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Spectral Problems Of Dissipative Singular q-Sturm–Liouville Operators in Limit-Circle Case

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Abstract. We consider the dissipative singular q-Sturm–Liouville operators acting in the Hilbert space $L^2_{w,q}(R_+)$, that the extensions of a minimal symmetric operator with deficiency indices $(2, 2)$ (in limit-circle case). We construct a self-adjoint dilation of the dissipative operator and its incoming and outgoing spectral representations, which make it possible to determine the scattering matrix of the dilation in terms of the Weyl–Titchmarsh function of a self-adjoint q-Sturm-Liouville operator. We also construct a functional model of the dissipative operator and determine its characteristic function in terms of the scattering matrix of the dilation (or of the Weyl–Titchmarsh function). Theorems on the completeness of the system of or root functions of the dissipative and accumulative q-Sturm–Liouville operators are proved.

1. Introduction and Notations

In this section, we describe some of the necessary q-notations and results (see [4-7, 9, 12-15]). Throughout the paper, q denotes a positive number such that $0 < q < 1$. For $\mu \in \mathbb{R} := (-\infty, \infty)$, a set $A \subseteq \mathbb{R}$ is called a μ-geometric set if $\mu t \in A$ for all $t \in A$. If $A \subseteq \mathbb{R}$ is a μ-geometric set, then it includes all geometric sequences $\{\mu^n t\}$ $(n = 0, 1, 2, ...)$, $t \in A$. Let f be a real or complex valued function defined on a q-geometric set A. The q-difference operator is defined by

$$D_q f(t) := \frac{f(t) - f(qt)}{t - qt}, \; t \in A \backslash \{0\}. \tag{1.1}$$

If $0 \in A$, the q-derivative at zero is defined to be

$$D_q f(0) := \lim_{n\to\infty} \frac{f(q^n t) - f(0)}{q^n t}$$

if the limit exists and does not depend on t. Since the formulation of the boundary-value problems requires the definition of $D_{q^{-1}}$ in a same manner to be

$$D_{q^{-1}} f(t) := \begin{cases} \frac{f(t)-f(q^{-1}t)}{t-q^{-1}t}, \; t \in A\backslash\{0\}, \\ D_q f(0), \; t = 0, \end{cases}$$

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provided that $D_q f(0)$ exists. As a converse of the q-difference operator, Jackson's q-integration [14], is given by

$$\int_0^x f(t)d_qt := x(1-q)\sum_{n=0}^{\infty} q^n f(q^n x),\ x \in A,$$

provided that the series is convergent, and

$$\int_a^b f(t)d_qt := \int_0^b f(t)d_qt - \int_0^a f(t)d_qt,\ a,b \in A.$$

When it is required, q will be replaced by q^{-1}. The following facts, which will be frequently used, can be verified directly from the definition:

$$D_{q^{-1}}f(t) = (D_q f)(q^{-1}t),\ (D_q^2 f)(q^{-1}t) = qD_q[D_q f(q^{-1}t)] = D_{q^{-1}}D_q f(t).$$

Related to this operator there exists a non-symmetric formula for the q-differentation of a product

$$D_q[f(t)g(t)] = g(t)D_q f(t) + f(qt)D_q g(t).$$

From now on, we shall consider only the functions q-regular at zero, that is, functions satisfying $\lim_{n\to\infty} f(q^n t) = f(0)$. The class of the functions being q-regular at zero includes the continuous functions. If f and g are q-regular at zero, then we have a rule of q-integration by parts given below

$$\int_0^a g(t)D_q f(t)d_qt = (fg)(a) - \int_0^a D_q g(t)f(qt)d_qt.$$

In [12], Hahn defined the q-integration for a function f over $[0,\infty)$ by

$$\int_0^{\infty} f(t)d_qt := (1-q)\sum_{n=-\infty}^{\infty} q^n f(q^n).$$

The q-difference calculus or quantum calculus was introduced at the beginning of the 19th century. Since then the subject of q-differential equations has developed and become a multidisciplinary subject ([4, 9, 12, 15]). There exist numerous physical models including q-derivatives, q-integrals q-exponential function, q-trigonometric function, q-Taylor formula, q-Beta and q-Gamma functions, Euler–Maclaurin formula and their related problems (see [9, 12]).

