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On Symmetric Meir-Keeler Contraction Type Couplings with an Application

Hassen Aydi[a,b,c], Hüseyin Işık[d], M.A. Barakat[e,f], Abdelbasset Felhi[g]

[a]Université de Sousse, Institut Supérieur d'Informatique et des Techniques de Communication, H. Sousse 4000, Tunisia
[b]Department Of Medical Research, China Medical University Hospital, China Medical University, Taichung 40402, Taiwan
[c]Department of Mathematics and Applied Mathematics, Sefako Makgatho Health Sciences University, Ga-Rankuwa, South Africa
[d]Department of Engineering Science, Bandırma Onyedi Eylül University, Bandırma 10200, Balıkesir, Turkey
[e]Department of Mathematics, College of Al Wajh, University of Tabuk, Saudi Arabia
[f]Department of Mathematics, Faculty of Sciences, Al-Azhar University, Assiut 71524, Egypt
[g]Institut préparatoire aux études d'ingénieurs de Bizerte, Carthage University, Tunisia

Abstract. Recently, a class of mappings named as couplings was introduced in [U.P.B. Sci. Bull. Series A, 79 (2017), 1-12]. Based on this concept, we introduce symmetric Meir-Keeler couplings and we ensure the existence of strong coupled fixed points. We present some concrete examples to support the obtained results. Furthermore, as an application of our results, we investigate the existence of a unique solution to a system of integral equations.

1. Introduction and Preliminaries

Let F be a self-mapping on a nonempty set X. If the system of equations for which a solution is sought is of the form $Tx = 0$, the function T could be represented as $Tx = Fx - x$. Then, a fixed point of F is a solution to $Tx = 0$. On that account, a wide variety of mathematical and practical problems can be solved by reducing them to an equivalent fixed point problem. This situation motivates researchers to study on extensions and generalizations of the well-known Banach fixed point theorem [4] considered as the source of metrical fixed point theory. One of the most interesting generalizations this phenomenon theorem has been given by Meir and Keeler [14], in 1969. Their main result is as follows.

Theorem 1.1. *Let (X, d) be a complete metric space and $T : X \to X$ be a given mapping. Suppose that for all $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that*

$$x, y \in X, \quad \varepsilon \le d(x, y) < \varepsilon + \delta(\varepsilon) \Rightarrow d(Tx, Ty) < \varepsilon. \tag{1}$$

Then T has a unique fixed point.

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Communicated by Dragan S. Djordjević
Corresponding author: Hassen Aydi
Email addresses: hassen.aydi@isima.rnu.tn (Hassen Aydi), isikhuseyin76@gmail.com (Hüseyin Işık), barakat14285@yahoo.com (M.A. Barakat), abdelbassetfelhi@gmail.com (Abdelbasset Felhi)

For papers dealing with Meir-Keeler contractions, see [1–3, 5, 11, 12]. On the other hand, the notion of a coupled fixed point was first introduced by Opoitsev [15] and investigated later by Guo and Lakshmikantham [8].

Definition 1.2. *[8, 15] An element $(x, y) \in X^2$ is called a coupled fixed point of the mapping $F : X^2 \to X$ if*

$$x = F(x, y) \quad \text{and} \quad y = F(y, x).$$

Definition 1.3. *[13] Let A and B be two nonempty subsets of a given set X. Any function $f : X \to X$ is said to be cyclic (with respect to A and B) if $f(A) \subset B$ and $f(B) \subset A$.*

Generalizing the concept of cyclic mappings, very recently Choudhury et al. [6] introduced the concept of couplings between two non-empty subsets in a metric space.

Definition 1.4. *[6] Let (X, d) be a metric space and let A and B be two non-empty subsets of X. A coupling with respect to A and B is a function $F : X \times X \to X$ such that $F(x, y) \in B$ and $F(y, x) \in A$ whenever $x \in A$ and $y \in B$.*

Definition 1.5. *[7] Let X be a nonempty set. An element $(x, y) \in X \times X$ is said a strong coupled fixed point of the mapping $F : X \times X \to X$ if (x, y) is a coupled fixed point and $x = y$, that is, if $x = F(x, x)$.*

The main result proved by Choudhury et al. [6] is given as follows.

