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Essential Norms of the Extended Cesàro Operators on Bergman Spaces with Exponential Weight in the Unit Ball

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Abstract. Let $\mathbb{B}_n = \{z \in \mathbb{C}^n : |z| < 1\}$ be the unit ball of the complex n-plane $\mathbb{C}^n$, g a holomorphic function in $\mathbb{B}_n$ and $A^2_{\alpha,\beta}(\mathbb{B}_n)$ the space of holomorphic functions that are L^2 with respect to a rapidly decreasing weight of form $\omega_{\alpha,\beta}(z) = (1-|z|)^\alpha e^{-\frac{\beta}{1-|z|}}$ on $\mathbb{B}_n$, where $\alpha \in \mathbb{R}$ and $\beta > 0$. In this paper, we compute the essential norm of the extended Cesàro operator T_g on $A^2_{\alpha,\beta}(\mathbb{B}_n)$. As a direct application, we obtain the essential norm for the one-variable case.

1. Introduction

Let $\mathbb{C}^n$ be the complex n-plane. If $z = (z_1, \dots, z_n)$, $w = (w_1, \dots, w_n) \in \mathbb{C}^n$, we write

$$\langle z, w\rangle = \sum_{j=1}^{n} z_j \overline{w_j}, \quad |z| = \langle z, z\rangle^{1/2}.$$

Let $\mathbb{B}_n = \{z \in \mathbb{C}^n : |z| < 1\}$ be the unit ball and $dv(z)$ denote the ordinary volume measure. The Bergman space with exponential weight, denoted by $A^2_{\alpha,\beta}(\mathbb{B}_n)$, consists of all holomorphic functions on $\mathbb{B}_n$ such that

$$\|f\|^2_{\alpha,\beta} = \int_{\mathbb{B}_n} |f(z)|^2 \omega_{\alpha,\beta}(z) dv(z) < +\infty,$$

where the rapidly decreasing weight $\omega_{\alpha,\beta}(z) = (1-|z|)^\alpha e^{-\frac{\beta}{1-|z|}}$, $\alpha \in \mathbb{R}$ and $\beta > 0$. Under the inner product

$$\langle f, g\rangle = \int_{\mathbb{B}_n} f(z)\overline{g(z)}(1-|z|)^\alpha e^{-\frac{\beta}{1-|z|}} dv(z),$$

$A^2_{\alpha,\beta}(\mathbb{B}_n)$ is a Hilbert space. Since each point evaluation is bounded on $A^2_{\alpha,\beta}(\mathbb{B}_n)$, there exists the reproducing kernel $K_{\alpha,\beta}(z, w)$. We know that $K_{\alpha,\beta}(z, w)$ is given by

$$K_{\alpha,\beta}(z, w) = \sum_{\gamma} \frac{z^\gamma \overline{w}^\gamma}{\|z^\gamma\|^2_{\alpha,\beta}}.$$

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Unfortunately, the explicit form of $K_{\alpha,\beta}(z,w)$ is unknown. One can see [15] for the one-variable theory of Bergman spaces with rapidly decreasing weights; see [3] for the several-variable theory.

Let $H(\mathbb{B}_n)$ be the space of all holomorphic functions on $\mathbb{B}_n$. For every $f \in H(\mathbb{B}_n)$, the radial derivative $\Re f$ of f is defined by

$$\Re f(z) = \sum_{j=1}^{n} z_j \frac{\partial f}{\partial z_j}(z).$$

For a fixed $g \in H(\mathbb{B}_n)$, the extended Cesàro operator T_g on some subspaces of $H(\mathbb{B}_n)$ is defined by

$$T_g(f)(z) = \int_0^1 f(tz)\Re g(tz)\frac{dt}{t}, \quad z \in \mathbb{B}_n.$$

This operator was first introduced by Hu in [8], and he explained the reasons why it was defined by such manner. The boundedness and compactness of T_g have been characterized for a large class of weights which satisfy certain conditions in terms of the symbol function g. We refer the readers to [1, 14, 15]. Recently, in [3] Cho and Park have obtained the following result.

Theorem 1.1. *Let $g \in H(\mathbb{B}_n)$. Then*

(1) T_g is bounded on $A^2_{\alpha,\beta}(\mathbb{B}_n)$ if and only if

$$\sup_{z\in\mathbb{B}_n}(1-|z|)^2|\Re g(z)| < +\infty.$$

(2) T_g is compact on $A^2_{\alpha,\beta}(\mathbb{B}_n)$ if and only if

$$\lim_{|z|\to 1}(1-|z|)^2|\Re g(z)| = 0.$$

Above mentioned result can be regarded as a prototype of the extended Cesàro operators on Bergman spaces with exponential weights in the several-variable theory. Here, we can also rethink Theorem 1.1 by the following way. For this, we need to introduce the weighted Bloch spaces.

