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Driving Function of the Vertical Slit

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Abstract. We consider the chordal Loewner equation and construct a family of vertical slits γ^p $(p > 0)$. We give the exact expression of its driving function λ, and its Hölder exponent near 0 in terms of p, which maps onto $(1/2, \infty)$ and has a natural connection with the known results. In addition, we extend the asymptotic behavior of the driving function λ to a general case.

1. Introduction

Let $\mathbb{H}$ be the upper half-plane. Suppose for any $T > 0$, $\gamma : [0, T] \to \overline{\mathbb{H}}$ is a simple curve with $\gamma(0) \in \mathbb{R}$ and $\gamma(0, T] \subset \mathbb{H}$. For each $t \in [0, T]$, the region $H_t = \mathbb{H} \setminus \gamma[0, t]$ is a simply connected subdomain of $\mathbb{H}$. There is a unique conformal map g_t from H_t onto $\mathbb{H}$ such that

$$g_t(z) = z + \frac{b(t)}{z} + O(\frac{1}{|z|^2}), \quad \text{as } z \to \infty.$$

If we change the parameterization of γ such that $b(t) = 2t$, then γ is said to be *parameterized by half-plane capacity*. In this case, $g_t(z)$ satisfies the equation

$$\frac{\partial}{\partial t} g_t(z) = \frac{2}{g_t(z) - \lambda(t)}, \quad g_0(z) = z, \tag{1.1}$$

where $\lambda(t) := \lim_{z \to \gamma(t)} g_t(z)$ is a continuous real-valued function. The equation (1.1) is called *(chordal) Loewner (differential) equation*, and g_t are called *Loewner chains*. λ is called the *driving function* or the *Loewner transform*, and γ is called the *trace* or the *Loewner curve*.

On the other hand, given a continuous function $\lambda : [0, T] \to \mathbb{R}$ and $z \in \mathbb{H}$, we can solve the initial value problem (1.1). Let T_z be the supremum of all t such that the solution is well defined up to time t with $g_t(z) \in \mathbb{H}$. Let

$$H_t := \{z \in \mathbb{H} : T_z > t\}.$$

Then g_t is the unique conformal transformation from H_t onto $\mathbb{H}$ with

$$g_t(z) = z + \frac{2t}{z} + O(\frac{1}{|z|^2}), \quad \text{as } z \to \infty.$$

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Let $K_t := \mathbb{H} \setminus H_t$. Then $\{K_t\}_{t\in[0,T]}$ is an increasing family of hulls (defined in Section 2), and we can say that the hulls K_t are *generated by the driving function* λ. In general, it is not true that $K_t = \gamma(0, t]$ for some simple curve γ with $\gamma(0) \in \mathbb{R}$ and $\gamma(0, t] \subset \mathbb{H}$. Marshall and Rohde [10] and Lind [7] proved that H_t is a slit half-plane for all t provided that $\|\lambda\|_{1/2} < 4$. Recall that Lip(1/2) is the space of Hölder continuous functions with exponent 1/2, and $\|\cdot\|_{1/2}$ denotes the seminorm in Lip(1/2).

Recently, many results were given about the exact solutions of Loewner equations. Kager, Nienhuis and Kadanoff [3] considered the driving functions of the forms $c\sqrt{t}$, ct and $c\sqrt{1-t}$, and gave the singular solutions by making use of the implicit functions. Prokhorov and Vasil'ev [11] showed that a tangential slit Γ (circular arc) is generated by a Hölder continuous driving function with exponent 1/3. Lau and Wu [4] constructed a family of tangential slits Γ^p ($p > 0$) by using Γ, and showed that the driving functions have the Hölder exponent $p/(2p + 1)$, which maps $(0, +\infty)$ onto the interval $(0, 1/2)$.

However, the exact solution is less clear when the Hölder exponent of the driving function lies in the interval $(1/2, 1)$. Based on this reason, we will construct a family of Loewner curves, whose driving functions have the Hölder exponent lying in the interval $(1/2, +\infty)$. Now, we introduce them as follows:

$$\gamma^p := \{i^{1-p}(1 + e^{i\theta})^p : \pi(1 - \frac{1}{2}\theta_p) \le \theta \le \pi\}, \tag{1.2}$$

where $\theta_p = 1/p$ if $p \ge 1$; $\theta_p = 1$ if $p \in (0, 1)$ (we define the branch in the domain $\mathbb{C} \setminus (-\infty, 0]$ such that $\arg 1 = 0$). The condition on the angle and the exponent p ensures that the simple curve $\gamma^p \setminus \{0\}$ is a vertical slit contained in the upper half plane. Our main theorem is

Theorem 1.1. *Let γ^p ($p > 0$) be the trace defined by* (1.2). *Then there exists $C > 0$ such that its driving function λ is of the form*

$$\lambda(t) = Ct^{\frac{p+1}{2p}} + o(t^{\frac{p+1}{2p}}), \quad as \ \ t \to 0.$$

We actually prove in Theorem 3.1 for a complete expression of $\lambda(t)$ in terms of a series, and the constant C is also given explicitly. We note that the Hölder exponent of λ decreases from ∞ to 1/2, and remark that the case in [3] is for $p = 1$, the case in [15] is for $p \in [1/3, \infty)$, and the case in [7], [10] and [16] corresponds to $p = \infty$ heuristically.