Theorem 1.6. *[6] Let A and B be two non-empty closed subsets of a complete metric space (X, d). Let $F : X^2 \to X$ be such that*

$$d(F(x, y), F(u, v)) \leq \frac{k}{2}[d(x, u) + d(y, v)], \tag{2}$$

where $x, v \in A$, $y, u \in B$ and $k \in (0, 1)$. Then $A \cap B \neq \emptyset$ and F has a unique strong coupled fixed point in $A \cap B$.

The mapping F satisfying (2) is said a Banach type coupling with respect to A and B. The aim of this paper is the investigation of strong coupled fixed points for couplings satisfying symmetric generalized Meir-Keeler contractions in the setting of metric spaces. Moreover, we provide some examples where we could not apply Theorem 1.6. Also, as an application of our results, we discuss the existence of a unique solution to a system of integral equations.

2. Main results

First, we introduce the following.

Definition 2.1. *Let A and B be two non-empty subsets of a metric space (X, d). A coupling $F : X^2 \to X$ is said a symmetric Meir-Keeler contraction type coupling with respect to A and B if for all $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that, for all $x, v \in A$ and $y, u \in B$,*

$$\begin{aligned} &\varepsilon \leq \frac{1}{2}[d(x, u) + d(y, v)] < \varepsilon + \delta(\varepsilon) \\ &\quad \Longrightarrow \frac{1}{2}[d(F(x, y), F(u, v)) + d(F(y, x), F(v, u))] < \varepsilon. \end{aligned} \tag{3}$$

We have the following useful lemma.

Lemma 2.2. *Let A and B be two non-empty subsets of a metric space (X, d) and $F : X^2 \to X$ be a symmetric Meir-Keeler contraction type coupling with respect to A and B. Then*

$$d(F(x, y), F(u, v)) + d(F(y, x), F(v, u)) \leq d(x, u) + d(y, v),$$

for all $x, v \in A$ and $y, u \in B$.

Proof. Let $x, v \in A$, $y, u \in B$. If $d(x,u) + d(y,v) > 0$, then for $\varepsilon = \frac{1}{2}[d(x,u) + d(y,v)]$, and since F is a symmetric Meir-Keeler contraction type coupling with respect to A and B, there exists $\delta(\varepsilon) > 0$ such that

$$\begin{aligned} &\varepsilon \leq \frac{1}{2}[d(x,u) + d(y,v)] < \varepsilon + \delta(\varepsilon) \\ &\quad\Longrightarrow \frac{1}{2}[d(F(x,y), F(u,v)) + d(F(y,x), F(v,u))] < \varepsilon. \end{aligned}$$

It follows that

$$d(F(x,y), F(u,v)) + d(F(y,x), F(v,u)) < d(x,u) + d(y,v).$$

Also, if $d(x,u) + d(y,v) = 0$, then $x = u$ and $y = v$ and so

$$d(F(x,y), F(u,v)) + d(F(y,x), F(v,u)) = 0 \leq d(x,u) + d(y,v).$$

□

Our main result is stated in the following.

Theorem 2.3. *Let A and B be two non-empty closed subsets of a complete metric space (X,d). Let $F : X^2 \to X$ be a symmetric Meir-Keeler contraction type coupling with respect to A and B. Then $A \cap B \neq \emptyset$ and F has a unique strong coupled fixed point in $A \cap B$.*

Proof. Let $x_0 \in A$ and $y_0 \in B$ be any two arbitrary elements. Choose

$$x_1 = F(y_0, x_0) \quad \text{and} \quad y_1 = F(x_0, y_0).$$

Mention that $x_1 \in A$ and $y_1 \in B$. By repeating this process, we get the two sequences $\{x_n\}$ and $\{y_n\}$ such that

$$x_{n+1} = F(y_n, x_n) \quad \text{and} \quad y_{n+1} = F(x_n, y_n), \quad \text{for all } n \in \mathbb{N}.$$

Clearly, we have for all $n \geq 0$ that $x_n \in A$ and $y_n \in B$. Let us choose

$$\lambda_n := \frac{1}{2}[d(x_n, y_{n+1}) + d(y_n, x_{n+1})], \quad \text{for } n \geq 0.$$

