcal{X}$ is a T_0-space if and only if $dim(V) \leq 1$.*

A *spectral space* is a topological space homeomorphic to the prime spectrum of a commutative ring equipped with the Zariski topology. By Hochster's characterization [6, p.52, Proposition 4], a topology τ on a set W is spectral if and only if the following axioms hold:

(1) W is a T_0-space.

(2) W is quasi-compact and has a basis of quasi-compact open subsets.

(3) The quasi-compact open subsets of W are closed under finite intersection and form an open base.

(4) W is a sober space (i.e., every irreducible closed subset of W has a generic point.)

Now, let R be a ring and M be an R-module. Then, it is well-known that X^R satisfies the above conditions (for example, see [3, Chap.II, 4.1 - 4.4]). If the radical natural map $\varphi : \mathcal{X} \to X^{\bar{R}}$ is surjective, then (2), (3) and (4) (Of course, if R is an EZ-ring, in this case) of Hochster's characterization hold for $\mathcal{X}$ by Theorem 4.11, Theorem 4.13 and Corollary 5.8, respectively. However, according to the Example 5.10, $\mathcal{X}$ is not always a T_0-space even if φ is surjective. Hence, if φ is surjective, then $\mathcal{X}$ is a spectral space if and only if it is a T_0-space with respect to the radical-Zariski topology.

Theorem 5.11. *Let M be an R-module. Then the following statements are equivalent:*

(1) $\mathcal{X}$ is a T_0-space.

(2) If $\mathcal{V}(L_1) = \mathcal{V}(L_2)$, then $L_1 = L_2$ for every elements $L_1, L_2 \in \mathcal{X}$.

(3) The radical natural map $\varphi : \mathcal{X} \to X^{\bar{R}}$ is injective.

(4) For any $p \in X^R$, $\mathrm{RSpec}_p(M) = \emptyset$ or $|\mathrm{RSpec}_p(M)| = 1$.

Moreover, If R is an EZ-ring and the corresponding radical natural map $\varphi : \mathcal{X} \to X^{\bar{R}}$ is surjective, then each of statements (1) - (4) *above is equivalent to the following statements:*

(5) $\mathcal{X}$ is a spectral space

(6) $\mathcal{X}$ is homeomorphic to $\mathrm{Spec}(\bar{R})$ *under φ*

Proof. (1) ⇒ (2) Let $\mathcal{X}$ be a T_0-space, $L_1, L_2 \in \mathcal{X}$ and $\mathcal{V}(L_1) = \mathcal{V}(L_2)$. Then, by Corollary 5.2 (1), $\overline{\{L_1\}} = \overline{\{L_2\}}$. and since $\mathcal{X}$ is a T_0-space, we have $L_1 = L_2$. (2) ⇒ (1) Let L_1 and L_2 be two distinct points of $\mathcal{X}$ and $\overline{\{L_1\}} = \overline{\{L_2\}}$. Then by Corollary 5.2 (1), we have $\mathcal{V}(L_1) = \mathcal{V}(L_2)$. Now, the assumption (2) gives the result. The equivalence of (2), (3) and (4) is proved in Proposition 4.1. For the "Moreover" statement, (1) ⇔ (5) is clear by the above argument. (3) ⇔ (6) By Corollary 4.4. □

Theorem 5.12. *Let M be an R-module and $\varphi : \mathcal{X} \to X^{\bar{R}}$ denote the corresponding radical natural map such that $\varphi(\mathcal{X})$ is a closed subset of $X^{\bar{R}}$. Then $\mathcal{X}$ is a spectral space if and only if φ is injective.*

