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Brück Conjecture and Homogeneous Differential Polynomial

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Abstract. In connection to Brück conjecture we prove a uniqueness theorem for entire functions concerning homogeneous differential polynomials.

1. Introduction, Definitions and Results

Let f, g and a be entire functions. If $f-a$ and $g-a$ have the same set of zeros with the same multiplicities, then f and g are said to share the function a CM (counting multiplicities). If a is a constant, then f and g are said to share the value a CM.

We denote by $M(r, f)$ the maximum modulus function of f. The order $\sigma(f)$ of f is defined as

$$\sigma(f) = \limsup_{r\to\infty} \frac{\log\log M(r,f)}{\log r}.$$

Also the hyper-order of f is defined as

$$\sigma_2(f) = \limsup_{r\to\infty} \frac{\log\log\log M(r,f)}{\log r}.$$

In 1977 L. A. Rubel and C. C. Yang [10] first considered the problem of value sharing by an entire function with its derivative. Inspired by their work a lot of researchers devoted themselves to explore such problems and extensions to different directions. In 1996 R. Brück [1] proposed the following conjecture:
Brück's Conjecture: Let f be a nonconstant entire function such that $\sigma_2(f)$ is not a positive integer or infinity. If f and $f^{(1)}$ share one finite value a CM, then $f^{(1)} - a = c(f - a)$ for some nonzero constant c.

R. Brück [1] himself resolved the conjecture for $a = 0$ but the case $a \neq 0$ is yet to be fully resolved.

For an entire function of finite order, G. G. Gundersen and L. Z. Yang [5] and L. Z. Yang [12] proved the following results.

Theorem 1.1. *[5] Let f be a nonconstant entire function of finite order. If f and $f^{(1)}$ share one finite value a CM, then $f^{(1)} - a = c(f - a)$ for some nonzero constant c.*

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Theorem 1.2. *[12] Let f be a nonconstant entire function of finite order. If f and $f^{(k)}$ share one finite value a CM, then $f^{(k)} - a = c(f - a)$ for some nonzero constant c, where k is a positive integer.*

In 2009 J. M. Chang and Y. Z. Zhu [2] considered the problem of a function sharing, instead of a value sharing, and proved the following result.

Theorem 1.3. *[2] Let f and a be two entire functions such that $\sigma(a) < \sigma(f) < \infty$. If f and $f^{(1)}$ share the function a CM, then $f^{(1)} - a = c(f - a)$ for some nonzero constant c.*

Considering $f = e^{2z} - (z-1)e^z$ and $a = e^{2z} - ze^z$, it is shown in [2] that the condition $\sigma(a) < \sigma(f)$ is crucial.

Brück's conjecture has also been generalised to linear differential polynomials by Z. Mao [9], H. Y. Xu and L. Z. Yang [11] and others.

In the paper we extend Theorem 1.3 to a homogeneous differential polynomial with polynomial coefficients.

Let f be an entire function and $a_1, a_2, \ldots, a_p$ be polynomials. An expression of the form

$$P[f] = \sum_{j=1}^{p} a_j \,(f)^{n_{j0}} \left(f^{(1)}\right)^{n_{j1}} \cdots \left(f^{(m_j)}\right)^{n_{jm_j}} \tag{1.1}$$

is called a homogeneous differential polynomial of degree n, where $n_{jk}(k = 0, 1, 2, \ldots, m_j\,; j = 1, 2, \ldots, p)$ are nonnegative integers satisfying $\sum_{k=0}^{m_j} n_{jk} = n$ for $j = 1, 2, \ldots, p$.

The number $\Gamma_j = \sum_{k=0}^{m_j}(k+1)n_{jk}$ is called the weight of the differential monomial $a_j\,(f)^{n_{j0}} \left(f^{(1)}\right)^{n_{j1}} \cdots \left(f^{(m_j)}\right)^{n_{jm_j}}$. Also the number $\Gamma_P = \max\{\Gamma_j : 1 \le j \le p\}$ is called the weight of $P[f]$ {see [4]}.

In the paper we denote by

$$Q[f] = b\,(f)^{q_0} \left(f^{(1)}\right)^{q_1} \cdots \left(f^{(l)}\right)^{q_l}, \tag{1.2}$$

where b is a polynomial, a differential monomial of degree n and weight Γ_Q.

Let $P[f]$ be given by (1.1). There exists(exist) one(more than one) term(terms) in $P[f]$ with $\Gamma_j = \Gamma_P$. Then we denote by $a = a(z)$ that coefficient a_j of these terms such that a_j has the maximum degree among those coefficients. If there exist more than one such a_j with maximum degree, then we denote by $a = a(z)$ any one of them.