Let $\alpha > 0$. The weighted Bloch space $\mathcal{B}^\alpha$ consists of all $f \in H(\mathbb{B}_n)$ such that

$$b(f) = \sup_{z\in\mathbb{B}_n}(1-|z|^2)^\alpha|\Re f(z)| < +\infty.$$

It is a Banach space with the norm $\|f\|_{\mathcal{B}^\alpha} = |f(0)| + b(f)$. As an important subspace of $\mathcal{B}^\alpha$, the little weighted Bloch space $\mathcal{B}^\alpha_0$ consists of all $f \in H(\mathbb{B}_n)$ such that

$$\lim_{|z|\to 1}(1-|z|^2)^\alpha|\Re f(z)| = 0.$$

The space $\mathcal{B}^\alpha_0$ is separable, since $\mathcal{B}^\alpha_0$ is the closure of the polynomials in $\mathcal{B}^\alpha$. One can see, for example, [21] for some information on the weighted Bloch spaces.

By the definitions of $\mathcal{B}^2$ and $\mathcal{B}^2_0$, Theorem 1.1 can be expressed into the following version.

Theorem 1.1′. *Let $g \in H(\mathbb{B}_n)$. Then*

(1) T_g is bounded on $A^2_{\alpha,\beta}(\mathbb{B}_n)$ if and only if $g \in \mathcal{B}^2$.

(2) T_g is compact on $A^2_{\alpha,\beta}(\mathbb{B}_n)$ if and only if $g \in \mathcal{B}^2_0$.

Motivated by such interesting observation and Theorem 1.1, here we compute the essential norm for this kind of operators. This paper can be regarded as a continuation of the investigations for the extended Cesàro operators on Bergman spaces with exponential weights in the unit ball.

Let X, Y be Banach spaces and $T : X \to Y$ a bounded linear operator. Recall that the essential norm of the bounded linear operator $T : X \to Y$, denoted by $\|T\|_e$, is defined as follows

$$\|T\|_e = \inf\left\{\|T-K\| : K \text{ is compact from } X \text{ to } Y\right\},$$

where $\|\cdot\|$ denotes the usual operator norm. From this definition and since the set of all compact operators is a closed subset of the space of bounded linear operators, it follows that the operator $T : X \to Y$ is compact if and only if $\|T\|_e = 0$. For some results in this topic, see, for example, [2, 4, 6, 7, 10–13, 16–20, 22].

In this paper, the letter C denotes a positive constant which may differ from one occurrence to the other. The notation $a \lesssim b$ means that there exists a positive constant C such that $a \le Cb$. If $a \lesssim b$ and $b \lesssim a$, then we write $a \asymp b$.

2. Prerequisites

Although the explicit form of $K_{\alpha,\beta}(z,w)$ is unknown, we have the following result (see [3]).

Lemma 2.1. *Let $\alpha \in \mathbb{R}$ and $\beta > 0$. Then for all $z \in \mathbb{B}_n$, it follows that*

$$K_{\alpha,\beta}(z,z) \asymp \left(1-|z|^2\right)^{-2n-\alpha-1} e^{\frac{2\beta}{1-|z|^2}}.$$

Let $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_n)$ be an n-tuple of nonnegative integers, then we write

$$|\gamma| = \sum_{j=1}^{n} \gamma_j \quad \text{and} \quad \partial^\gamma = \partial_1^{\gamma_1} \cdots \partial_n^{\gamma n},$$

where ∂_j denotes the partial differentiation with respect to the j-th component.

An advantage of the radial derivative is that it can be employed iteratively, that is, if $\mathfrak{R}^{k-1} f$ is defined for some $k \in \mathbb{N} \setminus \{1\}$, then $\mathfrak{R}^k f$ is naturally defined by

$$\mathfrak{R}^k f = \mathfrak{R}(\mathfrak{R}^{k-1} f).$$

We need the following estimate for the norms of the functions in $A^2_{\alpha,\beta}(\mathbb{B}_n)$. See [3] for a complete proof.