For $n \in \mathbb{N}$ and $\beta \in [0, 1)$, we say that the function f is in $C^{n,\beta}(0, 1)$ or $C^{n+\beta}(0, 1)$, if the n-th order derivatives of f is local β-Hölder continuous, i.e., for each $x \in (0, 1)$, $f^{(n)}$ is β-Hölder continuous in some neighbourhood of the point x. Making use of Theorem 1.1, we can obtain the following general case.

Theorem 1.2. *Let $f : [0, 1] \to \mathbb{R}$ be a function in $C^{1,\beta}(0, 1)$ for some $\beta \in (0, 1)$, and let the Loewner curve $\gamma(y) = f(y) + iy$. If there exist $a \neq 0$, $r > 1$ and $s > r + 1$ such that*

$$f'(y) = ay^r + O(y^{r+s}), \quad as \ \ y \to 0,$$

then its driving function λ has the expression:

$$\lambda(t) = 2arC\, t^{\frac{r+1}{2}} + o(t^{\frac{r+1}{2}}), \quad as \ \ t \to 0,$$

where C is the same as in Theorem 1.1.

Clearly, the γ^p ($0 < p < 1$) in Theorem 1.1 is the special case with $r = 1/p$. It is well known that for $\alpha > 1/2$, $\gamma \in C^{\alpha+1/2}$ if $\lambda \in C^\alpha$ ([9], [15]); the converse is partly proved by Rohde and Wang [12]. Our theorem is a supplement of this, since it gives a sufficient condition for the case that the exponent of λ is bigger than 1. The main technique of proof is to compare the Loewner curve $\gamma(y)$ with the special vertical slits γ^p, and to obtain the asymptotic expression of λ by using Theorem 4.3 in [8].

2. Preliminaries

We call a bounded subset $A \subset \mathbb{H}$ a *hull* if $A = \mathbb{H} \cap \overline{A}$ and $\mathbb{H} \setminus A$ is simply connected. For each hull A, there is a unique conformal transformation $g_A : \mathbb{H} \setminus A \to \mathbb{H}$ such that $\lim_{z\to\infty}(g_A(z) - z) = 0$ [6]. The *half-plane capacity* is defined by

$$\mathrm{hcap}(A) := \lim_{z\to\infty} z(g_A(z) - z).$$

In other words,

$$g_A(z) = z + \frac{\mathrm{hcap}(A)}{z} + O(\frac{1}{z^2}), \quad \text{as } z \to \infty.$$

The half-plane capacity can be defined in a number of equivalent ways [6], and there are various geometric interpretations ([5], [13]) for it. It is easy to check that $\mathrm{hcap}(A) > 0$ unless $A = \emptyset$. Moreover, the half-plane capacity has the following basic properties [6].

(i) *Scaling*: For $r > 0$, $\mathrm{hcap}(rA) = r^2\mathrm{hcap}(A)$.

(ii) *Translation*: For $c \in \mathbb{R}$, $\mathrm{hcap}(A + c) = \mathrm{hcap}(A)$.

(iii) *Monotonicity*: For $A \subset B$, $\mathrm{hcap}(B) = \mathrm{hcap}(A) + \mathrm{hcap}(g_A(B))$.

Suppose $\{K_t\}_{t\in[0,T]}$ is an increasing family of hulls generated by the Loewner equation (1.1). Then the half-plane capacity of the hull K_t is equal to $2t$. Let $g_t := g_{K_t}$ be the unique conformal transformation of $\mathbb{H} \setminus K_t$ onto $\mathbb{H}$ with $g_t(z) - z \to 0$ as $z \to \infty$. Let $K_{t,t+s}$ be the hull $g_t(K_{t+s} \setminus K_t) \cap \mathbb{H}$ for all $s > 0$. It is not hard to see that $\cap_{s>0}\overline{K_{t,t+s}}$ is the single point $\lambda(t)$. In particular, this implies that $\lambda(0) = \gamma(0)$ if $K_t = \gamma(0, t]$ for some simple curve γ.

To close this section, we list some simple but useful properties of the Loewner equation [8]. Assume that the hulls K_t are generated by the driving function $\lambda(t)$ in the equation (1.1).

