Filomat 36:9 (2022), 2939–2945
https://doi.org/10.2298/FIL2209939M

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Left-Right α-Fredholm and α-Weyl Operators with Application to the Weighted Spectrum

Rim Messaoud[a], Boulbaba Ghanmi[a], Saifeddine Ghnimi[a], Amira Missaoui[b]

[a]*Faculty of Sciences of Gafsa*
[b]*Faculty of Sciences of Sfax*

Abstract. The purpose of this paper is to introduce Fredholm operator, α-Fredholm operator and Weyl operator. Moreover, we introduce basic properties and application to operator matrix.

1. Introduction

BThroughout the paper, $\mathcal{H}$ is fixed (complex) non separable Hilbert space of dimension $h \geq \aleph_0$. Let $\mathcal{L}(\mathcal{H})$ denote the set of all bounded linear operators on $\mathcal{H}$. For each cardinal α, $\aleph_0 \leq \alpha \leq h$, $\mathcal{F}_\alpha$ denotes the two sided ideal in $\mathcal{L}(\mathcal{H})$ of all bounded linear operators of rank less than α. Let $\mathcal{I}_\alpha$ denote the norm closure of $\mathcal{F}_\alpha$ in $\mathcal{L}(\mathcal{H})$.
Clearly $\mathcal{F}_{\aleph_0} = K_0(H)$, the ideal of all finite dimensional operators, of course $\mathcal{I}_{\alpha_0}$ is the ideal of compact operators and $\mathcal{I}_h$ is the maximal closed. For each $T \in \mathcal{L}(\mathcal{H})$, we will denote the kernel and the range of T by $\mathcal{N}(T)$ and $\mathcal{R}(T)$, respectively, and the adjoint of T by T^*. The nullity, $n(T)$, of T is defined as the dimension of $\mathcal{N}(T)$. Sets of upper and lower α-Fredholm operators, respectively, are defined as:

$$\Phi_\alpha^+(\mathcal{H}) = \{T \in \mathcal{L}(\mathcal{H}) \text{ such that } \mathcal{R}(T) \text{ is } \alpha - \text{closed and } n(T) < \alpha\}$$

and

$$\Phi_\alpha^-(\mathcal{H}) = \{T \in \mathcal{L}(\mathcal{H}) \text{ such that } \mathcal{R}(T) \text{ is } \alpha - \text{closed and } n(T^*) < \alpha\}.$$

Operators in $\Phi_\alpha^\pm(\mathcal{H}) = \Phi_\alpha^+(\mathcal{H}) \cup \Phi_\alpha^-(\mathcal{H})$ is called $\alpha-$Semi$-$Fredholm. The set of $\alpha-$Fredholm is defined by $\Phi_\alpha(\mathcal{H}) = \Phi_\alpha^+(\mathcal{H}) \cap \Phi_\alpha^-(\mathcal{H})$.

The Weyl operator still conserves one of the basic properties for the operators between finite dimensional spaces: Fredholm alternative. Moreover, with some extra conditions (like finite ascent or descent), such opertors have very nice property: there are Drazin invertible (For more details about generalized invertibility). Now, the naturel question appears: it is necessary to observe only finite dimensional situation for kernel, or co-dimension of range, or ascent, etc.

The main results of the paper are present in remaining two sections. In the first section we define some basic properties in the way that we give a characterization of the approximate point spectrum of T, of weight α. In the end section we give application to operator matrix in the way that we establish criteria which ensure that the difference of the resolvent of M_c^1 and M_c^2 is compact.

2020 *Mathematics Subject Classification*. 47A53
Keywords. Fredholm operators, $\alpha-$Fredholm operator, Weyl operator
Received: 29 January 2018; Accepted: 03 June 2022
Communicated by Dragan S. Djordjević
Email addresses: rimessaoud@yahoo.fr (Rim Messaoud), bghanmi@gmail.com (Boulbaba Ghanmi), ghnimisaifeddine@yahoo.f (Saifeddine Ghnimi), missaouiamira@yahoo.fr (Amira Missaoui)

2. Preliminaries

The aim of this section consists in establishing some elementary concepts which will be used throughout the paper.