Proof. (⇒) Let $\mathcal{X}$ be a spectral space. Then it is a T_0-space and hence φ is injective by Theorem 5.11. (⇐) Since every closed subset of a spectral space is again a spectral one for the induced topology, we conclude that $Y = \varphi(\mathcal{X})$ is a spectral space for the induced topology. By Theorem 4.3 the bijection $\varphi : \mathcal{X} \to Y$ is continuous. Now, let N be a submodule of M and consider the closed subset $Y' = Y \cap V(\overline{(N : M)})$. We have $\varphi^{-1}(Y') = \varphi^{-1}(Y \cap V(\overline{(N : M)})) = \varphi^{-1}(Y) \cap \varphi^{-1}(V(\overline{(N : M)})) = \mathcal{X} \cap \mathcal{V}((N : M)M) = \mathcal{V}(N)$. Since φ is surjective, $\varphi(\mathcal{V}(N)) = \varphi(\varphi^{-1}(Y')) = Y'$. Thus $\varphi : \mathcal{X} \to Y$ is a homeomorphism and hence $\mathcal{X}$ is a spectral space. □

Theorem 5.13. *Let M be an R-module such that $\mathcal{X}$ is a non-empty finite set. Then $\mathcal{X}$ is a spectral space if and only if* $|\mathrm{RSpec}_p(M)| \leq 1$ *for every* $p \in \mathrm{RSpec}(R)$.

Proof. Since $|\mathcal{X}|$ is finite, then the conditions (2) and (3) described in Hochster's characterization of spectral spaces hold. For (4), let $Y = \{y_1, y_2, ..., y_k\}$ be an irreducible closed subset of $\mathcal{X}$. Since $Y = \{y_1\} \cup \{y_2\} \cup ... \cup \{y_k\} = \overline{\{y_1\}} \cup \overline{\{y_2\}} \cup ... \cup \overline{\{y_k\}}$, we have $Y = \overline{\{y_i\}}$ for some i as Y is irreducible. Hence $\mathcal{X}$ is a spectral space if and only if $\mathcal{X}$ is a T_0-space, which is equivalent to that $|\mathrm{RSpec}_p(M)| \leq 1$ for every $p \in \mathrm{RSpec}(R)$ by Theorem 5.11. □

References

[1] H. Ansari-Toroghy, R. Ovlyaee-Sarmazdeh, *On the prime spectrum of X-injective modules*, Comm. Algebra 38 (2010) 2606-2621.
[2] J. Brewer, F. Richman, Subrings of zero-dimensional rings. In: *Multiplicative Ideal Theory in Commutative Algebra. A Tribute to the Work of Robert Gilmer.* Brewer, J. W., Glaz, S., Heinzer, W. J., Olberding, B. M., Eds.; Springer, pp. 73-88, 2006.
[3] N. Bourbaki, *Commutative algebra, Chapters 1-7*, Springer-Verlag, 1989.
[4] J. Dauns, *Prime submodules*, Reine angew Math. 298 (1978) 156-181.
[5] Z. A. El-Bast, P. F. Smith, *Multiplication modules*, Comm. Algebra 16 (1988) 755-779.
[6] M. Hochster, *Prime ideal structure in commutative rings*, Trans. Amer. Math. Soc. 142 (1969) 43-60.
[7] C. P. Lu, *Prime submodules of modules*, Comment. Math. Univ. St. Paul. 33 (1984) 61-69.
[8] C. P. Lu, *The Zariski topology on the prime specrtum of a module*, Houston J. Math. 25 (1999) 417-432.
[9] C. P. Lu, *A module whose prime spectrum has the surjective natural map*, Houston J. Math. 33 (2007) 125-142.
[10] R. L. McCasland, M. E. Moore, *On radicals of submodules*, Comm. Algebra 19 (1991) 1327-1341.
[11] R. L. McCasland, M. E. Moore, P. F. Smith, *On the spectrum of a module over a commutative Ring*, Comm. Algebra 25 (1991) 79-103.
[12] R. L. McCasland, M. E. Moore, *Prime submodules*, Comm. Algebra 6 (1992) 1803-1817.
[13] M. E. Moore, S. J. Smith, *Prime and radical submodules of modules over commutative rings*, Comm. Algebra 30 (2002) 5037-5064.
[14] H. Sharif, Y. Sharifi, S. Namazi, *Rings satisfying the radical formula*, Acta Math. Hungar. 71 (1996) 103-108.