(i) *Scaling*: For $r > 0$, the scaled hulls rK_{t/r^2} are generated by $r\lambda(t/r^2)$.

(ii) *Translation*: For $c \in \mathbb{R}$, the shifted hulls $K_t + c$ are generated by $\lambda(t) + c$.

(iii) *Concatenation*: For $T > 0$, the mapped hulls $g_T(K_{T+t})$ are generated by $\lambda(T + t)$.

(iv) *Reflection*: The reflected hulls $R_I(K_t)$ are generated by $-\lambda(t)$, where R_I denotes reflection in the imaginary axis.

3. Proof of Theorem 1.1

For simplicity, we will use the following notation: $B(a, b) := \int_0^1 t^{a-1}(1 - t)^{b-1}dt,\ a, b > 0$. We give a more complete version of Theorem 1.1.

Theorem 3.1. *Let the slit γ^p ($p > 0$) be generated by the driving function λ in the Loewner equation (1.1). Then there exists $T > 0$ such that*

$$\lambda(t) = \Big(c\sum_{n=1}^{\infty} c_n t^{\frac{n}{p}} \Big)^{\frac{p+1}{2}}, \quad t \in [0, T],$$

where $c = \frac{(2p+1)^2}{4p(p+1)}\delta^{\frac{2p}{p+1}}$ with $\delta = \frac{4(p+1)B(\frac{1}{p}, 1-\frac{1}{2p})\sin\frac{\pi}{2p}}{p(2p+1)}$, and where

$$c_{n+1} = c_1 \sum_{j=1}^{n} \sum_{\substack{i_1+\cdots+i_j=n \\ i_1,\cdots,i_j\geq 1}} \frac{q(q+1)\cdots(q+j-1)}{j!} c_{i_1}\cdots c_{i_j}, \quad n \geq 1$$

with $c_1 = 16^{\frac{1}{p}}\frac{4p(p+1)}{(2p+1)^2}\delta^{-2}$. In particular, we have

$$\lambda(t) = (4^{1+\frac{1}{p}}\delta^{-1})t^{\frac{p+1}{2p}} + o(t^{\frac{p+1}{2p}}), \quad as\ t \to 0.$$

The technique of proof is similar to the proof of [4] due to Lau and Wu. We will also divide the proof into two lemmas to obtain the functional equation in Lemma 3.4.

Let $\gamma(t) : 0 \leq t \leq T$ be the parametric representation of γ^p by the half-plane capacity, i.e., $\text{hcap}\gamma(0,t] = 2t$ for $t \in [0,T]$. Let g_t be the solution of the Loewner equation which maps $\mathbb{H} \setminus \gamma(0,t]$ onto $\mathbb{H}$, and let $\lambda(t) = g_t(\gamma(t))$. Since g_t is well-defined in $\mathbb{R} \setminus \{0\}$, the two functions $\alpha(t) = g_t(0-)$ and $\beta(t) = g_t(0+)$ are also well defined. It is easy to see that $\alpha(t) < \lambda(t) < \beta(t)$ for each $t > 0$. When there is no confusion, we will suppress the variable t and just write λ, α and β for brevity.

Let f_t be the inverse of g_t. Then we will give an integral expression of f_t as follows.

Lemma 3.2. *Let $w(z) = (-iz)^{-\frac{1}{p}}$ for $z \in \mathbb{H}$ (we take the branch such that $\ln 1 = 0$). Let $h_t = w \circ f_t$ be defined on $\mathbb{H}$. Then*

$$h_t(z) - h_t(z_0) = -\frac{i^{\frac{1}{p}}}{p} \int_{z_0}^{z} (\xi - \alpha)^{-\frac{1}{2p}-1}(\xi - \lambda)(\xi - \beta)^{-\frac{1}{2p}-1} d\xi \tag{3.1}$$

for any fixed $z_0 \in \mathbb{H}$.

Proof. We write $w(z) = \psi \circ \phi(z)$, $z \in \mathbb{H}$, where $\psi(z) = 1/z$, $\phi(z) = (-iz)^{\frac{1}{p}}$. Note that γ^p is a circular arc when $p = 1$, and denote it by γ^1. Let $\hat{\gamma}$ be the subarc of γ^1 on $\{\pi(1 - \frac{1}{2}\theta_p) \leq \theta \leq \pi\}$, and let $x_0(t) = \text{Re}\, w(\gamma(t))$ for $t \in (0,T]$. Then we have

$$w(\gamma^p) = \psi \circ \phi(\gamma^p) = \psi(-i\hat{\gamma}) = \{x - \frac{1}{2}i : \ x \geq x_0(T)\}.$$

Clearly $w(\mathbb{H}) = \{re^{i\theta} : \ r > 0, -\frac{\pi}{2p} < \theta < \frac{\pi}{2p}\}$. It follows that for $p \geq 1/2$, w maps $\mathbb{H} \setminus \gamma(0,t]$ conformally onto the domain

$$M_t = \{re^{i\theta} : \ r > 0, -\frac{\pi}{2p} < \theta < \frac{\pi}{2p}\} \setminus \{x - \frac{1}{2}i : \ x \geq x_0(t)\}.$$

But for $0 < p < 1/2$, w is multivalued (as $w(\mathbb{H})$ wraps around). We will divide the proof into two cases.