Definition 2.1. *A subspace A of a Hilbert space H is called $\alpha-$closed if there is a closed subspace E of H that $E \subset A$ and* $\dim(A \cap E^{\perp}) < \alpha$.

Definition 2.2. *Let H be a Hilbert space and α be a cardinal number. A subset $A \subset H$ is called $\alpha-$bounded if for each $\varepsilon > 0$ there exists a set of points $(f_m)_{m\in I}$, $f_m \in A$, $\mid I \mid < \alpha$ and with*

$$A \subset \cup_{m\in I} B(h_m, \varepsilon)$$

where $\mid I \mid < \alpha$ is the cardinal of I and $B(h_m, \varepsilon)$ denotes the ball about f_m, with center f_m and radius ε.

Remark 2.3. *Let H be a Hilbert space and $A \subset H$. We said A is totally bounded if and if the closure of A is compact. Then, if $\alpha = \aleph_0$, the Definition 2.2 coincide with the definition of A to be totally bounded.*

Definition 2.4. *Let X be non separable space such* $\dim(X) \geq \aleph_0$. *Let α be a cardinal number, $\aleph_0 \leq \alpha \leq \dim(X)$. Then, $A \in \phi_\alpha(X)$ if and only if*

i) $\dim N(A) < \alpha$,

ii) $\dim R(A)^{\perp} < \alpha$,

iii) *$R(A)$ is $\alpha-$closed.*

Definition 2.5. *Let $\mathcal{H}, \mathcal{K}$, be two Hilbert spaces and let α be a cardinal number. An operator $K \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ is called $\alpha-$compact if $K(B)$ is $\alpha-$bounded in $\mathcal{K}$ for every bounded subset $B \in \mathcal{H}$. The family of $\alpha-$compact from $\mathcal{H}$ to $\mathcal{K}$ is denoted by $\mathcal{K}_\alpha(\mathcal{H}, \mathcal{H})$ is closed two-sided ideal of $\mathcal{L}(\mathcal{H})$ (see [5]). In the case when $\alpha = \aleph_0$, we have $\mathcal{K}_{\aleph_0}(\mathcal{H}) = \mathcal{K}(\mathcal{H})$ the ideal of all compact operators.*

3. Basic properties

The set of operators that are left-invertible modulo $\mathcal{K}_\alpha(\mathcal{H})$ will be denoted by Φ_ℓ^α. To be perfectly:

$$\Phi_\ell^\alpha(\mathcal{H}) = \Big\{T \in \mathcal{L}(\mathcal{H}) : \exists\, K \in \mathcal{K}_\alpha(\mathcal{H}) \text{ and } S \in \mathcal{L}(\mathcal{H}); ST = I + K\Big\}.$$

Remark 3.1.

i) *In the case $\alpha = \aleph_0$, we have $\Phi_\ell^{\aleph_0}(\mathcal{H}) = \Phi_\ell(\mathcal{H})$, the left Fredholm operators(see,[1],[2]).*

ii) *It is easy to see that , if $T \in \Phi_\ell^\alpha(\mathcal{H})$ and $K \in \mathcal{K}_\alpha(\mathcal{H})$, then $T + K \in \mathcal{K}_\alpha(\mathcal{H})$.*

Lemma 3.2. *Let $\mathcal{H}$ be Hilbert space and let α be a cardinal number such that $\alpha \leq \dim(\mathcal{H})$. Then, $\Phi_\ell^\alpha(\mathcal{H})$ is open.*