From (3) and Lemma 2.2, we have

$$\begin{aligned} \lambda_{n+1} &= \frac{1}{2}[d(x_{n+1}, y_{n+2}) + d(y_{n+1}, x_{n+2})] \\ &= \frac{1}{2}[d(F(y_n, x_n), F(x_{n+1}, y_{n+1})) + d(F(x_n, y_n), F(y_{n+1}, x_{n+1}))] \\ &\leq \frac{1}{2}[d(x_n, y_{n+1}) + d(y_n, x_{n+1})] = \lambda_n. \end{aligned}$$

The sequence $\{\lambda_n\}$ is nonincreasing and is bounded below, so it converges to some $\varepsilon \geq 0$. Suppose that $\varepsilon > 0$. Mention that $\varepsilon = \inf\{\lambda_n : n = 0, 1, 2, \cdots\}$. Thus there exists $p \in \mathbb{N}$ such that

$$\varepsilon \leq \lambda_p = \frac{1}{2}[d(x_p, y_{p+1}) + d(y_p, x_{p+1})] < \varepsilon + \delta(\varepsilon),$$

where $\delta(\varepsilon) > 0$ is any arbitrary constant. Using (3), we get

$$\lambda_{p+1} = \frac{1}{2}[d(F(y_p, x_p), F(x_{p+1}, y_{p+1})) + d(F(x_p, y_p), F(y_{p+1}, x_{p+1})) < \varepsilon,$$

which is a contradiction. Then $\varepsilon = 0$, and so

$$\lim_{n\to\infty}[d(x_n, y_{n+1}) + d(y_n, x_{n+1})] = 0. \tag{4}$$

Let $\eta_n = d(x_n, y_n)$, for $n \geq 0$. From (3) and Lemma 2.2, we have

$$\begin{aligned}
\eta_{n+1} &= \frac{1}{2}[d(x_{n+1}, y_{n+1}) + d(y_{n+1}, x_{n+1})] \\
&= \frac{1}{2}[d(F(y_n, x_n), F(x_n, y_n)) + d(F(x_n, y_n), F(y_n, x_n)) \\
&\leq \frac{1}{2}[d(x_n, y_n) + d(y_n, x_n)] \\
&= \eta_n.
\end{aligned}$$

The sequence $\{\eta_n\}$ is nonincreasing, so it converges to some $a \geq 0$. Similarly, we may prove that $a = 0$, and so

$$\lim_{n\to\infty} d(x_n, y_n) = 0. \tag{5}$$

We have

$$d(x_n, x_{n+1}) + d(y_n, y_{n+1}) \leq 2d(x_n, y_n) + d(y_n, x_{n+1}) + d(x_n, y_{n+1}), \quad \forall\, n \in \mathbb{N}.$$

It follows from (4) and (5),

$$\lim_{n\to\infty}[d(x_n, x_{n+1}) + d(y_n, y_{n+1})] = 0.$$

Then for each $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, one can write

$$\begin{cases} d(x_n, y_n) < \varepsilon, \\ d(x_n, x_{n+1}) + d(y_n, y_{n+1}) < 2\delta(\varepsilon), \\ d(x_n, y_{n+1}) + d(x_n, y_{n+1}) < 2[\delta(\varepsilon) + \varepsilon]. \end{cases} \tag{6}$$

Notice that the condition (3) is also true if we replace $\delta(\varepsilon)$ by $\delta'(\varepsilon) = \min\{\delta(\varepsilon), \varepsilon\}$. So, without restriction of the generality, we can suppose that $\delta(\varepsilon) \leq \varepsilon$. Let $n \geq N$. We will prove by induction on p that

$$\frac{1}{2}[d(x_n, y_{n+p}) + d(y_n, x_{n+p})] < \delta(\varepsilon) + \varepsilon, \quad \forall\, p \in \mathbb{N}. \tag{7}$$

This inequality is true for $p = 1$ due to (6). Suppose that (7) is true for all $k = 1, 2, \cdots, p$. Then, we have

$$\frac{1}{2}[d(x_n, y_{n+p}) + d(y_n, x_{n+p})] < \delta(\varepsilon) + \varepsilon.$$