Case 1: $p \geq 1/2$. $h_t = w \circ f_t$ maps $\mathbb{H}$ conformally onto M_t. Note that the boundary of the domain M_t is a quadrilateral for each $t \in (0,T]$. Applying the Christoffel-Schwarz formula to any fixed $z_0 \in \mathbb{H}$, we can express h_t as

$$h_t(z) - h_t(z_0) = C_0 \int_{z_0}^{z} (\xi - \alpha)^{-\frac{1}{2p}-1}(\xi - \lambda)(\xi - \beta)^{-\frac{1}{2p}-1} d\xi.$$

To obtain the constant C_0, we observe that

$$h_t'(z) = C_0(z - \alpha)^{-\frac{1}{2p}-1}(z - \lambda)(z - \beta)^{-\frac{1}{2p}-1}.$$

Since $h_t'(z) = (w \circ f_t)'(z) = (-\frac{1}{p})\Big(-if_t(z)\Big)^{-\frac{1}{p}-1}(-if_t'(z))$, we can obtain

$$f_t'(z) = (-C_0 p)(-i)^{\frac{1}{p}}\Big(f_t(z)\Big)^{\frac{1}{p}+1}(z - \alpha)^{-\frac{1}{2p}-1}(z - \lambda)(z - \beta)^{-\frac{1}{2p}-1}.$$

Noting that

$$f_t(z) = z - \frac{2t}{z} + O(\frac{1}{z^2}), \quad \text{as } z \to \infty, \tag{3.2}$$

we can conclude that $f_t'(z) \to 1$ and $f_t(z)/z \to 1$ as $z \to \infty$. It follows that $-C_0 p(-i)^{\frac{1}{p}} = 1$, i.e., $C_0 = -i^{\frac{1}{p}}/p$.

Case 2: $0 < p < 1/2$. We need to adjust M_t as a polygon in some Riemann surface to apply the Christoffel-Schwarz formula. Let $S := \mathbb{R}^+ \times \mathbb{R}$ be the Reimann surface in the following sense:

(i) $S = \cup_{m,n \in \mathbb{Z}}(U_m \cup V_n)$, where $U_m = \mathbb{R}^+ \times (2m\pi, 2(m+1)\pi)$, $V_n = \mathbb{R}^+ \times ((2n+1)\pi, (2n+3)\pi)$. For each $m, n \in \mathbb{Z}$, define

$$\phi_m : \ U_m \to \mathbb{C}, \ (r,\theta) \mapsto re^{i\theta}, \qquad \varphi_n : \ V_n \to \mathbb{C}, \ (r,\theta) \mapsto re^{i\theta}.$$

(ii) If U_m and V_n intersect for some $m, n \in \mathbb{Z}$, then the transition map

$$\Phi_{m,n} = \phi_m \circ \varphi_n^{-1} : \varphi_n(U_m \cap V_n) \to \phi_m(U_m \cap V_n)$$

is a conformal map from $U_m \cap V_n$ onto itself.

Define the map $w^* : \mathbb{H} \to S$, $w^*(re^{i\theta}) = (r^{-\frac{1}{p}}, \frac{\pi}{2p} - \frac{\theta}{p})$ for $r > 0$, $\theta \in (0, \pi)$. When there is no confusion, we will follow the notations in Case 1, and we still denote by M_t the Riemann surface $w^*(\mathbb{H} \setminus \gamma[0, t])$, and denote w^* by w. Obviously, w is 1-1 from $\mathbb{H} \setminus \gamma[0, t]$ onto M_t, and the boundary of M_t consists of three rays. It follows from [1] and [2] that the Christoffel-Schwarz formula (3.1) still holds for this case, and the same proof can be carried through. □

Lemma 3.3. *With the above notations, we have the following identities for $\lambda(t)$, $\alpha(t)$ and $\beta(t)$:*

$$\lambda = (\frac{1}{2p} + 1)(\alpha + \beta), \quad \frac{1}{4p}(\alpha + \beta)^2 + \alpha\beta = -4t, \tag{3.3}$$

and

$$(\beta - \alpha)^{1+\frac{1}{p}} = \delta\lambda, \tag{3.4}$$

where δ be defined in Theorem 3.1.