Further, let $N = \{j : 1 \le j \le p \text{ and } \Gamma_j \ne \Gamma_P\}$ and $\chi_j = \frac{\deg a_j - \deg a}{\Gamma_P - \Gamma_j}$ if $j \in N$ and $\chi_j = 0$ if $j \in \{1, 2, \ldots, p\} \setminus N$. We note that if $j \in N$, then $\deg a_j$ is not necessarily less than or equal to $\deg a$, but if $j \in \{1, 2, \ldots, p\} \setminus N$, then we have $\deg a_j \le \deg a$.

We now state the main result of the paper.

Theorem 1.4. *Let f, α_1, α_2 be three entire functions such that $\sigma(\alpha_j) < \sigma(f) < \infty$ for $j = 1, 2$. Suppose that $P[f]$ and $Q[f]$ are given by* (1.1) *and* (1.2) *respectively such that* $\deg b \le \deg a$ *and* $\Gamma_P > \Gamma_Q$.

Let $\sigma(f) > 1 + \max_{1 \le j \le p}\{\chi_j, 0\}$ and $A = A(z)$ be a polynomial such that f satisfies the following differential equation

$$P[f] - \alpha_1 = e^A \left(Q[f] - \alpha_2\right).$$

Then A is a constant.

Remark 1.5. *If $\sigma(f) < 1$, then $\dfrac{P[f] - \alpha_1}{Q[f] - \alpha_2} = e^A$ easily implies that A is a constant.*

Remark 1.6. *If $P[f]$ is a differential monomial, then the proof of Theorem 1.4 reveals that the hypothesis on the order of f can be removed.*

Remark 1.7. *Following example shows that the hypothesis on the order of f is crucial for a homogeneous differential polynomial.*

Example 1.8. *[9] Let* $f = e^{-\frac{z^2}{2}} + z^2, \alpha_1 = \alpha_2 = z^2, P[f] = \frac{1}{3}f^{(2)} + \frac{z}{3}f^{(1)} + \frac{1}{3}f$ *and* $Q[f] = f$. *Then* $\sigma(f) = 2 = 1 + \max_{1\le j\le 3}\{\chi_j, 0\}$ *and* $P[f] - \alpha_1 = \frac{2}{3}e^{\frac{z^2}{2}}(Q[f] - \alpha_2)$.

For an entire function f we denote by $\nu(r, f)$ the central index of f {see p. 50 [8]}.

2. Lemmas

In this section we present some necessary lemmas.

Lemma 2.1. {*p.51 [8]*} *If f is an entire function of order $\sigma(f)$, then*

$$\sigma(f) = \limsup_{r\to\infty} \frac{\log^+ \nu(r, f)}{\log r}.$$

Lemma 2.2. {*p.9 [8]*} *Let* $P(z) = b_n z^n + b_{n-1}z^{n-1} + \cdots + b_0$ $(b_n \neq 0)$ *be a polynomial of degree n. Then for every* $\varepsilon(> 0)$ *there exists* $R(> 0)$ *such that for all* $|z| = r > R$ *we get*

$$(1 - \varepsilon)|b_n|r^n \le |P(z)| \le (1 + \varepsilon)|b_n|r^n.$$

Lemma 2.3. {*p.51 [8]*} *Let f be a transcendental entire function. Then there exists a set* $E_1 \subset (1, \infty)$ *with finite logarithmic measure such that for* $|z| = r \notin [0, 1] \cup E_1$ *and* $|f(z)| = M(r, f)$ *we get*

$$\frac{f^{(j)}(z)}{f(z)} = (1 + o(1))\left\{\frac{\nu(r, f)}{z}\right\}^j$$

for $j = 1, 2, 3, \ldots, k$, *where k is a positive integer.*

Lemma 2.4. {*[6, 7] see also [3]*} *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *be an entire function,* $\mu(r, f) = \max\{|a_n|r^n : n = 0, 1, 2, \ldots\}$ *be the maximum term and* $\nu(r, f) = \max\{n : \mu(r, f) = |a_n|r^n\}$ *be the central index. Then*

(*i*) $\log \mu(r, f) = \log |a_0| + \int_0^r \frac{\nu(t, f)}{t}dt$, *where* $a_0 \neq 0$;