From (6),

$$\begin{aligned}
&\frac{1}{2}[d(x_n, y_{n+p+1}) + d(y_n, x_{n+p+1})] \\
&\quad \leq \frac{1}{2}[d(x_n, x_{n+1}) + d(y_n, y_{n+1})] + \frac{1}{2}[d(x_{n+1}, y_{n+p+1}) + d(y_{n+1}, x_{n+p+1})] \\
&\quad \leq \delta(\varepsilon) + \frac{1}{2}[d(F(y_n, x_n), F(x_{n+p}, y_{n+p})) + d(F(x_n, y_n), F(y_{n+p}, x_{n+p}))].
\end{aligned}$$

We distinguish the following two cases.
Case 1. $\frac{1}{2}[d(y_n, x_{n+p}) + d(x_n, y_{n+p})] < \varepsilon$. Using Lemma 2.2, we obtain

$$\begin{aligned}
&d(F(y_n, x_n), F(x_{n+p}, y_{n+p})) + d(F(x_n, y_n), F(y_{n+p}, x_{n+p})) \\
&\quad \leq d(y_n, x_{n+p}) + d(x_n, y_{n+p}).
\end{aligned}$$

Therefore, in this case

$$\begin{aligned}\frac{1}{2}[d(x_n, y_{n+p+1}) + d(x_n, y_{n+p+1})] &\le \delta(\varepsilon) + \frac{1}{2}[d(y_n, x_{n+p}) + d(x_n, y_{n+p})] \\ &< \delta(\varepsilon) + \varepsilon.\end{aligned}$$

Case 2. $\varepsilon \le \frac{1}{2}[d(y_n, x_{n+p}) + d(x_n, y_{n+p})] < \varepsilon + \delta(\varepsilon)$. In this case, using (3), we obtain

$$\frac{1}{2}[d(F(y_n, x_n), F(x_{n+p}, y_{n+p})) + d(F(x_n, y_n), F(y_{n+p}, x_{n+p}))] < \varepsilon.$$

Thus,

$$\frac{1}{2}[d(x_n, y_{n+p+1}) + d(x_n, y_{n+p+1})] < \delta(\varepsilon) + \varepsilon.$$

Consequently, the inequality (7) holds for $k = p + 1$. Hence (7) is true for all $p \in \mathbb{N}$. On the hand, using (6) and (7), for all $n \ge N$ and for all $p \in \mathbb{N}$, we have

$$\begin{aligned}d(x_n, x_{n+p}) + d(y_n, y_{n+p}) &\le 2d(x_n, y_n) + d(x_n, y_{n+p}) + d(y_n, x_{n+p}) \\ &< 2\varepsilon + 2[\delta(\varepsilon) + \varepsilon] \\ &\le 6\varepsilon.\end{aligned}$$

So $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences. Since A and B are closed subsets of the complete metric space (X, d), there exists $x \in A$ and $y \in B$ such that

$$\lim_{n\to\infty} d(x_n, x) = \lim_{n\to\infty} d(y_n, y) = 0.$$

Also, the continuity of the metric function d together with (5) imply that $\lim_{n\to\infty} d(x_n, y_n) = d(x, y) = 0$. Hence $x = y$, then $A \cap B \neq \emptyset$. Now, we will show that x is a strong coupled fixed point in $A \cap B$. From (3) and Lemma 2.2, we have

$$\begin{aligned}d(x, F(x, y)) &\le d(x, x_{n+1}) + d(x_{n+1}, F(x, y)) \\ &= d(x, x_{n+1}) + d(F(x, y), F(y_n, x_n)) \\ &\le d(x, x_{n+1}) + d(F(x, y), F(y_n, x_n)) + d(F(y, x), F(x_n, y_n)) \\ &\le d(x, x_{n+1}) + d(y_n, x) + d(x_n, y).\end{aligned}$$