Annaby and Mansour [4, 7] investigated a q-Sturm–Liouville eigenvalue problem and formulated a self-adjoint q-Sturm–Liouville operator in a Hilbert space. They discussed the properties of the eigenvalues and the eigenfunctions as well. Annaby et al. [5, 6] constructed the q-Titchmarsh–Weyl theory for singular q-Sturm–Liouville problems and defined q-limit-point and q-limit-circle singularities.

An important class of non-self-adjoint operators is the class of dissipative operators. The spectral analysis of non-self-adjoint (dissipative) operators is based on ideas of the functional model and dilation theory rather than on traditional resolvent analysis and Riesz integrals. The functional model of non-self-adjoint dissipative operators plays an important role within both the abstract operator theory and its more specialized applications in other disciplines. The construction of functional models for dissipative operators, natural analogues of spectral decompositions for self-adjoint operators is based on Sz. Nagy-Foiaş dilation theory [20] and Lax–Phillips scattering theory [16]. The characteristic function occupies a central place in this theory; it carries complete information regarding the spectral properties of a dissipative operator. For example, the question of completeness of a system of eigenvectors and associated vectors (or root vectors) is answered in terms of factorization of the characteristic function. The adequacy of this approach to dissipative Sturm–Liouville and q-Sturm–Liouville operators has been demonstrated, for example, in [1-3, 10, 11].

In this paper we consider the dissipative singular q-Sturm–Liouville operator acting in the space $\mathcal{L}^2_{w,q}(\mathbb{R}_+)$, that the extension of a minimal symmetric operator in Weyl's limit-circle case at singular end point ∞. We construct a self-adjoint dilation of the dissipative operator and its incoming and outgoing spectral representations, which makes it possible to determine the scattering matrix of dilation (in terms of the Weyl–Titchmarsh function of a self-adjoint q-Sturm–Liouville operator) according to the scheme of Lax and Phillips [16]. With the help of the incoming spectral representation, we construct a functional model of the dissipative operator and specify its characteristic function in terms of the Weyl–Titchmarsh function of a self-adjoint q-Sturm–Liouville operator (or in terms of the scattering matrix of the self-adjoint dilation). Finally, on the basis of the results obtained regarding the theory of the characteristic function, we prove theorems on completeness of the system of eigenfuncions and associated functions (or root functions) of dissipative and accumulative q-Sturm–Liouville operators. The results of the present paper are new even in the case $w = r = 1$.

2. The dissipative operator and self-adjoint dilation of the dissipative operator

We consider the following singular q-Sturm–Liouville expression

$$(\tau y)(t) = \frac{1}{w(t)}[-\frac{1}{q}D_{q^{-1}}(r(t)D_q y(t)) + u(t)y(t)],\ t \in \mathbb{R}_+ := [0, \infty), \tag{2.1}$$

where r, w and u are real-valued functions defined on $\mathbb{R}_+$ and are continuous at zero such that $r(t) \neq 0$, $w(t) > 0$ for all $t \in \mathbb{R}_+$, and D_q is the q-difference operator given by (2.1).

We pass from the expression (2.1) to operators by introducing the Hilbert space $\mathcal{L}^2_{w,q}(\mathbb{R}_+)$ which consists of all complex-valued functions y satisfying

$$\int_0^\infty w(t)\left|y(t)\right|^2 d_q t < +\infty$$

and with the inner product

$$(y, z) = \int_0^\infty w(t)y(t)\overline{z(t)}d_q t.$$

Let $\mathfrak{D}_{\max}$ denote the linear set of all functions $y \in \mathcal{L}^2_{w,q}(\mathbb{R}_+)$ such that y and $D_q y$ are are continuous at zero and $\tau y \in \mathcal{L}^2_{w,q}(\mathbb{R}_+)$. The *maximal operator* $T_{\max}$ on $\mathfrak{D}_{\max}$ is defined by the equality $T_{\max} y = \tau y$.