(*ii*) *for* $r < R$

$$M(r, f) \le \mu(r, f)\left\{\nu(R, f) + \frac{R}{R - r}\right\}.$$

Lemma 2.5. *Let f be a transcendental entire function and* $E \subset (1, \infty)$ *be a set of finite logarithmic measure. Then there exists a set* $\Omega \subset [1, \infty)$ *of infinite logarithmic measure such that* $E \cap \Omega = \emptyset$ *and*

$$\sigma(f) = \lim_{\substack{r\to\infty \\ r\in\Omega}} \frac{\log \nu(r, f)}{\log r}.$$

Moreover, let α_1 and α_2 be two entire functions such that $\sigma(\alpha_j) < \sigma(f) < \infty$ *for* $j = 1, 2$. *Then there exists a sequence* $\left\{z_k = r_k e^{i\theta_k}\right\}$ *with* $|f(z_k)| = M(r_k, f), \theta_k \in [0, 2\pi), \lim_{k\to\infty} \theta_k = \theta_0 \in [0, 2\pi)$ *and* $r_k \in \Omega$ *such that for any given* $\varepsilon(> 0)$ *and for sufficiently large r_k following hold:*

(*i*) $r_k^{\sigma(f)-\varepsilon} < \nu(r_k, f) < r_k^{\sigma(f)+\varepsilon}$,

(ii) $\dfrac{M(r_k, \alpha_j)}{M(r_k, f)} < \exp\left\{-\dfrac{1}{2} r_k^{\frac{2}{3}\sigma(f)}\right\}$ *for* $j = 1, 2$.

Proof. *Since by Lemma 2.1,* $\sigma(f) = \limsup_{r\to\infty} \dfrac{\log \nu(r, f)}{\log r}$*, there exists a strictly increasing unbounded sequence* $\{\xi_n\}$ *such that*

$$\sigma(f) = \lim_{n\to\infty} \frac{\log \nu(\xi_n, f)}{\log \xi_n}.$$

Let $\delta(< \infty)$ *be the logarithmic measure of* E*. We now choose a subsequence* $\{s_n\}$ *of* $\{\xi_n\}$ *such that*

$$(2 + 2e^{\delta})s_k < s_{k+1}$$

for $k = 1, 2, 3, \ldots$ *and*

$$\sigma(f) = \lim_{k\to\infty} \frac{\log \nu(s_k, f)}{\log s_k}. \tag{2.1}$$

Suppose that $\Omega'_k = \left[s_k, (2 + 2e^{\delta})s_k\right]$ *and* $\Omega' = \bigcup_{k=1}^{\infty} \Omega'_k$*. Since* $(2 + 2e^{\delta})s_k < s_{k+1}$*, we see that* $\Omega'_k \cap \Omega'_{k+1} = \emptyset$ *for* $k = 1, 2, 3, \ldots$.

If $\mu_l(\Omega')$ *denotes the logarithmic measure of* Ω'*, then*

$$\mu_l(\Omega') = \sum_{k=1}^{\infty} \int_{s_k}^{(2+2e^{\delta})s_k} \frac{dt}{t} = \sum_{k=1}^{\infty} \log(2 + 2e^{\delta}) = \infty.$$

Let $\Omega = \Omega' \setminus E = \bigcup_{k=1}^{\infty} \left(\Omega'_k \setminus E\right) = \bigcup_{k=1}^{\infty} \Omega_k$*, where* $\Omega_k = \Omega'_k \setminus E$*. Since* $\mu_l(\Omega') = \infty$ *and* $\mu_l(E) < \infty$*, we see that* $\mu_l(\Omega) = \infty$.

We now verify that $\Omega_k \neq \emptyset$ *for* $k = 1, 2, 3, \ldots$*. If* $\Omega_k = \emptyset$ *for some* k*, then* $[s_k, (2 + 2e^{\delta})s_k] \subset E$ *and so* $\delta = \mu_l(E) \geq \int_{s_k}^{(2+2e^{\delta})s_k} \dfrac{dt}{t} > \log 2 + \delta$*, a contradiction.*

Now for $r \in \Omega'_k$ *we have* $\nu(s_k, f) \leq \nu(r, f)$ *and* $\log r \leq \log s_k \left\{1 + \frac{\log(2+2e^{\delta})}{\log s_k}\right\}$*. Therefore by* (2.1) *we get*

$$\begin{aligned} \sigma(f) &\geq \limsup_{\substack{r\to\infty \\ r\in\Omega}} \frac{\log \nu(r, f)}{\log r} \geq \liminf_{\substack{r\to\infty \\ r\in\Omega}} \frac{\log \nu(r, f)}{\log r} \\ &\geq \lim_{k\to\infty} \frac{\log \nu(s_k, f)}{\log s_k} \cdot \frac{1}{\lim\limits_{k\to\infty} \left[1 + \dfrac{\log(2 + 2e^{\delta})}{\log s_k}\right]} = \sigma(f) \end{aligned}$$

and so

$$\sigma(f) = \lim_{\substack{r\to\infty \\ r\in\Omega}} \frac{\log \nu(r, f)}{\log r}. \tag{2.2}$$