Proof. Let $T \in \Phi_\ell^\alpha(\mathcal{H})$. Then, there exist $S \in \mathcal{L}(\mathcal{H})$ and $K \in \mathcal{K}_\alpha(\mathcal{H})$ such that $ST = I + K$. Take $r =\parallel S \parallel^{-1}$. Let $R \in \mathcal{L}(\mathcal{H})$ such that $\parallel R \parallel \leq r$, we get

$$S(T + R) = I + K + SR.$$

Since $\parallel SR \parallel \leq \parallel S \parallel \parallel R \parallel < \parallel S \parallel \parallel S \parallel^{-1} = 1$. Then $I + SR$ is invertible in $\mathcal{L}(\mathcal{H})$. Therefore,

$$(I + SR)^{-1} S(T + R) = I + (I + SR).$$

Since $(I + SR)^{-1}K \in \mathcal{K}_\alpha(\mathcal{H})$. Then, $T + R \in \Phi_\ell^\alpha(\mathcal{H})$. We conclude the $\Phi_\ell^\alpha(\mathcal{H})$ is open set. □

Lemma 3.3. *Let $T \in \mathcal{L}(\mathcal{H})$ and $S \in \mathcal{L}(\mathcal{H})$.*

i) *If $T \in \Phi_\ell^\alpha(\mathcal{H})$ and $S \in \Phi_\ell^\alpha(\mathcal{H})$, then $TS \in \Phi_\ell^\alpha(\mathcal{H})$.*

ii) *If $TS = ST \in \Phi_\ell^\alpha(\mathcal{H})$, then $T \in \Phi_\ell^\alpha(\mathcal{H})$ and $S \in \Phi_\ell^\alpha(\mathcal{H})$.*

Proof.

i) Let $T \in \Phi_\ell^\alpha(\mathcal{H})$ and $S \in \Phi_\ell^\alpha(\mathcal{H})$. This implies that both are left invertible modulo $\mathcal{K}_\alpha(\mathcal{H})$. Hence, there exist operators $T_1, S_1 \in \mathcal{L}(\mathcal{H})$ and $K_1, K_2 \in \mathcal{K}(\mathcal{H})$ such that $T_1T = I - K_1$ and $S_1S = I - K_2$. Therefore,

$$\begin{aligned}(S_1T_1)TS &= S_1(I - K_1)S,\\ &= S_1S - S_1K_1S,\\ &= I - (K_2 + S_1K_1S).\end{aligned}$$

Thus, $TS \in \mathcal{K}_\alpha(\mathcal{H})$

ii) If $TS = ST \in \Phi_\ell^\alpha(\mathcal{H})$, then there exist $V \in \mathcal{L}(\mathcal{H})$ and $K \in \mathcal{K}_\alpha(\mathcal{H})$ such that

$$VTS = I - K = VST. \tag{1}$$

By equation (1), it is easy to see that $T \in \Phi_\alpha(\mathcal{H})$ and $S \in \Phi_\alpha(\mathcal{H})$. □

We recall the following lemma (see,[3],[4])

Lemma 3.4. *Let T be a closed linear transformation with domain $\mathcal{D}(T)$ dense in a Hilbert space $\mathcal{H}$ and let $\varepsilon > 0$. Then, there exists a closed subspace $\mathcal{K}$ of $\mathcal{H}$ such that*

$$\mathcal{N}(T) \subset \mathcal{K} \subset \mathcal{D}(T); \| Tf \| < \varepsilon \| f \|, \ \forall\ f \in \mathcal{K}, f \neq 0 \tag{2}$$

and

$$\| Tf \| \geq \varepsilon \| f \|, \ \forall\ f \in \mathcal{K}^\perp \cap \mathcal{D}(T). \tag{3}$$

The subspace $\mathcal{K}$ given in the lemma 3.4 is not unique. Edgar. G, Ernest. J and Lee. G in [4], prove that all the subspace satisfying the condition of lemma 3.4 have the same dimension. Let δ_ε denote the common dimension of all subspace $\mathcal{K}$ satisfying the conditions (2) and (3).