For each $x, y \in \mathfrak{D}_{\max}$ we define the *q-Wronski determinant* (or *q-Wronskian*) as follows:

$$\mathcal{W}_q(x, y)(t) = x(t)(rD_q y)(t) - (rD_q)x(t)y(t),\ t \in \mathbb{R}_+.$$

Given any functions $y, z \in \mathfrak{D}_{\max}$, we get the following q-Green's formula (or Lagrange's identity) ([4, 6, 7])

$$\int_0^t (\tau y)(\xi))\overline{z(\xi)}d_q\xi - \int_0^t y(\xi)\overline{(\tau z)(\xi)}d_q\xi = [y, z](t) - [y, z](0),\ t \in \mathbb{R}_+, \tag{2.2}$$

where $[y, z](t)$ is the Lagrange bracket defined by

$$[y, z](t) = r(q^{-1}t)[y(t)\overline{(D_{q^{-1}}z)(t)} - (D_{q^{-1}}y)(t)\overline{z(t)}],\ t \in \mathbb{R}_+.$$

It follows directly from (2.2) that limit $[y, z](\infty) := \lim_{t\to\infty}[y, z](t)$ exists and it is finite for all $y, z \in \mathfrak{D}_{\max}$. For an arbitrary function $y \in \mathfrak{D}_{\max}$, $y(0)$ and $(rD_{q^{-1}}y)(0)$ can be defined as $y(0) := \lim_{t\to 0^+} y(t)$ and $(rD_{q^{-1}}y)(0) := \lim_{t\to 0^+}(rD_{q^{-1}}y)(t)$. These limits exist and they are finite (since y and $rD_{q^{-1}}y$ are continuous at zero). Let us consider, in $\mathcal{L}^2_{w,q}(\mathbb{R}_+)$, the linear dense set $\mathfrak{D}_{\min}$ consisting of precisely the vectors $y \in \mathfrak{D}_{\max}$ with

$$y(0) = (rD_{q^{-1}})y(0) = 0,\ \ [y, z](\infty) = 0,\ \forall z \in \mathfrak{D}_{\max}. \tag{2.3}$$

Let the restriction of the operator $T_{\max}$ to $\mathfrak{D}_{\min}$ be represented by $T_{\min}$. It can be concluded from (2.3) that $T_{\min}$ is symmetric. The *minimal operator* $T_{\min}$ is a closed symmetric operator with deficiency indices $(2,2)$ or $(1,1)$, and $T_{\max} = T^{*}_{\min}$ (see [1, 4-6, 8, 17]).

We suppose that Weyl's limit-circle case is valid for the expression τ, i.e. the symmetric operator $T_{\min}$ has deficiency indices $(2,2)$ ([1, 4-6, 8, 17]).

We mean by $\varphi(t)$ and $\psi(t)$ the solutions (real-valued) of the equation

$$\tau y = 0,\ t \in \mathbb{R}_{+} \tag{2.4}$$

with the following initial conditions

$$\varphi(0) = 1,\ (rD_{q^{-1}}\varphi)(0) = 0,\ \psi(0) = 0,\ (rD_{q^{-1}}\psi)(0) = 1. \tag{2.5}$$

The Wronskian of the two solutions of (2.4) is independent of t, and the two solutions of this equation are linearly independent if and only if their Wronskian is non-zero. It can be derived from the conditions (2.5) and the constancy of the Wronskian that ([4, 6, 7]) $\mathcal{W}_q[\varphi,\psi](t) = \mathcal{W}_q[\varphi,\psi](0) = 1$ $(t \in \mathbb{R}_{+})$. As a result, φ and ψ construct a fundamental system of solutions of (2.4). Since $T_{\min}$ has deficiency indices $(2,2)$, φ and ψ belong to $\mathcal{L}^2_{w,q}(\mathbb{R}_{+})$, and furthermore $\varphi,\psi \in \mathfrak{D}_{\max}$.