Suppose that for all $\alpha \in \left[\frac{3}{2}, (2 + 2e^{\delta})\right]$ *we have* $\alpha s_k \notin \Omega_k$*. This implies* $\left[\frac{3}{2}s_k, (2 + 2e^{\delta})s_k\right] \setminus E = \emptyset$ *and so* $\left[\frac{3}{2}s_k, (2 + 2e^{\delta})s_k\right] \subset E$ *for some* $k = 1, 2, \ldots$*. Hence*

$$\delta = \mu_l(E) \geq \int_{\frac{3}{2}s_k}^{(2+2e^{\delta})s_k} \frac{dt}{t} = \log\left[\frac{2}{3}(2 + 2e^{\delta})\right] > \log\frac{4}{3} + \delta,$$

a contradiction.

Hence we choose $\alpha_k \in \left[\frac{3}{2}, 2 + 2e^{\delta}\right]$ *such that* $\alpha_k s_k \in \Omega_k$ *for* $k = 1, 2, 3, \ldots$. *Without loss of generality we suppose that* $f(0) \neq 0$. *Then by Lemma 2.4 we get*

$$\begin{aligned}\log \mu(\alpha_k s_k, f) &= \log |f(0)| + \int_0^{\alpha_k s_k} \frac{\nu(t, f)}{t} dt \\ &\geq \log |f(0)| + \int_{s_k}^{\alpha_k s_k} \frac{\nu(t, f)}{t} dt \\ &\geq \log |f(0)| + \nu(s_k, f) \log \alpha_k \\ &\geq \log |f(0)| + \nu(s_k, f) \log \frac{3}{2}\end{aligned}$$

and so

$$\nu(s_k, f) \leq \frac{1}{\log \frac{3}{2}} \left[\log \mu(\alpha_k s_k, f) - \log |f(0)|\right]. \tag{2.3}$$

Using Cauchy's inequality we get

$$\mu(r, f) \leq M(r, f). \tag{2.4}$$

From (2.3) *and* (2.4) *we get for all sufficiently large* k

$$\nu(s_k, f) \leq \frac{2}{\log \frac{3}{2}} \log M(\alpha_k s_k, f). \tag{2.5}$$

We put $r_k = \alpha_k s_k$. *Then* $\{r_k\}$ *is an increasing unbounded sequence in* Ω. *From* (2.5) *we get*

$$\frac{\log \nu(s_k, f)}{\log s_k} \leq \frac{\log \frac{2}{\log \frac{3}{2}}}{\log s_k} + \frac{\log \log M(r_k, f)}{\log r_k \left[1 - \frac{\log \alpha_k}{\log r_k}\right]}.$$

This implies by (2.1) *that*

$$\sigma(f) = \lim_{k \to \infty} \frac{\log \log M(r_k, f)}{\log r_k}. \tag{2.6}$$

Since $\{r_k\} \subset \Omega$, *then from* (2.2) *we obtain*

$$\sigma(f) = \lim_{k \to \infty} \frac{\log \nu(r_k, f)}{\log r_k}$$

from which (*i*) *follows.*

Let $\eta = \sigma(f) - \max\{\sigma(\alpha_1), \sigma(\alpha_2)\} > 0$. *By* (2.6) *there exists a positive integer* p_1 *such that for* $k \geq p_1$ *we get*

$$M(r_k, f) > \exp\left\{r_k^{\sigma(f) - \frac{\eta}{3}}\right\}. \tag{2.7}$$

Also there exists a positive integer p_2 *such that for* $k \geq p_2$ *and* $j = 1, 2$ *we get*

$$M(r_k, \alpha_j) < \exp\left\{r_k^{\sigma(\alpha_j) + \frac{\eta}{3}}\right\}. \tag{2.8}$$

Let $p = \max\{p_1, p_2\}$. *Then from* (2.7) *and* (2.8) *we obtain for* $k \geq p$ *and* $j = 1, 2$

$$\frac{M(r_k, \alpha_j)}{M(r_k, f)} < \exp\left\{r_k^{\sigma(\alpha_j) + \frac{\eta}{3}} - r_k^{\sigma(f) - \frac{\eta}{3}}\right\}. \tag{2.9}$$