Definition 3.5. (*see,[4]*) *Let $T \in \mathcal{L}(\mathcal{H})$. We define the approximate nullity $\delta(A)$ of A to be*

$$\delta(A) = \min_{\varepsilon>0} \delta_\varepsilon.$$

Definition 3.6. *[4] Let $\mathcal{H}$ be a Hilbert space and let α be a cardinal number such that $\aleph_0 \leq \alpha \leq \dim(\mathcal{H})$. The approximate point spectrum of T,of weight α, denoted Π_α, is defined by*

$$\Pi_\alpha(T) = \Big\{\lambda \in C \text{ such that } \delta(A - \lambda) \geq \alpha\Big\}.$$

The following proposition gives a characterization of the approximate point spectrum of T of weight α.

Theorem 3.7. *Let T be an operator and $\mathcal{H}$ a Hilbert space of infinite dimension h, and let α be a cardinal, $\aleph_0 \leq \alpha \leq h$. Then, the following conditions are equivalent*

i) *$\lambda \in \Pi_\alpha(T)$.*

ii) *$A - \lambda$ is not left-invertible modulo $\mathcal{K}_\alpha(\mathcal{H})$.*

iii) *$A - \lambda$ is not right-invertible modulo $\mathcal{K}_\alpha(\mathcal{H})$.*

iv) *Either λ is an eigenvalue of A of multiplicity at least α, or the range of $A - \lambda$ is not $\alpha -$ closed.*

We first prove the following theorem:

Lemma 3.8. *Let $T \in \mathcal{L}(\mathcal{H})$. If $0 \in \rho(T)$, then for $\lambda \neq 0$, we have*

$$\lambda \in \Pi_\alpha(T) \text{ if and only if } \frac{1}{\lambda} \in \Pi_\alpha(T^{-1}).$$

Proof. For $\lambda \in C^n$, we can write

$$T - \lambda = -\lambda(T^{-1} - \frac{1}{\lambda})T = -\lambda T(T^{-1} - \frac{1}{\lambda}).$$

Using lemma 3.3, we infer that

$$T - \lambda \in \Phi_\ell^\alpha(\mathcal{H}) \Leftrightarrow T^{-1} - \frac{1}{\lambda} \in \Phi_\ell^\alpha(\mathcal{H}).$$

By Theorem 3.7, we conclude

$$\Pi_\alpha(T^{-1}) = \Big\{\frac{1}{\lambda} : \lambda \in \Pi_\alpha(T)\Big\}. \tag{4}$$

□

Corollary 3.9. *Let $T \in \mathcal{L}(\mathcal{H})$ and $S \in \mathcal{L}(\mathcal{H})$ such that $0 \in \rho(T) \cap \rho(S)$. If $T^{-1} - S^{-1} \in \mathcal{K}_\alpha(\mathcal{H})$. Then,*

$$\Pi_\alpha(T) = \Pi_\alpha(S).$$

Proof. The proof is immediately by lemma 3.4 and remark 3.1. □

Theorem 3.10. *Let $T, S \in \mathcal{L}(\mathcal{H})$ and α be a cardinal number such that $\aleph_0 \leq \alpha \leq h$. If $TS \in \mathcal{K}_\alpha(\mathcal{H})$, then*

$$\Pi_\alpha(T+S)\backslash\{0\} \subset [\Pi_\alpha(T) \cup \Pi_\alpha(S)]\backslash\{0\}.$$

If, further, T and S commute, then

$$\Pi_\alpha(T+S)\backslash\{0\} = [\Pi_\alpha(T) \cup \Pi_\alpha(S)]\backslash\{0\}.$$

Proof. Let $\lambda \notin \Pi_\alpha(T) \cup \Pi_\alpha(S) \cup \{0\}$. Then

$$T - \lambda \in \Phi_\ell^\alpha(\mathcal{H}) and S - \lambda \in \Phi_\ell^\alpha(\mathcal{H}).$$