Lemma 2.2. *Let $k \in \mathbb{N}$. Then for all $f \in A^2_{\alpha,\beta}(\mathbb{B}_n)$, it follows that*

$$\|f\|^2_{\alpha,\beta} \asymp \sum_{m=0}^{k-1} \sum_{|\gamma|=m} |\partial^\gamma f(0)|^2 + \|\mathfrak{R}^k f\|^2_{\alpha+4k,\beta}.$$

In particular, we have

Corollary 2.1. *For all $f \in A^2_{\alpha,\beta}(\mathbb{B}_n)$, it follows that*

$$\|f\|^2_{\alpha,\beta} \asymp |f(0)|^2 + \|\mathfrak{R} f\|^2_{\alpha+4,\beta}.$$

Lemma 2.3. *Let $g \in H(\mathbb{B}_n)$. Then for all $f \in H(\mathbb{B}_n)$ and $z \in \mathbb{B}_n$, it follows that*

$$\mathfrak{R}(T_g f)(z) = f(z)\mathfrak{R}g(z).$$

Proof. From an elementary computation, the result follows. □

Lemma 2.4. *Let $k \in \mathbb{N}$ and $g \in H(\mathbb{B}_n)$. Then for all $z \in \mathbb{B}_n$, it follows that*

$$\mathfrak{R}^k(T_g f)(z) = \sum_{j=0}^{k-1} C^j_{k-1} \mathfrak{R}^j f(z) \mathfrak{R}^{k-j} g(z).$$

Proof. Since $\mathfrak{R}^k(T_g f) = \mathfrak{R}^{k-1}(\mathfrak{R} T_g f)$, the result follows from Lemma 2.3 and the Leibnitz formula. □

The following result will be used in the proof of main results.

Lemma 2.5. *Let $k \in \mathbb{N}$, $g \in H(\mathbb{B}_n)$ and T_g be bounded on $A^2_{\alpha,\beta}(\mathbb{B}_n)$. Then there exists a positive constant C independent of $f \in A^2_{\alpha,\beta}(\mathbb{B}_n)$ and $a \in \mathbb{B}_n$ such that*

$$\Big|\sum_{j=0}^{k-1} C_{k-1}^{j}\mathfrak{R}^j f(a)\mathfrak{R}^{k-j}g(a)\Big| \le C\|T_g f\|_{\alpha,\beta}\|K_{\alpha+4k,\beta}(a,\cdot)\|_{\alpha+4k,\beta}. \tag{1}$$

Proof. Since T_g is bounded on $A^2_{\alpha,\beta}(\mathbb{B}_n)$, by Lemma 2.2 we have that $\mathfrak{R}^k(T_g f) \in A^2_{\alpha+4k,\beta}(\mathbb{B}_n)$ for $f \in A^2_{\alpha,\beta}(\mathbb{B}_n)$. Then, from Lemma 2.4, it follows that

$$\sum_{j=0}^{k-1} C_{k-1}^{j}\mathfrak{R}^j f(z)\mathfrak{R}^{k-j}g(z) \in A^2_{\alpha+4k,\beta}(\mathbb{B}_n).$$

On the other hand, by the reproducing kernel, we have

$$\sum_{j=0}^{k-1} C_{k-1}^{j}\mathfrak{R}^j f(a)\mathfrak{R}^{k-j}g(a) = \int_{\mathbb{B}_n}\sum_{j=0}^{k-1} C_{k-1}^{j}\mathfrak{R}^j f(z)\mathfrak{R}^{k-j}g(z)K_{\alpha+4k,\beta}(a,z)\omega_{\alpha+4k,\beta}(z)dv(z). \tag{2}$$

From (2), Hölder inequality and Lemma 2.2, it follows that

$$\begin{aligned}
\Big|\sum_{j=0}^{k-1} C_{k-1}^{j}\mathfrak{R}^j f(a)\mathfrak{R}^{k-j}g(a)\Big| &\le \int_{\mathbb{B}_n}\Big|\sum_{j=0}^{k-1} C_{k-1}^{j}\mathfrak{R}^j f(z)\mathfrak{R}^{k-j}g(z)K_{\alpha+4k,\beta}(a,z)\Big|\omega_{\alpha+4k,\beta}(z)dv(z)\\
&\le \Big(\int_{\mathbb{B}_n}\Big|\sum_{j=0}^{k-1} C_{k-1}^{j}\mathfrak{R}^j f(z)\mathfrak{R}^{k-j}g(z)\Big|^2\omega_{\alpha+4k,\beta}(z)dv(z)\Big)^{\frac12}\\
&\quad\times\Big(\int_{\mathbb{B}_n}\Big|K_{\alpha+4k,\beta}(a,z)\Big|^2\omega_{\alpha+4k,\beta}(z)dv(z)\Big)^{\frac12}\\
&= \Big\|\mathfrak{R}^k T_g f\Big\|_{\alpha+4k,\beta}\Big\|K_{\alpha+4k,\beta}(a,\cdot)\Big\|_{\alpha+4k,\beta}\\
&\le C\Big\|T_g f\Big\|_{\alpha,\beta}\Big\|K_{\alpha+4k,\beta}(a,\cdot)\Big\|_{\alpha+4k,\beta}.
\end{aligned}$$

This finishes the proof of the lemma. □

The following is the special case of Lemma 2.5.