Now for all sufficiently large values of k we get

$$\begin{aligned} & r_k^{\sigma(f)-\frac{\eta}{3}} - r_k^{\sigma(\alpha_j)+\frac{\eta}{3}} \\ = & \frac{1}{2} r_k^{\sigma(f)-\frac{\eta}{3}} \left[2 - 2 r_k^{\sigma(\alpha_j)-\sigma(f)+\frac{2\eta}{3}} \right] \\ \geq & \frac{1}{2} r_k^{\sigma(f)-\frac{\eta}{3}} \left[2 - 2 r_k^{-\frac{\eta}{3}} \right] \\ > & \frac{1}{2} r_k^{\sigma(f)-\frac{\eta}{3}} \\ \geq & \frac{1}{2} r_k^{\frac{2}{3}\sigma(f)}. \end{aligned}$$

Therefore from (2.9) *we get for all sufficiently large values of k*

$$\frac{M(r_k, \alpha_j)}{M(r_k, f)} < \exp\left\{ -\frac{1}{2} r_k^{\frac{2}{3}\sigma(f)} \right\} \ \textit{for} \ \ j = 1, 2,$$

which is (*ii*).

Now we choose $\theta_k \in [0, 2\pi)$ *in such a manner that* $|f(r_k e^{i\theta_k})| = M(r_k, f)$*. If necessary, considering a subsequence of* θ_k *we get* $\lim_{k\to\infty} \theta_k = \theta_0 \in [0, 2\pi)$*. This proves the lemma.* □

3. Proof of Theorem 1.4

Proof. Let $P[f] = \sum_{j=1}^{p} P_j[f]$, where $P_j[f] = a_j\,(f)^{n_{j0}} \left(f^{(1)}\right)^{n_{j1}} \cdots \left(f^{(m_j)}\right)^{n_{jm_j}}$ for $j = 1, 2, \ldots, p$.

By Lemma 2.3 there exists $E_1 \subset (1, \infty)$ with finite logarithmic measure such that for $|z| = r \notin E_1 \cup [0, 1]$ and $|f(z)| = M(r, f)$, we get

$$\frac{f^{(j)}(z)}{f(z)} = \left(\frac{\nu(r, f)}{z} \right)^j (1 + o(1)), \tag{3.1}$$

for $j = 1, 2, \ldots, u$, where $u = \max\{l, m_j : j = 1, 2, \ldots, p\}$.

Again we suppose that

$$\frac{P[f] - \alpha_1}{Q[f] - \alpha_2} = e^A, \tag{3.2}$$

where A is a polynomial.

Now for all z with $|z| = r \notin E_1 \cup [0, 1]$ and $|f(z)| = M(r, f)$ we get by (3.1) for $j = 1, 2, \ldots, p$

$$\begin{aligned} \frac{P_j[f]}{f^n} &= a_j \left(\frac{f^{(1)}(z)}{f(z)} \right)^{n_{j1}} \left(\frac{f^{(2)}(z)}{f(z)} \right)^{n_{j2}} \cdots \left(\frac{f^{(m_j)}(z)}{f(z)} \right)^{n_{jm_j}} \\ &= a_j \left(\frac{\nu(r, f)}{z} \right)^{\Gamma_j - n} (1 + o(1)), \end{aligned} \tag{3.3}$$

where $\Gamma_j = \Gamma_{P_j}$ for $j = 1, 2, \ldots, p$.

Similarly for all z with $|z| = r \notin E_1 \cup [0, 1]$ and $|f(z)| = M(r, f)$ we get

$$\frac{Q[f]}{f^n} = b(1 + o(1)) \left(\frac{\nu(r, f)}{z} \right)^{\Gamma_Q - n}. \tag{3.4}$$

From (3.3) we get for all z with $|z| = r \notin E_1 \cup [0, 1]$ and $|f(z)| = M(r, f)$

$$\frac{P[f]}{f^n} = \sum_{j=1}^{p} a_j(1+o(1))\left(\frac{\nu(r,f)}{z}\right)^{\Gamma_j - n}. \tag{3.5}$$

In Lemma 2.5 we choose $E = E_1 \cup [0,1]$. Then by Lemma 2.5 there exists a set $\Omega \subset [1,\infty)$ of infinite logarithmic measure such that $E \cap \Omega = \emptyset$. Also there exists a sequence $\{z_k = r_k e^{i\theta_k}\}$ with $r_k \in \Omega$ such that $|f(z_k)| = M(r_k, f), \theta_k \in [0, 2\pi)$ and $\lim_{k\to\infty} \theta_k = \theta_0 \in [0, 2\pi)$. Further for given $\varepsilon(0 < \varepsilon < 1)$ and for sufficiently large r_k we get

$$r_k^{\sigma-\varepsilon} < \nu(r_k, f) < r_k^{\sigma+\varepsilon} \tag{3.6}$$

and

$$\frac{M(r_k,\alpha_j)}{M(r_k,f)} < \exp\left\{-\frac{1}{2} r_k^{\frac{2}{3}\sigma}\right\} \tag{3.7}$$

for $j = 1, 2$, where $\sigma = \sigma(f)$.