Proof. Noting that $h_t(\infty) = 0$, by letting $z_0 \to \infty$, and making a change of variable $w = \xi^{-1}$, we have

$$h_t(z) = \int_0^{\frac{1}{z}} \Phi(w)dw, \quad \text{where} \quad \Phi(w) = \frac{i^{\frac{1}{p}}(1 - \lambda w)}{p(1 - \alpha w)^{1+\frac{1}{2p}}(1 - \beta w)^{1+\frac{1}{2p}} w^{1-\frac{1}{p}}}.$$

We expand the first three terms of $\Phi(w)$ and obtain

$$\Phi(w) = \frac{i^{\frac{1}{p}}}{p} w^{\frac{1}{p}-1}(1 + a_1 w + a_2 w^2 + o(w^2)), \quad \text{as } w \to 0,$$

where $a_1 = (1 + \frac{1}{2p})(\alpha + \beta) - \lambda$ and $a_2 = (1 + \frac{1}{2p})(1 + \frac{1}{4p})(\alpha^2 + \beta^2) + (1 + \frac{1}{2p})^2\alpha\beta - (1 + \frac{1}{2p})(\alpha + \beta)\lambda$. Integrating $\Phi(w)$ and noting that $f_t(z) = i\big(h_t(z)\big)^{-p}$, we conclude that

$$f_t(z) = z - \frac{pa_1}{p+1} - \Big(\frac{pa_2}{2p+1} - \frac{pa_1^2}{2p+2}\Big)\frac{1}{z} + o(\frac{1}{z}), \quad \text{as } z \to \infty.$$

Hence it follows from (3.2) that $a_1 = 0$, $a_2 = t(4p + 2)/p$. The first identity in (3.3) follows. By equating the two expressions of a_2, and use the first identity in (3.3) to substitute away the λ, we obtain the second identity in (3.3).

To prove (3.4), we use (3.1) to express $h_t(z)$ as

$$h_t(z) - h_t(z_0) = -\frac{i^{\frac{1}{p}}}{p(\beta - \alpha)^{\frac{1}{p}}} \int_{\frac{z_0-\alpha}{\beta-\alpha}}^{\frac{z-\alpha}{\beta-\alpha}} \frac{\xi - \frac{\lambda-\alpha}{\beta-\alpha}}{\xi^{1+\frac{1}{2p}}(\xi - 1)^{1+\frac{1}{2p}}} d\xi.$$

Letting $z = \lambda$ and $z_0 \to \infty$, then we can obtain

$$h_t(\lambda) = \frac{i^{\frac{1}{p}}}{p(\beta - \alpha)^{\frac{1}{p}}} \int_r^{\infty} \frac{\xi - r}{\xi^{1+\frac{1}{2p}}(\xi - 1)^{1+\frac{1}{2p}}} d\xi, \tag{3.5}$$

where $r = \frac{\lambda-\alpha}{\beta-\alpha} \in (0, 1)$. Define the complex-valued functions F, G and H on $\mathbb{H}$ by

$$F(\xi) = \frac{1}{\xi^{1+\frac{1}{2p}}(\xi - 1)^{1+\frac{1}{2p}}}, \quad G(\xi) = \frac{1}{\xi^{\frac{1}{2p}}(\xi - 1)^{1+\frac{1}{2p}}}, \quad H(\xi) = \frac{1}{\xi^{1+\frac{1}{2p}}(\xi - 1)^{\frac{1}{2p}}},$$

where we define the branch such that $\ln 1 = 0$. Observing that $F(\xi) = G(\xi) - H(\xi)$, we have

$$(\xi - r)F(\xi) = H(\xi) + (1-r)F(\xi) = (1-r)G(\xi) + rH(\xi).$$

Hence it follows from (3.5) that

$$h_t(\lambda) = \frac{i^{\frac{1}{p}}(\beta-\lambda)}{p(\beta-\alpha)^{1+\frac{1}{p}}} \int_r^\infty G(\xi)d\xi + \frac{i^{\frac{1}{p}}(\lambda-\alpha)}{p(\beta-\alpha)^{1+\frac{1}{p}}} \int_r^\infty H(\xi)d\xi.$$

Using the principle of integration by parts, we obtain

$$\int_r^\infty (G(\xi)+H(\xi))d\xi = \frac{2p(\beta-\alpha)^{\frac{1}{p}}}{(\lambda-\alpha)^{\frac{1}{2p}}(\lambda-\beta)^{\frac{1}{2p}}}.$$