Corollary 2.2. *Let $g \in H(\mathbb{B}_n)$ and T_g be bounded on $A^2_{\alpha,\beta}(\mathbb{B}_n)$. Then there exists a positive constant C independent of $f \in A^2_{\alpha,\beta}(\mathbb{B}_n)$ and $a \in \mathbb{B}_n$ such that*

$$\Big|f(a)\mathfrak{R}g(a)\Big| \le C\Big\|T_g f\Big\|_{\alpha,\beta}\Big\|K_{\alpha+4,\beta}(a,\cdot)\Big\|_{\alpha+4,\beta}.$$

The next result provides a useful function called usually the test function.

Lemma 2.6. *Let $w \in \mathbb{B}_n$. Then the function k_w belongs to $A^2_{\alpha,\beta}(\mathbb{B}_n)$, and $\sup_{w\in\mathbb{B}_n}\|k_w\|_{\alpha,\beta} \asymp 1$, where*

$$k_w(z) = \Big(1-|w|^2\Big)^{-\frac{2n+\alpha+1}{2}} e^{-\frac{\beta}{1-|w|^2}} e^{\frac{2n}{1-\langle z,w\rangle}}, \quad z \in \mathbb{B}_n.$$

Proof. One can refer to Lemma 3.4 in [3]. □

3. Essential norm of T_g on $A^2_{\alpha,\beta}(\mathbb{B}_n)$

For the essential norm of T_g on $A^2_{\alpha,\beta}(\mathbb{B}_n)$, we have the following result.

Theorem 3.1. *Let $g \in H(\mathbb{B}_n)$ and the operator T_g be bounded on $A^2_{\alpha,\beta}(\mathbb{B}_n)$. Then*

$$\|T_g\|_e \asymp A = \limsup_{|z|\to 1}(1-|z|)^2|\Re g(z)|.$$

Proof. Let Ω be a compact subset of $\mathbb{B}_n$. Then for $z \in \Omega$, the function $k_w(z)$ satisfies

$$|k_w(z)| \lesssim (1-|w|)^{-\frac{2n+\alpha+1}{2}} e^{-\frac{\beta}{1-|w|^2}} e^{\frac{2\beta}{1-\max\{|z|:z\in\Omega\}}} \to 0 \tag{3}$$

as $|w| \to 1$. From Lemma 2.6 and (3), it follows that k_w is uniformly bounded, and $k_w \to 0$ uniformly on every compact subset of $\mathbb{B}_n$ as $|w| \to 1$. If K is compact on $A^2_{\alpha,\beta}(\mathbb{B}_n)$, then

$$\begin{aligned}\|T_g - K\| &\geq \limsup_{|w|\to 1}\|T_g k_w - K k_w\|_{\alpha,\beta}\\ &\geq \limsup_{|w|\to 1}\|T_g k_w\|_{\alpha,\beta} - \limsup_{|w|\to 1}\|K k_w\|_{\alpha,\beta}\\ &= \limsup_{|w|\to 1}\|T_g k_w\|_{\alpha,\beta}.\end{aligned} \tag{4}$$

By a direct computation, we have

$$k_w(w) = \left(1-|w|^2\right)^{-\frac{2n+\alpha+1}{2}} e^{\frac{\beta}{1-|w|^2}}. \tag{5}$$

Since $K_{\alpha+4,\beta}(w,\cdot)$ is the reproducing kernel of $A^2_{\alpha+4,\beta}(\mathbb{B}_n)$, we have

$$\begin{aligned}\left\|K_{\alpha+4,\beta}(w,\cdot)\right\|^2_{\alpha+4,\beta} &= \left\langle K_{\alpha+4,\beta}(w,\cdot), K_{\alpha+4,\beta}(w,\cdot)\right\rangle = K_{\alpha+4,\beta}(w,w)\\ &\asymp \left(1-|w|^2\right)^{-2n-\alpha-5} e^{\frac{2\beta}{1-|w|^2}}.\end{aligned}$$