Lemma 2.1. *For arbitrary functions $y, z \in \mathfrak{D}_{\max}$, we have the equality* **(***the Plücker identity***)**

$$[y,z](t) = [y,\varphi](t)[\bar{z},\psi](t) - [y,\psi](t)[\bar{z},\varphi](t),\ t \in \mathbb{R}_{+}\cup\{\infty\}. \tag{2.6}$$

Proof. Since the functions φ and ψ are real-valued and since $[\varphi,\psi](t) = 1$ $(t \in \mathbb{R}_{+}\cup\{\infty\})$, one obtains

$$\begin{aligned}
&[y,\varphi](t)[\bar{z},\psi](t) - [y,\psi](t)[\bar{z},\varphi](t)\\
&= r(q^{-1}t)(yD_{q^{-1}}\varphi - D_{q^{-1}}y\varphi)(t)r(q^{-1}t)(\bar{z}D_{q^{-1}}\psi - \overline{D_{q^{-1}}z}\psi)(t)\\
&-r(q^{-1}t)(yD_{q^{-1}}\psi - D_{q^{-1}}y\psi)(t)r(t)(\bar{z}D_{q^{-1}}\varphi - \overline{D_{q^{-1}}z}\varphi)(t)\\
&= (r(q^{-1}t))^2(yD_{q^{-1}}\varphi\bar{z}D_{q^{-1}}\psi - yD_{q^{-1}}\varphi\overline{D_{q^{-1}}z}\psi - D_{q^{-1}}y\varphi\bar{z}D_{q^{-1}}\psi\\
&+D_{q^{-1}}y\varphi\overline{D_{q^{-1}}z}\psi - yD_{q^{-1}}\psi\bar{z}D_{q^{-1}}\varphi + yD_{q^{-1}}\psi\overline{D_{q^{-1}}z}\varphi\\
&+D_{q^{-1}}y\psi\bar{z}D_{q^{-1}}\varphi - D_{q^{-1}}y\psi\overline{D_{q^{-1}}z}\varphi)(t)\\
&= r(q^{-1}t)(-y\overline{D_{q^{-1}}z} + D_{q^{-1}}y\bar{z})(t)r(q^{-1}t)(D_{q^{-1}}\varphi\psi - \varphi D_{q^{-1}}\psi)(t) = [y,z](t).
\end{aligned}$$

The Lemma 2.1 is proved. □

Let us consider the operator $T_{\beta\gamma}$ with domain $\mathfrak{D}(T_{\beta\gamma})$ consisting of vectors $y \in \mathfrak{D}_{\max}$ which satisfy the boundary conditions

$$(rD_{q^{-1}})y(0) - \beta y(0) = 0,\ \beta \in \mathbb{C} \tag{2.7}$$

$$[y,\varphi](\infty) - \gamma[y,\psi](\infty) = 0,\ \Im\gamma = 0 \text{ or } \gamma = \infty. \tag{2.8}$$

Here for $\gamma = \infty$, condition (2.8) should be replaced by $[y,\psi](\infty) = 0$.

We shall remind that the linear operator $\mathbf{T}$ (with dense domain $\mathfrak{D}(\mathbf{T})$) acting on some Hilbert space $\mathbf{H}$ is called *dissipative* (*accumulative*) if $\Im(\mathbf{T}f,f) \geq 0$ ($\Im(\mathbf{T}f,f) \leq 0$) for all $f \in \mathfrak{D}(\mathbf{T})$.