We can apply lemma 3.4, we infer that $TS - \lambda(T + S - \lambda) = (T - \lambda)(S - \lambda) \in \Phi_\ell^\alpha(\mathcal{H})$.
Since $TS \in \mathcal{K}_\alpha(\mathcal{H})$. Using remark 3.1, we have $T + S - \lambda \in \Phi_\ell^\alpha(\mathcal{H})$.
Therefore,

$$\Pi_\alpha(T+S)\backslash\{0\} \subset [\Pi_\alpha(T) \cup \Pi_\alpha(S)]\backslash\{0\}.$$

Conversely, let $\lambda \notin \Pi_\alpha(T+S) \cup \{0\}$. Then, $T + S - \lambda \in \Phi_\ell^\alpha(\mathcal{H})$.
Since $T = T \in \Phi_\ell^\alpha(\mathcal{H})$. Then, we have

$$TS - \lambda(T + S - \lambda) = ST - \lambda(T + S - \lambda) \in \Phi_\ell^\alpha(\mathcal{H}),$$

i.e, $(T - \lambda)(S - \lambda) = (S - \lambda)(T - \lambda) \in \Phi_\ell^\alpha(\mathcal{H})$.
We can apply Remark 3.1 we infer that

$$(T - \lambda) \in \Phi_\ell^\alpha(\mathcal{H}) \text{ and } (S - \lambda) \in \Phi_\ell^\alpha(\mathcal{H}).$$

This prove that:

$$[\Pi_\alpha(T) \cup \Pi_\alpha(S)]\backslash\{0\} \subseteq \Pi_\alpha(T+S)\backslash\{0\}.$$

Therefore,

$$\Pi_\alpha(T+S)\backslash\{0\} = [\Pi_\alpha(T) \cup \Pi_\alpha(S)]\backslash\{0\}.$$

□

Lemma 3.11. *Let $T \in \mathcal{L}(\mathcal{H})$ and $\mathcal{H}$ be a direct sum of closed subspaces $\mathcal{L}(\mathcal{H}_1)$and $\mathcal{L}(\mathcal{H}_2)$ which are $T-$invariant. If $T_1 = T_{|\mathcal{H}_1} : \mathcal{H}_1 \longrightarrow \mathcal{H}_1$ and $T_2 = T_{|\mathcal{H}_2} : \mathcal{H}_2 \longrightarrow \mathcal{H}_2$. Then*

$$\Pi_\alpha(T) = \Pi_\alpha(T_1) \cup \Pi_\alpha(T_2). \tag{5}$$

Proof. We shall show that $\Pi_\alpha(T) \subset \Pi_\alpha(T_1) \cup \Pi_\alpha(T_2)$. The operator T has the following matrix form with respect to the decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$:

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} : \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{bmatrix} \longrightarrow \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{bmatrix}.$$

Let $\lambda \notin \Pi_\alpha(T_1) \cup \Pi_\alpha(T_2)$. Therefore,

$$\lambda \notin \Pi_\alpha(T_1) \textit{ and } \lambda \notin \Pi_\alpha(T_2).$$

which implies that

$$T_1 - \lambda \in \Phi_\ell^\alpha(\mathcal{H}_1) \textit{ and } T_2 - \lambda \in \Phi_\ell^\alpha(\mathcal{H}_2).$$

More precisely, $\exists\, T_1^1 \in \mathcal{L}(\mathcal{H}_1)$, $T_2^1 \in \mathcal{L}(\mathcal{H}_2)$,$F_1 \in \mathcal{F}(\mathcal{H}_1)$ and $F_2 \in \mathcal{F}(\mathcal{H}_2)$ such that,

$$\begin{cases} T_1^1(T_1 - \lambda) = I - F_1 \\ \\ T_2^1(T_2 - \lambda) = I - F_2 \end{cases}$$

Let

$$T^0 = \begin{bmatrix} T_1^1 & 0 \\ 0 & T_2^1 \end{bmatrix}, F^0 = \begin{bmatrix} F_1 & 0 \\ 0 & F_2 \end{bmatrix}.$$

Since,

$$T - \lambda = \begin{bmatrix} T_1 - \lambda & 0 \\ 0 & T_2 - \lambda \end{bmatrix}.$$