Now by Lemma 2.2 we get from (3.4) and (3.6) for sufficiently large $|z_k| = r_k$

$$\begin{aligned} \left|\frac{Q[f](z_k)}{f^n(z_k)}\right| &= (1+o(1))\left|b(z_k)\left(\frac{\nu(r_k,f)}{z_k}\right)^{\Gamma_Q - n}\right| \\ &\geq M_1 r_k^{\{\deg b + (\sigma-1-\varepsilon)(\Gamma_Q - n)\}}, \end{aligned} \tag{3.8}$$

where M_1 is a positive constant.

Again for sufficiently large $|z_k| = r_k$ we get from (3.7)

$$\left|\frac{\alpha_j(z_k)}{f^n(z_k)}\right| = \frac{|\alpha_j(z_k)|}{\{M(r_k,f)\}^n} \leq \frac{M(r_k,\alpha_j)}{M(r_k,f)} < \exp\left\{-\frac{1}{2} r_k^{\frac{2}{3}\sigma}\right\}, \tag{3.9}$$

for $j = 1, 2$.

Hence for sufficiently large $|z_k| = r_k$ we get from (3.8) and (3.9)

$$\begin{aligned} \left|\frac{Q[f](z_k)}{f^n(z_k)}\right| - \left|\frac{\alpha_2(z_k)}{f^n(z_k)}\right| &> M_1 r_k^{\{\deg b + (\sigma-1-\varepsilon)(\Gamma_Q - n)\}} - \exp\left\{-\frac{1}{2} r_k^{\frac{2}{3}\sigma}\right\} \\ &> M_2 r_k^{\{\deg b + (\sigma-1-\varepsilon)(\Gamma_Q - n)\}}, \end{aligned} \tag{3.10}$$

where M_2 is a positive constant.

From (3.2) we obtain

$$e^{A(z)} = \frac{\frac{P[f]}{f^n} - \frac{\alpha_1}{f^n}}{\frac{Q[f]}{f^n} - \frac{\alpha_2}{f^n}} = F(z), \text{ say.}$$

So $A(z) = \log F(z) = \log|F(z)| + i\mathrm{Arg}F(z)$, where $\mathrm{Arg}F(z)$ is the principal argument of $F(z)$. Therefore for sufficiently large $|z_k| = r_k$ we get

$$\begin{aligned} |A(z_k)| &\leq |\log|F(z_k)|| + |\mathrm{Arg}F(z_k)| \\ &\leq \left|\log \frac{\left|\frac{P[f](z_k)}{f^n(z_k)}\right| + \left|\frac{\alpha_1(z_k)}{f^n(z_k)}\right|}{\left|\frac{Q[f](z_k)}{f^n(z_k)}\right| - \left|\frac{\alpha_2(z_k)}{f^n(z_k)}\right|}\right| + 2\pi. \end{aligned} \tag{3.11}$$

Also by Lemma 2.2 we get for all sufficiently large $|z_k| = r_k$

$$\frac{1}{2}|\beta| r_k^{\deg A} \leq |A(z_k)|, \tag{3.12}$$

where β is the leading coefficient of $A(z)$.

So for sufficiently large $|z_k| = r_k$ we get from (3.11) and (3.12)

$$\frac{1}{2}|\beta| r_k^{\deg A} \leq \left| \log \frac{\left|\frac{P[f](z_k)}{f^n(z_k)}\right| + \left|\frac{\alpha_1(z_k)}{f^n(z_k)}\right|}{\left|\frac{Q[f](z_k)}{f^n(z_k)}\right| - \left|\frac{\alpha_2(z_k)}{f^n(z_k)}\right|} \right| + 2\pi. \tag{3.13}$$

Let $\Gamma_1 = \Gamma_2 = \cdots = \Gamma_{t+1} = \Gamma_P = \Gamma$, say, and $\Gamma_j < \Gamma$ for $j = t+2, t+3, \ldots, p$.