Noting that the first identity in (3.3), we can conclude that

$$h_t(\lambda) = \frac{2(\beta-\lambda)^{1-\frac{1}{2p}}}{(\beta-\alpha)(\lambda-\alpha)^{\frac{1}{2p}}} + \frac{2i^{\frac{1}{p}}(p+1)\lambda}{p(2p+1)(\beta-\alpha)^{1+\frac{1}{p}}} \int_r^\infty H(\xi)d\xi. \tag{3.6}$$

Observing that h_t maps $\mathbb{R}$ onto the boundary of the domain M_t, we can see that $\mathrm{Im} h_t(\lambda) = -\frac{1}{2}$. Then this identity together with (3.6) implies

$$\frac{1}{2} = \frac{2(p+1)\lambda}{p(2p+1)(\beta-\alpha)^{1+\frac{1}{p}}}\Big(\mathrm{Re}\int_r^\infty H(\xi)d\xi \cdot \sin\frac{\pi}{2p} + \mathrm{Im}\int_r^\infty H(\xi)d\xi \cdot \cos\frac{\pi}{2p}\Big).$$

In order to obtain the last identity of this lemma, we need only to prove that the expression in brackets in the right side equals $B(\frac{1}{p}, 1-\frac{1}{2p})\sin\frac{\pi}{2p}$. Next, we will prove it in the following paragraph.

Let $\epsilon \in (0,1)$ be very small. Without loss of generality, we can assume that $r < 1-\epsilon$. We choose the following integral paths:

$$\begin{aligned} &\Lambda_1: \ \xi(x) = x, && r \le x \le 1-\epsilon; \\ &\Lambda_2: \ \xi(x) = 1+\epsilon e^{ix}, && 0 \le x \le \pi; \\ &\Lambda_3: \ \xi(x) = x, && 1+\epsilon \le x < \infty. \end{aligned}$$

Let the integral path $\Lambda = \Lambda_1 + \Lambda_2^- + \Lambda_3$, where Λ_2^- denotes that the parameter x starts from π. Then it follows that

$$\int_r^\infty H(\xi)d\xi = \int_{\Lambda_1} H(\xi)d\xi + \int_{\Lambda_2^-} H(\xi)d\xi + \int_{\Lambda_3} H(\xi)d\xi.$$

Noting that $(-1)^{-\frac{\pi}{2p}} = \cos\frac{\pi}{2p} - i\sin\frac{\pi}{2p}$, we obtain

$$\mathrm{Re}\int_{\Lambda_1} H(\xi)d\xi \cdot \sin\frac{\pi}{2p} + \mathrm{Im}\int_{\Lambda_1} H(\xi)d\xi \cdot \cos\frac{\pi}{2p} = 0.$$

Calculating the integration on the paths Λ_2^- and Λ_3, and letting $\epsilon \to 0$, we have

$$\int_{\Lambda_2^-} H(\xi)d\xi = \epsilon^{1-\frac{1}{2p}} \int_\pi^0 \frac{ie^{i(1-\frac{1}{2p})x}}{(1+\epsilon e^{ix})^{1+\frac{1}{2p}}}dx \to 0,$$

$$\int_{\Lambda_3} H(\xi)d\xi = \int_0^{\frac{1}{1+\epsilon}} t^{\frac{1}{p}-1}(1-t)^{-\frac{1}{2p}}dt \to B(\frac{1}{p}, 1-\frac{1}{2p}).$$

Combining the above three expressions gives

$$\mathrm{Re}\int_r^\infty H(\xi)d\xi\cdot\sin\frac{\pi}{2p}+\mathrm{Im}\int_r^\infty H(\xi)d\xi\cdot\cos\frac{\pi}{2p}=B(\frac{1}{p},1-\frac{1}{2p})\sin\frac{\pi}{2p}.$$

Therefore we complete the proof of this lemma. □

In order to prove Theorem 3.1, we need the following functional equation due to Lau and Wu [4], which is associated with the driving function $\lambda(t)$ of γ^p.

Lemma 3.4. *[4] Let $\varphi:[0,T]\to[0,1)$ be a continuous function such that $\varphi(0)=0$, and satisfies*

$$\varphi(t)(1-\varphi(t))^q=c_1t^q,\quad t\in[0,T] \tag{3.7}$$

for some $q,c_1>0$. Then $\varphi(t)=\sum_{n=1}^\infty c_nt^{qn}$ with

$$c_{n+1}=c_1\sum_{j=1}^n\sum_{\substack{i_1+\cdots+i_j=n\\ i_1,\cdots,i_j\geq 1}}\frac{q(q+1)\cdots(q+j-1)}{j!}c_{i_1}\cdots c_{i_j},\quad n\geq 1.$$

Proof of Theorem 3.1. Using (3.3) and (3.4) to substitute away the α and β, we have the following functional equation

$$\frac{4p(p+1)}{(2p+1)^2}\lambda^2-(\delta\lambda)^{\frac{2p}{p+1}}+16t=0,\quad t\in[0,T].$$