Then,

$$\begin{aligned} T^0(T-\lambda) &= \begin{bmatrix} T_1^1 & 0 \\ 0 & T_2^1 \end{bmatrix} \begin{bmatrix} T_1 - \lambda & 0 \\ 0 & T_2 - \lambda \end{bmatrix} \\ &= \begin{bmatrix} T_1^1(T_1 - \lambda) & 0 \\ 0 & T_2^1(T_2 - \lambda) \end{bmatrix} \\ &= \begin{bmatrix} I - F_1 & 0 \\ 0 & I - F_2 \end{bmatrix} = I - F^0 \end{aligned}$$

where $F^0 \in \mathcal{K}_\alpha(\mathcal{H})$, then $\lambda \notin \Pi_\alpha(T)$.
To prove the inverse inclusion of Eq.(5), let $\lambda \notin \Pi_\alpha(T)$.
Therefore,

$$T - \lambda \in \Phi_\ell^\alpha(\mathcal{H}).$$

Which implies that exists $F \in \mathcal{F}_\alpha(\mathcal{H})$ and $S \in \mathcal{L}(\mathcal{H})$ such that

$$S(T - \lambda) = I - F \tag{6}$$

with $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ and

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}; F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}.$$

Then, equation (6) is equivalent to

$$\begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}\begin{bmatrix} T_1-\lambda & 0 \\ 0 & T_2-\lambda \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}\begin{bmatrix} T_1-\lambda & 0 \\ 0 & T_2-\lambda \end{bmatrix}.$$

Which give

$$\begin{bmatrix} S_{11}(T_1-\lambda) & S_{12}(T_2-\lambda) \\ S_{21}(T_1-\lambda) & S_{22}(T_2-\lambda) \end{bmatrix} = \begin{bmatrix} I-F_{11} & -F_{12} \\ -F_{21} & I-F_{22} \end{bmatrix}.$$

This proved that $S_{11}(T_1-\lambda) = I - F_{11}$; $S_{11} \in \mathcal{L}(\mathcal{H}_1)$ and $F_{11} \in \mathcal{F}_\alpha(\mathcal{H}_1)$. Also, $S_{22}(T_2-\lambda) = I - F_{22}$; $S_{22} \in \mathcal{L}(\mathcal{H}_2)$ and $F_{22} \in \mathcal{F}_\alpha(\mathcal{H}_2)$ and in this case, we obtain

$$\lambda \notin \Pi_\alpha(T_1) \cup \Pi_\alpha(T_2).$$

This proof is complete. □

Corollary 3.12. *Let $T \in \mathcal{L}(\mathcal{H})$, $n \in \mathcal{N}$ and let $\mathcal{H}$ be a direct sum of n closed subspaces $\mathcal{H}_1, \mathcal{H}_2, ...\mathcal{H}_n$ which are $T-$invariant. If*

$$T_1 = T_{|\mathcal{H}_1} : \mathcal{H}_1 \longrightarrow \mathcal{H}_1$$

$$T_2 = T_{|\mathcal{H}_2} : \mathcal{H}_2 \longrightarrow \mathcal{H}_2$$

$$\vdots$$

$$T_n = T_{|\mathcal{H}_n} : \mathcal{H}_n \longrightarrow \mathcal{H}_n.$$

Then,

$$\Pi_\alpha(T) = \cup_{k=1}^{n} \Pi_\alpha(T_k).$$

Proof. Follows from the previous Theorem. □

4. Application to operator matrix

Let $\mathcal{H}$ be non separable Hilbert space such that $\dim(\mathcal{H}) \geq \aleph_0$. Let $A_1, A_2, B_1, B_2, C \in \mathcal{H}$. In the sum space $\mathcal{H} \oplus \mathcal{H}$, we consider two operators defined by 2×2 block operators matrix

$$M_c^1 = \begin{bmatrix} A_1 & C \\ 0 & B_1 \end{bmatrix}; M_c^2 = \begin{bmatrix} A_2 & C \\ 0 & B_2 \end{bmatrix}.$$