Then, from (5) it follows that

$$\left\|K_{\alpha+4,\beta}(w,\cdot)\right\|_{\alpha+4,\beta} \asymp \left(1-|w|^2\right)^{-\frac{2n+\alpha+1}{2}-2} e^{\frac{\beta}{1-|w|^2}} = \left(1-|w|^2\right)^{-2} k_w(w). \tag{6}$$

So, by Corollary 2.2 and (6) we have

$$\left|k_w(w)\Re g(w)\right| \leq C\|T_g k_w\|_{\alpha,\beta}\|K_{\alpha+4,\beta}(w,\cdot)\|_{\alpha+4,\beta} \lesssim \|T_g k_w\|_{\alpha,\beta}\left(1-|w|^2\right)^{-2} k_w(w).$$

Hence, we have

$$\left(1-|w|^2\right)^2|\Re g(w)| \lesssim \|T_g k_w\|_{\alpha,\beta}. \tag{7}$$

Since $1-|w|^2 \asymp 1-|w|$, by (4) and (7) we obtain

$$\|T_g - K\| \geq \limsup_{|w|\to 1}(1-|w|)^2|\Re g(w)|. \tag{8}$$

This shows that

$$\|T_g\|_e \gtrsim \limsup_{|w|\to 1}(1-|w|)^2|\Re g(w)| = A. \tag{9}$$

For a holomorphic function $f = \sum_m a_m z^m$ on $\mathbb{B}_n$, let

$$T_j f(z) = \sum_{|m|=0}^{j} a_m z^m, \quad R_j f(z) = \sum_{|m|=j+1}^{\infty} a_m z^m.$$

Then, the operator T_j is compact on $A^2_{\alpha,\beta}(\mathbb{B}_n)$, and

$$\|T_g\|_e = \|T_g(T_j + R_j)\|_e \le \|T_g T_j\|_e + \|T_g R_j\|_e = \|T_g R_j\|_e \le \|T_g R_j\|. \tag{10}$$

Thus, (10) shows that $\|T_g\|_e \le \liminf_{j\to\infty} \|T_g R_j\|$. Hence, by Corollary 2.1 we have

$$\begin{aligned}
\|T_g\|_e^2 &\le \liminf_{j\to\infty} \|T_g R_j\|^2 = \liminf_{j\to\infty} \sup_{\|f\|_{\alpha,\beta}\le 1} \|T_g R_j f\|^2_{\alpha,\beta} \\
&= \liminf_{j\to\infty} \sup_{\|f\|_{\alpha,\beta}\le 1} \int_{\mathbb{B}_n} \left|\Re(T_g R_j f)(z)\right|^2 \omega_{\alpha+4,\beta}(z)dv(z) \\
&= \liminf_{j\to\infty} \sup_{\|f\|_{\alpha,\beta}\le 1} \int_{\mathbb{B}_n} \left|R_j f(z)\Re g(z)\right|^2 \omega_{\alpha+4,\beta}(z)dv(z) \\
&\le A^2 \liminf_{j\to\infty} \sup_{\|f\|_{\alpha,\beta}\le 1} \int_{\mathbb{B}_n} \left|R_j f(z)\right|^2 \omega_{\alpha,\beta}(z)dv(z) \\
&\le A^2 \liminf_{j\to\infty} \sup_{\|f\|_{\alpha,\beta}\le 1} \|f\|^2_{\alpha,\beta} \\
&= A^2,
\end{aligned}$$

which shows that $\|T_g\|_e \le A$. From this and (9), the desired result follows. □

Remark 3.1. *It is easy to see that result (2) in Theorem 1.1′ can be regarded as a corollary of Theorem 3.1.*

As an application, we have the following result.

Corollary 3.1. *Let $g \in H(\mathbb{D})$ and the operator T_g be bounded on $A^2_{\alpha,\beta}(\mathbb{D})$. Then*

$$\|T_g\|_e \asymp B = \limsup_{|z|\to 1} (1-|z|)^2 |g'(z)|.$$

Proof. Let $\delta \in (0,1)$. If $g \in H(\mathbb{D})$, then $\Re g(z) = zg'(z)$. By this, we have

$$\sup_{|z|\ge\delta} (1-|z|)^2 |g'(z)| \asymp \sup_{|z|\ge\delta} (1-|z|)^2 |zg'(z)|.$$

From this, the desired result follows. □

Corollary 3.2. *Let $g \in H(\mathbb{D})$ and the operator T_g be bounded on $A^2_{\alpha,\beta}(\mathbb{D})$. Then T_g is compact on $A^2_{\alpha,\beta}(\mathbb{D})$ if and only if*

$$\lim_{|z|\to 1} (1-|z|)^2 |g'(z)| = 0.$$

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