Let $\varphi(t)=c^{-1}\lambda^{\frac{2}{p+1}}$ with $c=\frac{(2p+1)^2}{4p(p+1)}\delta^{\frac{2p}{p+1}}$, and let $q=1/p$. Simplifying the above equation, we arrive

$$\varphi(t)(1-\varphi(t))^q=c_1t^q,$$

where $c_1=16^{\frac{1}{p}}\frac{4p(p+1)}{(2p+1)^2}\delta^{-2}$ has the expression in Theorem 3.1. It follows from Lemma 3.4 that $\varphi(t)=\sum_{n=1}^\infty c_nt^{qn}$ as stated. Hence we have

$$\lambda(t)=\Big(c\sum_{n=1}^\infty c_nt^{\frac{n}{p}}\Big)^{\frac{p+1}{2}},\quad t\in[0,T].$$

Therefore we complete the proof. □

4. Proof of Theorem 1.2

For simplicity, we will use the following notations in this section: $g(\epsilon)\lesssim h(\epsilon)$ means $g(\epsilon)\leq ch(\epsilon)$ for some positive constant c; $g(\epsilon)\asymp h(\epsilon)$ means $g(\epsilon)\lesssim h(\epsilon)$ and $h(\epsilon)\lesssim g(\epsilon)$; $g(\epsilon)\sim h(\epsilon)$ means $\lim_{\epsilon\to 0}g(\epsilon)/h(\epsilon)=1$. To prove Theorem 1.2, we need the following two lemma.

Lemma 4.1. *For $x+iy\in\gamma^p$ $(p>0)$, we have $x=\frac{p}{2}y^{1+\frac{1}{p}}+O(y^{1+\frac{3}{p}})$ as $y\to 0$.*

Proof. For $p=1$, γ is the circular arc, $(x-1)^2+y^2=1$, so that $x=\frac{1}{2}y^2+O(y^4)$ as $y\to 0$. For $p\neq 1$, we have $x+iy=i^{1-p}(u+iv)^p$, where $u+iv\in\gamma$. Hence by using the binomial expansion,

$$x+iy=i^{1-p}(\frac{1}{2}v^2+O(v^4)+vi)^p=v^p\Big(i+\frac{1}{2}pv-\frac{p(p-1)i}{8}v^2+O(v^3)\Big).$$

Comparing the real and imaginary parts, we obtain $x=\frac{p}{2}y^{1+\frac{1}{p}}+O(y^{1+\frac{3}{p}})$. □

Lemma 4.2. *Let $f : [0,1] \to \mathbb{R}$ be a function in $C^{1,\beta}(0,1)$ for some $\beta \in (0,1)$, and let the Loewner curve $\gamma(y) = f(y) + iy$. If there exists $\kappa \in \mathbb{R}$ such that $\lim_{y\to 0} f'(y) = \kappa$, then*

$$\lim_{y\to 0} \frac{\mathrm{hcap}\gamma(0,y]}{y^2} = \frac{1}{b^2(\theta)\sin^2\theta},$$

where

$$b(\theta) = 2(\frac{\pi}{\theta} - 1)^{\frac{1}{2}-\frac{\theta}{\pi}} \quad \textit{with} \quad \theta = \mathrm{arccot}\,\kappa.$$

Proof. Define the function $t(y)$ by

$$t(y) = \frac{1}{2}\mathrm{hcap}\gamma(0,y], \quad y \in [0,1].$$

It is well known that $t(y)$ is a strictly increasing continuous function with $t(0) = 0$ (see [6] in detail). Let $y(t)$ be the inverse of $t(y)$, and let $\Gamma(t) : 0 \le t \le T$ be the parametric representation of $\gamma(y)$ by the half-plane capacity. Then it is easy to check that $\Gamma(t) = \gamma(y(t))$. Noting that $\gamma(y) \in C^{1,\beta}(0,1)$, and using the result of [12], we conclude that its driving function $\lambda(t)$ is in $C^{\beta+1/2}(0,t(1))$. Therefore it follows from [15] that $\Gamma(t)$ is in $C^1(0,t(1))$. Hence $y(t)$ is also in $C^1(0,t(1))$ and $y'(t) > 0$ in the interval $(0,t(1))$. Then it follows that

$$\lim_{t\to 0} \arg\,\Gamma'(t) = \lim_{t\to 0} \arg\,(f'(y(t)) + i)y'(t) = \theta,$$

where θ is defined in the above. Noting that $\theta \in (0,\pi)$, and making use of Theorem 1.2 in [16], we can obtain

$$\lim_{t\to 0} \frac{\Gamma(t)}{\sqrt{t}} = b(\theta)e^{i\theta},$$

where $b(\theta)$ is defined in the above. From $y(t) = \mathrm{Im}\Gamma(t)$, it follows that