Passing to limit as $n \to \infty$ in the above inequality, we have

$$d(x, F(x, y)) \le 0,$$

that is, $d(x, F(x, x)) = 0$ so $x = F(x, x)$. Assume that $F(x, x) = x$ and $F(y, y) = y$ with $x \neq y$ and $x, y \in A \cap B$. From (3) and for $\varepsilon = d(x, y) > 0$, there exists $\delta(\varepsilon) > 0$ such that

$$\begin{aligned}&\varepsilon \le \frac{1}{2}[d(x, y) + d(x, y)] < \varepsilon + \delta(\varepsilon) \\ &\implies d(x, y) = \frac{1}{2}[d(F(x, x), F(y, y)) + d(F(x, x), F(y, y))] < \varepsilon = d(x, y),\end{aligned}$$

which is a contradiction. This implies that $x = y$, i.e., the strong coupled fixed point is unique. □

Remark 2.4. *Note that if F satisfies symmetric Banach contractive type coupling, that is, there exists a constant $k \in (0, 1)$ such that for $x, v \in A$ and $y, u \in B$,*

$$d(F(x, y), F(u, v)) + d(F(y, x), F(v, u)) \le k[d(x, u) + d(y, v)],$$

then F is a symmetric Meir-Keeler contraction type coupling. Indeed, by taking $\delta(\varepsilon) = (\frac{1}{k} - 1)\varepsilon$, it is easy to see that

$$\varepsilon \leq \frac{1}{2}[d(x,u) + d(y,v)] < \varepsilon + \delta(\varepsilon)$$
$$\Longrightarrow \frac{1}{2}[d(F(x,y), F(u,v)) + d(F(y,x), F(v,u))] < \varepsilon.$$

for all $x, v \in A$ and $y, u \in B$.

The following examples illustrate Theorem 4 where Theorem 1.6 is not applicable.

Example 2.5. *Let $X = \mathbb{R}$ and $d(x,y) = |x - y|$ for all $x, y \in X$. Consider the closed subsets $A = \{0\}$ and $B = [0,1]$. Define $F : X \times X \to X$ as*

$$F(x,y) = \begin{cases} \frac{y-x}{2}, & \text{if } x < y, \\ 0, & \text{if } x \geq y. \end{cases}$$

For all $(x,y) \in A \times B$, we have $0 \leq y$. If $0 < y$, we get $F(x,y) = \frac{y}{2} \in B$ and $F(y,x) = 0 \in A$. While, if $y = 0$, then $F(x,y) = 0 = F(y,x) \in A \cap B$. Then F is a coupling with respect to A and B.

Let $\varepsilon > 0$ and let $x, v \in A$ and $y, u \in B$, such that $\varepsilon \leq \frac{1}{2}[d(x,u) + d(y,v)] < \varepsilon + \delta(\varepsilon)$ where $\delta(\varepsilon) > 0$ will be chosen later. So $x = v = 0$ and $y, u \in [0,1]$. We distinguish the following cases.

Case 1. If $y \in (0,1]$ and $u = 0$, we have

$$\begin{aligned} &d(F(x,y), F(u,v)) + d(F(y,x), F(v,u)) \\ &= d(\frac{y}{2}, 0) + d(0,0) \\ &= \frac{1}{2}y = \frac{1}{2}[d(x,u) + d(y,v)]. \end{aligned}$$

Case 2. If $y, u \in (0,1]$, we have

$$\begin{aligned} &d(F(x,y), F(u,v)) + d(F(y,x), F(v,u)) \\ &= d(\frac{y}{2}, 0) + d(\frac{u}{2}, 0) \\ &= \frac{1}{2}(y + u) \\ &= \frac{1}{2}[d(x,u) + d(y,v)]. \end{aligned}$$

Case 3. If $y = u = 0$, we have

$$d(F(x,y), F(u,v)) + d(F(y,x), F(v,u)) = 0 = \frac{1}{2}[d(x,u) + d(y,v)].$$

Case 4. If $y = 0$ and $u \in (0,1]$, we have

$$\begin{aligned} &d(F(x,y), F(u,v)) + d(F(y,x), F(v,u)) \\ &= d(0,0) + d(\frac{u}{2}, 0) \\ &= \frac{1}{2}u = \frac{1}{2}[d(x,u) + d(y,v)]. \end{aligned}$$

We conclude that

$$\begin{aligned} &\frac{1}{2}[d(F(x,y), F(u,v)) + d(F(y,x), F(v,u))] \\ &= \frac{1}{4}[d(x,u) + d(y,v)] \\ &< \frac{1}{2}(\varepsilon + \delta(\varepsilon)) < \varepsilon, \end{aligned}$$

which holds if we choose $\delta(\varepsilon) < \varepsilon$. Hence (3) holds. All hypotheses of Theorem 4 are verified, so F has a unique strong coupled fixed point, which is 0.