Theorem 2.2. *If $\Im\beta \geq 0$, $\Im\gamma = 0$ or $\gamma = \infty$, then the operator $T_{\beta\gamma}$ is dissipative in the space $\mathcal{L}^2_{w,q}(\mathbb{R}_{+})$.*

Proof. Let $y \in \mathfrak{D}(T_{\beta\gamma})$. Then we have

$$(T_{\beta\gamma}y,y) - (y,T_{\beta\gamma}y) = [y,y](\infty) - [y,y](0). \tag{2.9}$$

Using Lemma 2.1 and condition (2.8) we obtain that

$$[y,y](\infty) = 0. \tag{2.10}$$

Further, we obtain from the condition (2.7) that

$$[y,y](0) = -2i\Im\beta \left|y(0)\right|^2 . \tag{2.11}$$

Substituting (2.10), (2.11) in (2.9) one gets

$$\Im(T_{\beta\gamma}y,y) = \Im\beta \left|y(0)\right|^2 \geq 0 \text{ for } \Im\beta \geq 0, \tag{2.12}$$

and this completes the proof. □

It follows from (2.12) that all the eigenvalues of dissipative operator $T_{\beta\gamma}$ lie in the closed upper half plane $\Im\lambda \geq 0$.

Theorem 2.3. *If $\Im\beta > 0, \Im\gamma = 0$ or $\gamma = \infty$, then the dissipative operator $T_{\beta\gamma}$ has not any real eigenvalue.*

Proof. Suppose that the operator $T_{\beta\gamma}$ has a real eigenvalue λ_0. Let $y_0(t) := y(t,\lambda_0)$ be the corresponding eigenfunction. Since $(T_{\beta\gamma}y_0, y_0) = \lambda_0 \left\|y_0\right\|^2$, we get from (2.12) that $\Im\beta \left|y_0(0)\right|^2 = \Im\lambda_0 \left\|y_0\right\|^2 = 0$ and $y_0(0) = 0$. By the boundary condition (2.7), we have $(rD_{q^{-1}})y_0(0) = 0$, and then by the uniqueness theorem of the Cauchy problem for the equation $\tau y = \lambda y$, $t \in \mathbb{R}_+$, we have $y_0(t) \equiv 0$. The theorem is proved. □

According to equality (2.12) if $\Im\beta \leq 0, \Im\gamma = 0$ or $\gamma = \infty$, ($\Im\beta = 0$ or $\beta = \infty$, $\Im\gamma = 0$ or $\gamma = \infty$) then $T_{\beta\gamma}$ is an accumulative (self-adjoint) operator in $\mathcal{L}^2_{w,q}(\mathbb{R}_+)$. Here for $\beta = \infty$, condition (2.7) should be replaced by $y(0) = 0$. The proof of the next result is analogous to that of Theorem 2.3.

Corollary 2.4. *If $\Im\beta < 0, \Im\gamma = 0$ or $\gamma = \infty$, then the accumulative operator $T_{\beta\gamma}$ has not any real eigenvalue.*

In the sequel we shall study the dissipative operators $T_{\beta\gamma}$ ($\Im\beta > 0, \Im\gamma = 0$ or $\gamma = \infty$) generated by the expression (2.1) and the boundary conditions (2.7) and (2.8).

We deal with the Hilbert spaces $\mathcal{L}^2(\mathbb{R}_-)$, ($\mathbb{R}_- := (-\infty, 0]$) and $\mathcal{L}^2(\mathbb{R}_+)$ consisting of all functions σ_- and σ_+, respectively, such that

$$\int_{-\infty}^{0} |\sigma_-(t)|^2 \, dt < \infty, \ \int_0^\infty |\sigma_+(t)|^2 \, dt < \infty$$

with the inner product

$$(\sigma_-, \rho_-)_{\mathcal{L}^2(\mathbb{R}_-)} = \int_{-\infty}^0 \sigma_-(t)\overline{\rho_-(t)}dt, \ (\sigma_+, \rho_+)_{\mathcal{L}^2(\mathbb{R}_+)} = \int_0^\infty \sigma_+(t)\overline{\rho_+(t)}dt,$$