Without loss of generality we suppose that the degrees of no two polynomials of $a_1, a_2, \ldots, a_{t+1}$ are same. Also without loss of generality we assume that $\deg a_{t+1} > \deg a_t > \deg a_j$ for $j = 1, 2, \ldots, t-1$. Then from (3.5) we get for all sufficiently large $|z_k| = r_k$

$$\begin{aligned} \frac{P[f](z_k)}{f^n(z_k)} &= a_t(z_k)(1+o(1))\left\{1 + \sum_{j=1}^{t-1} \frac{a_j(z_k)}{a_t(z_k)}\right\}\left(\frac{\nu(r_k, f)}{z_k}\right)^{\Gamma - n} \\ &\quad + \sum_{j=t+1}^{p} a_j(z_k)(1+o(1))\left(\frac{\nu(r_k, f)}{z_k}\right)^{\Gamma_j - n} \\ &= F_1(z_k) + F_2(z_k), \text{ say.} \end{aligned} \tag{3.14}$$

Since by Lemma 2.2 $\dfrac{a_j(z_k)}{a_t(z_k)} \to 0$ as $k \to \infty$ for $j = 1, 2, \ldots, t-1$, we see that for sufficiently large $|z_k| = r_k$

$$F_1(z_k) = a_t(z_k)(1+o(1))\left(\frac{\nu(r_k, f)}{z_k}\right)^{\Gamma - n}. \tag{3.15}$$

Now

$$F_2(z_k) = \frac{a_{t+1}(z_k)}{z_k^{\Gamma - n}}(1+o(1))\left[(\nu(r_k, f))^{\Gamma - n} + \sum_{j=t+2}^{p} \frac{a_j(z_k)}{a_{t+1}(z_k)} z_k^{\Gamma - \Gamma_j} (\nu(r_k, f))^{\Gamma_j - n}\right]. \tag{3.16}$$

Let $\deg a_s = d_s$ for $s = 1, 2, \ldots, p$. Since $\sigma > 1 + \max\limits_{1 \leq j \leq p}\{\chi_j, 0\} \geq 1 + \dfrac{d_j - d_{t+1}}{\Gamma - \Gamma_j}$ for $j = t+2, \ldots, p$, we choose ε such that

$$0 < \varepsilon < \min_{t+2 \leq j \leq p} \frac{d_{t+1} - d_j + (\sigma - 1)(\Gamma - \Gamma_j)}{\Gamma + \Gamma_j - 2n}.$$

Then by Lemma 2.2 and (3.6) we get for $j = t+2, \ldots, p$

$$\left|\frac{a_j(z_k)}{a_{t+1}(z_k)} z_k^{\Gamma - \Gamma_j} (\nu(r_k, f))^{\Gamma_j - n}\right| \leq M_3 r_k^{\{d_j - d_{t+1} + \Gamma - \Gamma_j + (\sigma + \varepsilon)(\Gamma_j - n)\}}$$

and so

$$\begin{aligned} \frac{\left|\frac{a_j(z_k)}{a_{t+1}(z_k)} z_k^{\Gamma - \Gamma_j} (\nu(r_k, f))^{\Gamma_j - n}\right|}{(\nu(r_k, f))^{\Gamma - n}} &\leq M_3 r_k^{\{d_j - d_{t+1} + \Gamma - \Gamma_j + (\sigma + \varepsilon)(\Gamma_j - n) - (\sigma - \varepsilon)(\Gamma - n)\}} \\ &\to 0 \text{ as } k \to \infty, \end{aligned}$$

because $d_j - d_{t+1} + \Gamma - \Gamma_j + (\sigma + \varepsilon)(\Gamma_j - n) - (\sigma - \varepsilon)(\Gamma - n) < 0$, where M_3 is a positive constant.

Hence for all sufficiently large values of $|z_k| = r_k$ we get for $j = t+2, \ldots, p$

$$\frac{a_j(z_k)}{a_{t+1}(z_k)} z_k^{\Gamma - \Gamma_j} (\nu(r_k, f))^{\Gamma_j - n} = o\left((\nu(r_k, f))^{\Gamma - n}\right).$$

Therefore from (3.16) we obtain

$$F_2(z_k) = \frac{a_{t+1}(z_k)}{z_k^{\Gamma-n}}(1+o(1))(\nu(r_k,f))^{\Gamma-n}. \tag{3.17}$$

So from (3.14), (3.15) and (3.17) we get for all sufficiently large values of $|z_k| = r_k$

$$\frac{P[f](z_k)}{f^n(z_k)} = (a_t(z_k) + a_{t+1}(z_k))\,(1+o(1))\left(\frac{\nu(r_k,f)}{z_k}\right)^{\Gamma-n}. \tag{3.18}$$

Hence by Lemma 2.2 we get from (3.6) and (3.18) for all sufficiently large values of $|z_k| = r_k$

$$M_4 r_k^{\{\deg a_{t+1}+(\sigma-1-\varepsilon)(\Gamma-n)\}} \leq \left|\frac{P[f](z_k)}{f^n(z_k)}\right| \leq M_5 r_k^{\{\deg a_{t+1}+(\sigma-1+\varepsilon)(\Gamma-n)\}}, \tag{3.19}$$

where M_4 and M_5 are positive constants.