The main purpose of this section is to establish criteria which ensure that the difference of the resolvent of M_c^1 and M_c^2 is compact. We suppose for $i \in \{1, 2\}$, $\mathcal{D}(B_i) \subset C$ and $\rho(A_i) \cap \rho(B_i)$ is not empty. It is easy to see that if $z \in \rho(A_i) \cap \rho(B_i)$, $z \in \rho(M_c^i)$, $i = 1, 2$ and we get

$$(M_c^i - z)^{-1} = \begin{bmatrix} (A_i - z)^{-1} & -(A_i - z)^{-1}C(B_i - z)^{-1} \\ 0 & (B_i - z)^{-1} \end{bmatrix}.$$

Theorem 4.1. *Assume that $\rho(A_1) \cap \rho(A_2) \cap \rho(B_1) \cap \rho(B_2)$ is empty. If*

$$(A_1 - z)^{-1}(A_1 - A_2)(A_2 - z)^{-1} \in \mathcal{F}_\alpha(\mathcal{H})$$

and

$$(B_1 - z)^{-1}(B_1 - A_2)(B_2 - z)^{-1} \in \mathcal{F}_\alpha(\mathcal{H}).$$

Then,

$$\Pi_\alpha(M_c^1) = \Pi_\alpha(M_c^2).$$

Proof. It is easy to see that if $z \in \rho(A_i) \cap \rho(B_i)$, then $z \in \rho(M_c^i)$, $i = 1, 2$ and we get

$$(M_c^i - z)^{-1} = \begin{bmatrix} (A_i - z)^{-1} & -(A_i - z)^{-1}C(B_i - z)^{-1} \\ 0 & (B_i - z)^{-1} \end{bmatrix}$$

on the other hand, If $z \in \rho(A_1) \cap \rho(A_2) \cap \rho(B_1) \cap \rho(B_2)$, then $z \in \rho(M_c^1) \cap \rho(M_c^2)$ and

$$(M_c^1 - z)^{-1} - (M_c^2 - z)^{-1} = \begin{bmatrix} R_1(z) & R_3(z) \\ 0 & R_2(z) \end{bmatrix}$$

where
$R_1(z) = (A_1 - z)^{-1}(A_1 - A_2)(A_2 - z)^{-1}$,
$R_2(z) = (B_1 - z)^{-1}(B_1 - A_2)(B_2 - z)^{-1}$,
$R_3(z) = -R_1(z)C(B_2 - z)^{-1} - (A_1 - z)^{-1}$. Then,

$$(M_c^1 - z)^{-1} - (M_c^2 - z)^{-1} \in \mathcal{F}_\alpha(\mathcal{H} \oplus \mathcal{H}).$$

According to lemma 3.11, we obtain

$$\Pi_\alpha(M_c^1) = \Pi_\alpha(M_c^2).$$

□

5. Conclusion

In this paper, Fredholm operator, α-Fredholm operator and Weyl operator are introduced and investigated. Some specially properties and application to operators matrix are given.

References

[1] Aiena, P. Semi-Fredholm operators, perturbation theory and localized SVEP, XXEscuela venezolana de Mtematicas, E.D. Ivic, Merida(Venezuela) (2007).
[2] Baloudi, H; Jeribi, A. Left-right Fredholm and weyl spectra of the sum of two Bounded operators and application, Mediterranean journal of mathematics 11(3), 939-953 (2014) .
[3] Ernest, J. Left Invertibility of closed operators, Tôhoku Math. Journ. **24**, 529-537 (1972).
[4] G. Edgar, J. Ernest and S. G. Lee: Weighing Operator spectra, Indiana University Mathematics Journal 21(1), (1971).
[5] Athmouni, N; Baloudi, H , Jeribi, A; Kacem,G. On weighted and pseudo-weighted spectra of bounded operators, Commun. Korean Math. Soc. 33 (2018), No. 3, pp. 809–821.