Adding the spaces $\mathcal{L}^2(\mathbb{R}_-)$ and $\mathcal{L}^2(\mathbb{R}_+)$ to the Hilbert space $H := \mathcal{L}^2_{w,q}(\mathbb{R}_+)$, we obtain an orthogonal sum Hilbert space as $\mathcal{H} = \mathcal{L}^2(\mathbb{R}_-) \oplus H \oplus \mathcal{L}^2(\mathbb{R}_+)$, and we call it as the *main Hilbert space of the dilation*. In the space $\mathcal{H}$, we consider the operator $\mathcal{S}_{\beta\gamma}$ generated by the expression

$$\mathcal{S}\langle\sigma_-, y, \sigma_+\rangle = \langle i\frac{d\sigma_-}{d\xi}, \tau y, i\frac{d\sigma_+}{d\varsigma}\rangle \tag{2.13}$$

on the set $\mathfrak{D}\left(\mathcal{S}_{\beta\gamma}\right)$ of vectors $\langle\sigma_-, y, \sigma_+\rangle$ satisfying the conditions $\sigma_\mp \in \mathcal{W}_2^1(\mathbb{R}_\mp)$, $y \in \mathfrak{D}_{\max}$ and

$$(rD_{q^{-1}}y)(0) - \beta y(0) = \beta\sigma_-(0), (rD_{q^{-1}}y)(0) - \overline{\beta}y(0) = \beta\sigma_+(0), \tag{2.14}$$

$$[y,\varphi](\infty) - \gamma[y,\psi](\infty) = 0, \tag{2.15}$$

where $\alpha^2 := 2\Im\beta$, $\alpha > 0$, and $\mathcal{W}_2^1(\mathbb{R}_\mp)$ is the Sobolev space consisting of all functions $f \in \mathcal{L}^2(\mathbb{R}_\mp)$ such that f are locally absolutely continuous functions on $\mathbb{R}_\pm$ and $f' \in \mathcal{L}^2(\mathbb{R}_\pm)$. Then we have

Theorem 2.5. *The operator $\mathcal{S}_{\beta\gamma}$ is self-adjoint in $\mathcal{H}$ and it is a self-adjoint dilation of the dissipative operator $T_{\beta\gamma}$.*

Proof. Suppose that $f, g \in \mathfrak{D}\left(\mathcal{S}_{\beta\gamma}\right)$, $f = \langle\sigma_-, y, \sigma_+\rangle$ and $g = \langle\rho_-, z, \rho_+\rangle$. Then, integrating by parts and using (2.2), we get that

$$\left(\mathcal{S}_{\beta\gamma}f, g\right)_{\mathcal{H}} = \int_{-\infty}^{0} i\sigma_-'\overline{\rho}_- d\xi + (\tau y, z)_H + \int_0^\infty i\sigma_+'\overline{\rho}_+ d\varsigma = i\sigma_-(0)\overline{\rho}_-(0)$$

$$-i\sigma_{+}(0)\,\overline{\rho}_{+}(0)+[y,z](\infty)-[y,z](0)+\left(f,\mathcal{S}_{\beta\gamma}g\right)_{\mathcal{H}}. \tag{2.16}$$

Next, using the boundary conditions (2.14), (2.15) for the components of the vectors f and g and Lemma 2.1 a straightforward calculation shows that $i\sigma_{-}(0)\overline{\rho_{-}(0)}-i\sigma_{+}(0)\overline{\rho_{+}(0)}+[y,z](\infty)-[y,z](0)=0$. Thus, $\mathcal{S}_{\beta\gamma}$ is symmetric. Therefore, to prove that $\mathcal{S}_{\beta\gamma}$ is self-adjoint, it suffices for us to show that $\mathcal{S}_{\beta\gamma}^{*}\subseteq\mathcal{S}_{\beta\gamma}$. Take $g=\langle\rho_{-},z,\rho_{+}\rangle\in\mathfrak{D}(\mathcal{S}_{\beta\gamma}^{*})$. Let $\mathcal{S}_{\beta\gamma}^{*}g=g^{*}$, $g^{*}=\langle\rho_{-}^{*},z^{*},\rho_{+}^{*}\rangle\in\mathcal{H}$, so that

$$\left(\mathcal{S}_{\beta\gamma}f,g\right)_{\mathcal{H}}=(f,g^{*})_{\mathcal{H}},\ \forall f\in\mathfrak{D}\left(\mathcal{S}_{\beta\gamma}\right). \tag{2.17}$$