Therefore from (3.9) and (3.19) we get for all large values of $|z_k| = r_k$

$$\left|\frac{P[f](z_k)}{f^n(z_k)}\right| + \left|\frac{\alpha_1(z_k)}{f^n(z_k)}\right| \leq M_6 r_k^{\{\deg a_{t+1}+(\sigma-1+\varepsilon)(\Gamma-n)\}} \tag{3.20}$$

and

$$\left|\frac{P[f](z_k)}{f^n(z_k)}\right| + \left|\frac{\alpha_1(z_k)}{f^n(z_k)}\right| \geq M_7 r_k^{\{\deg a_{t+1}+(\sigma-1-\varepsilon)(\Gamma-n)\}}, \tag{3.21}$$

where M_6 and M_7 are positive constants.

Now by Lemma 2.2 we get from (3.4) and (3.6) for sufficiently large $|z_k| = r_k$

$$\left|\frac{Q[f](z_k)}{f^n(z_k)}\right| \leq M_8 r_k^{\{\deg b+(\sigma-1+\varepsilon)(\Gamma_Q-n)\}}, \tag{3.22}$$

where M_8 is a positive constant.

Therefore by (3.9) and (3.22) we get for sufficiently large $|z_k| = r_k$

$$\left|\frac{Q[f](z_k)}{f^n(z_k)}\right| - \left|\frac{\alpha_2(z_k)}{f^n(z_k)}\right| \leq M_9 r_k^{\{\deg b+(\sigma-1+\varepsilon)(\Gamma_Q-n)\}}, \tag{3.23}$$

where M_9 is a positive constant.

Now from (3.21) and (3.23) we get for all sufficiently large $|z_k| = r_k$

$$\frac{\left|\frac{P[f](z_k)}{f^n(z_k)}\right| + \left|\frac{\alpha_1(z_k)}{f^n(z_k)}\right|}{\left|\frac{Q[f](z_k)}{f^n(z_k)}\right| - \left|\frac{\alpha_2(z_k)}{f^n(z_k)}\right|} \geq \frac{M_7}{M_9} r_k^{\{\deg a_{t+1}-\deg b+(\sigma-1-\varepsilon)(\Gamma-n)-(\sigma-1+\varepsilon)(\Gamma_Q-n)\}}, \tag{3.24}$$

where $\deg a_{t+1} - \deg b + (\sigma-1-\varepsilon)(\Gamma-n) - (\sigma-1+\varepsilon)(\Gamma_Q-n) > 0$ for sufficiently small $\varepsilon(>0)$.

Also for sufficiently large values of $|z_k| = r_k$ we obtain from (3.10) and (3.20)

$$\frac{\left|\frac{P[f](z_k)}{f^n(z_k)}\right| + \left|\frac{\alpha_1(z_k)}{f^n(z_k)}\right|}{\left|\frac{Q[f](z_k)}{f^n(z_k)}\right| - \left|\frac{\alpha_2(z_k)}{f^n(z_k)}\right|} \leq \frac{M_6}{M_2} r_k^{\{\deg a_{t+1}-\deg b+(\sigma-1+\varepsilon)(\Gamma-n)-(\sigma-1-\varepsilon)(\Gamma_Q-n)\}}, \tag{3.25}$$

where $\deg a_{t+1} - \deg b + (\sigma-1+\varepsilon)(\Gamma-n) - (\sigma-1-\varepsilon)(\Gamma_Q-n) > 0$ for $\varepsilon(>0)$.

Now in view of (3.24) we get from (3.13) and (3.25) for all sufficiently large values of $|z_k| = r_k$

$$\frac{1}{2}|\beta| r_k^{\deg A} \leq \{\deg a_{t+1} - \deg b + (\sigma-1+\varepsilon)(\Gamma-n) - (\sigma-1-\varepsilon)(\Gamma_Q-n)\}\log r_k + O(1),$$

which implies $\deg A = 0$ and so A is a constant. This proves the theorem. □

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