Note that Theorem 1.6 is not applicable. Indeed, taking $x = u = v = 0$ and $y = 1$, we have

$$\begin{aligned} d(F(0,1), F(0,0)) &= d(\frac{1}{2}, 0) \\ &= \frac{1}{2} > \frac{k}{2} \\ &= \frac{k}{2}[d(0,0) + d(1,0)], \end{aligned}$$

for each $k \in [0,1)$.

Example 2.6. *Let $X = \mathbb{R}$ and $d(x,y) = |x-y|$ for all $x, y \in X$. Consider the closed subsets $A = [-1,0]$ and $B = [0,1]$. Define $F : X \times X \to X$ as*

$$F(x,y) = \frac{y}{2+x^2}.$$

For all $x \in A = [-1,0]$ and $y \in B = [0,1]$, we have

$$F(x,y) = \frac{y}{2+x^2} \in [0,1] \quad \text{and} \quad F(y,x) = \frac{x}{2+y^2} \in [-1,0].$$

Then F is a coupling with respect to A and B.
Let $\varepsilon > 0$. Take $x, v \in A$ and $y, u \in B$ such that $\varepsilon \leq \frac{1}{2}[d(x,u) + d(y,v)] < \varepsilon + \delta(\varepsilon)$, where $0 < \delta(\varepsilon) < \varepsilon$. We have

$$\begin{aligned} &\frac{1}{2}[d(F(x,y), F(u,v)) + d(F(y,x), F(v,u))] \\ &= \frac{1}{2}[|\frac{y}{2+x^2} - \frac{v}{2+u^2}| + |\frac{x}{2+y^2} - \frac{u}{2+v^2}|] \\ &\leq \frac{1}{4}[|y-v| + |x-u|] \\ &< \frac{1}{2}[\varepsilon + \delta(\varepsilon)] \\ &\leq \varepsilon. \end{aligned}$$

All hypotheses of Theorem 4 are verified, so F has a unique strong coupled fixed point, which is 0.

Note that Theorem 1.6 is not applicable. Indeed, taking $x = u = 0$, we have

$$\begin{aligned} d(F(0,y), F(0,v)) &= d(\frac{y}{2}, \frac{v}{2}) = \frac{1}{2}|y-v| \\ &\not\leq \frac{k}{2}|y-v| = \frac{k}{2}[d(x,u) + d(y,v)], \end{aligned}$$

for each $k \in [0,1)$.

Example 2.7. *Let $X = \mathbb{R}$ and $d(x,y) = |x-y|$ for all $x, y \in X$. Consider the closed subsets $A = \{0\}$ and $B = \mathbb{R}$. Define $F : X \times X \to X$ as*

$$F(x,y) = \frac{(x+3)y}{6}.$$

For all $x = 0$ and $y \in \mathbb{R}$, we have

$$F(x,y) = \frac{3y}{6} \in B = \mathbb{R} \quad \text{and} \quad F(y,x) = \frac{(y+3)x}{6} = 0 \in A.$$

Then F is a coupling with respect to A and B.
Let $\varepsilon > 0$. Choose $x, v \in A$ and $y, u \in B$ such that

$$\varepsilon \leq \frac{1}{2}[d(x,u) + d(y,v)] = \frac{1}{2}(|u| + |y|) < \varepsilon + \delta(\varepsilon)$$

where $0 < \delta(\varepsilon) < \varepsilon$. We have

$$\begin{aligned}
&\frac{1}{2}[d(F(x,y), F(u,v)) + d(F(y,x), F(v,u))] \\
&\quad = \frac{1}{2}[|\frac{3y}{6} - 0| + |0 - \frac{3u}{6}|] \\
&\quad \leq \frac{1}{4}[|y| + |u|] \\
&\quad < \frac{1}{2}[\varepsilon + \delta(\varepsilon)] \leq \varepsilon.
\end{aligned}$$