$$\lim_{t\to 0} \frac{y(t)}{\sqrt{t}} = b(\theta)\sin\theta.$$

This implies what we need to prove. Hence we complete the proof. □

Proof of Theorem 1.2. Let γ^p $(p > 0)$ be defined in (1.2). Then it follows from Lemma 4.1 that $\gamma^{\frac{1}{r}}$ has the parametric representation $\Gamma_r(y) = g(y) + yi,\ 0 \le y \le 1$, where the real-valued function g is sufficiently smooth and of the form

$$g(y) = \frac{1}{2r}y^{1+r} + O(y^{1+3r}), \quad \text{as } y \to 0.$$

It follows from the assumption that

$$f(y) - f(0) = ay^{1+r} + O(y^{1+r+s}), \quad \text{as } y \to 0.$$

By translation property which we list in Section 2, we can assume that $f(0) = 0$. Moreover, by scaling property and reflection property, we can assume that $a = (2r)^{-1}$. Hence it follows that $f(y) - g(y) = O(y^\rho)$, where $\rho = 1 + r + \min\{2r, s\}$. Define

$$t = t(y) := \frac{1}{2}\mathrm{hcap}\gamma(0,y], \quad \tau = \tau(y) := \frac{1}{2}\mathrm{hcap}\Gamma_r(0,y].$$

When there is no confusion, we will suppress the variable y and just write t and τ for brevity. It is easy to see that $t(y)$ and $\tau(y)$ are strictly increasing. Moreover, making use of Lemma 4.2, we have

$$t(y) \sim \frac{1}{4}y^2 \sim \tau(y), \quad \text{as } y \to 0. \tag{4.1}$$

Let $\hat{\gamma}(t)$, $\hat{\Gamma}_r(\tau)$ be the parametric representation of γ, Γ_r by the half-plane capacity respectively. Then it is easy to check that $\hat{\gamma}(t(y)) = f(y) + yi$ and $\hat{\Gamma}_r(\tau(y)) = g(y) + yi$. Therefore it follows that

$$d_H(\hat{\gamma}(0,t], \hat{\Gamma}_r(0,\tau]) \lesssim y^\rho, \tag{4.2}$$

where $d_H(A, B)$ denotes the Hausdorff distance between A and B. Let $p_1 = \hat{\gamma}(t(y))$, $p_2 = \hat{\Gamma}_r(\tau(y))$, and let $p = \hat{\gamma}(t(y + y^\rho))$. Then it follows from the assumption of f that

$$|p - p_1| = \sqrt{y^{2\rho} + (f(y + y^\rho) - f(y))^2} \lesssim y^\rho \lesssim y \asymp \operatorname{diam} \hat{\gamma}(0,\ t(y)].$$

This implies that

$$|p - p_2| \leq |p - p_1| + |p_1 - p_2| \lesssim y^\rho \lesssim y \asymp \operatorname{diam} \hat{\Gamma}_r(0, \tau(y)].$$

Denote $G_y = \hat{\mathbb{C}} \setminus d\,\overline{\mathbb{D}}$, where $\hat{\mathbb{C}}$ denotes the extended complex plane, and where $d = \max\{|p_1|, |p_2|\}$. Without loss of generality, we can assume that $d = |p_1|$. Otherwise, we can choose $p = \hat{\Gamma}_r(\tau(y + y^\rho))$. Let $\lambda(t)$, $\lambda_r(\tau)$ be the driving functions of γ, Γ_r respectively. Noting that (4.2), and applying Theorem 4.3 in [8], we have

$$|\lambda(t) - \lambda_r(\tau)| \lesssim y^{\frac{\rho}{2}}(c_0 + \omega(p, \infty, G_y)),$$

where c_0 is a positive constant, and where $\omega(p, \infty, G_y)$ denotes the hyperbolic distance from p to ∞ in the domain G_y. By calculations, we obtain

$$\lim_{y\to 0} \frac{|p| - d}{y^\rho} = \lim_{y\to 0} \frac{\sqrt{(y + y^\rho)^2 + (f(y + y^\rho))^2} - \sqrt{y^2 + (f(y))^2}}{y^\rho} = 1.$$

Observing that

$$\omega(p, \infty, G_y) = \ln \frac{|p| + d}{|p| - d} = \ln(1 + \frac{2d}{|p| - d}),$$

we can easily check that $\omega(p, \infty, G_y) \lesssim -\ln y$. Letting $y \to 0$, noting that $\rho/2 > r + 1$, and making use of (4.1) and Theorem 1.1, we can conclude that $\lambda(t) \sim Ct^{\frac{r+1}{2}}$ with the constant C given in Theorem 3.1. □

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