All hypotheses of Theorem 4 are verified, so F has a unique strong coupled fixed point, which is 0.
Note that Theorem 1.6 is not applicable. Indeed, taking $x = v = u = 0$, we have

$$\begin{aligned}
d(F(0,y), F(0,0)) &= d(\frac{y}{2}, 0) = \frac{1}{2}|y| \\
&\nleq \frac{k}{2}|y| = \frac{k}{2}[d(x,u) + d(y,v)],
\end{aligned}$$

for each $k \in [0,1)$.

3. An Application

Consider the following, a system of integral equations:

$$\begin{cases} x(s) = \int_a^b H(s, r, x(r), y(r))\, dr \\ y(s) = \int_a^b H(s, r, y(r), x(r))\, dr \end{cases} \tag{8}$$

where $s \in I = [a,b]$, $H : I \times I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $b > a \geq 0$.

In this section, we prove the existence of a unique solution to the system of integral equations (8) that belongs to the space $C(I, \mathbb{R})$ of the continuous functions defined on I and with real values.

Let $X := C(I, \mathbb{R})$ be endowed with the metric d defined by

$$d(x,y) = \sup_{s \in I} |x(s) - y(s)|,$$

for all $x, y \in X$. Define an operator $F : X \times X \to X$ by

$$F(x,y)(s) = \int_a^b H(s, r, x(r), y(r))\, dr, \quad x, y \in X, \quad s \in I.$$

Then the existence of a unique solution to (8) is equivalent to the existence of a strong coupled fixed point of F.

Theorem 3.1. *Assume that the following conditions are satisfied:*

(i) $H : I \times I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is continuous,* (*ii*) *for each* $(x,y),(u,v) \in X^2$ *and* $s,r \in I$, *we have*

$$\left|H(s,r,x(r),y(r)) - H(s,r,u(r),v(r))\right|$$
$$\leq \gamma(s,r)\,k[|x(r)-u(r)| + |y(r)-v(r)|],$$

where $k \in (0,1)$ *and* $\gamma : I \times I \to [0,+\infty)$ *is a continuous function satisfying*

$$\sup_{s\in I} \int_a^b \gamma(s,r)\,dr \leq \frac{1}{2}.$$

Then, the system of integral equations (8) has a unique solution in X.

Proof. Let $(x,y),(u,v) \in X \times X$. Using (*ii*), we obtain

$$\begin{aligned}
&\left|F(x,y)(s) - F(u,v)(s)\right| \\
&\leq \int_a^b \left|H(s,r,x(r),y(r)) - H(s,r,u(r),v(r))\right| dr \\
&\leq \int_a^b \gamma(s,r)\,k[|x(r)-u(r)| + |y(r)-v(r)|]dr \\
&\leq k \int_a^b \gamma(s,r)\,[d(x,u) + d(y,v)]dr \\
&\leq k \sup_{s\in I} \left(\int_a^b \gamma(s,r)\,dr\right)[d(x,u) + d(y,v)] \\
&\leq \frac{k}{2}[d(x,u) + d(y,v)].
\end{aligned}$$

for all $s \in I$. Therefore, we get

$$\left|F(x,y)(s) - F(u,v)(s)\right| \leq \frac{k}{2}[d(x,u) + d(y,v)]. \tag{9}$$

Similarly, one can also get

$$\left|F(y,x)(s) - F(v,u)(s)\right| \leq \frac{k}{2}[d(x,u) + d(y,v)]. \tag{10}$$

By (9) and (10), we have

$$d(F(x,y),F(u,v)) + d(F(y,x),F(v,u)) \leq k[d(x,u) + d(y,v)],$$

for all $(x,y),(u,v) \in X \times X$. Then, by Remark 2.4, F is a symmetric Meir-Keeler contraction type coupling by taking $A = B = X$. Consequently, from Theorem 2.3, F has a strong coupled fixed point, that is, the equation (8) has a unique solution in X. □

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