By choosing the components for $f\in\left(\mathcal{S}_{\beta\gamma}\right)$ suitably in (2.17), it is not difficult to show that $\rho_{\mp}\in\mathcal{W}_{2}^{1}(\mathbb{R}_{\mp})$, $z\in\mathfrak{D}_{\max}$ and $g^{*}=\mathcal{S}g$, where the operator $\mathcal{S}$ is defined by (2.13). Consequently, (2.17) takes the form $(\mathcal{S}f,g)_{\mathcal{H}}=(f,\mathcal{S}g)_{\mathcal{H}},\ \forall f\in\mathfrak{D}\left(\mathcal{S}_{\beta\gamma}\right)$. Therefore, the sum of the integral terms in the bilinear form $(\mathcal{S}f,g)_{\mathcal{H}}$ must be equal to zero:

$$i\sigma_{-}(0)\,\overline{\rho_{-}(0)}-i\sigma_{+}(0)\,\overline{\rho_{+}(0)}+[y,z](\infty)-[y,z](0)=0 \tag{2.18}$$

for all $f=\langle\sigma_{-},y,\sigma_{+}\rangle\in\mathfrak{D}\left(\mathcal{S}_{\beta\gamma}\right)$. Further, solving the boundary conditions (2.14) for $y(0)$ and $(rD_{q^{-1}}y)(0)$, we find that

$$y(0)=-\frac{i}{\alpha}(\sigma_{+}(0)-\sigma_{-}(0)\,,(rD_{q^{-1}}y)(0)=\alpha\sigma_{-}(0)-\frac{i\beta}{\alpha}(\sigma_{+}(0)-\sigma_{-}(0)).$$

Therefore, using (2.14), (2.15) we find that (2.18) is equivalent to the equality

$$i\sigma_{-}(0)\,\overline{\rho_{-}(0)}-i\sigma_{+}(0)\,\overline{\rho_{+}(0)}=[y,z](0)-[y,z](\infty)$$

$$=-\frac{i}{\alpha}(\sigma_{+}(0)-\sigma_{-}(0))(\overline{pz'})(0)-\alpha[\sigma_{-}(0)-\frac{i\beta}{\alpha^{2}}(\sigma_{+}(0)-\sigma_{-}(0)]\overline{z}(0)$$

$$-[y,\varphi](\infty)[\overline{z},\varphi](\infty)+[y,\psi](\infty)[\overline{z},\psi](\infty)=-\frac{i}{\alpha}(\sigma_{+}(0)-\sigma_{-}(0))(\overline{pz'})(0)$$

$$-\alpha[\sigma_{-}(0)-\frac{i\beta}{\alpha^{2}}(\sigma_{+}(0)-\sigma_{-}(0))]\overline{z}(0)-([\overline{z},\varphi](\infty)-\gamma[\overline{z},\psi](\infty))[y,\psi](\infty).$$

Since the values $\sigma_{\pm}(0)$ can be arbitrary complex numbers, a comparison of the coefficient of $\sigma_{\pm}(0)$ on the left and right of the last equality gives us that the vector $g=\langle\rho_{-},z,\rho_{+}\rangle$ satisfies the boundary conditions $(rD_{q^{-1}})z(0)-\beta z(0)=\alpha\rho_{-}(0)$, $(rD_{q^{-1}})z(0)-\overline{\beta}z(0)=\alpha\rho_{+}(0)$, $[z,\varphi](\infty)-\gamma[z,\psi](\infty)=0$. Consequently, the inclusion $\mathcal{S}_{\beta\gamma}^{*}\subseteq\mathcal{S}_{\beta\gamma}$ is established, and hence $\mathcal{S}_{\beta\gamma}=\mathcal{S}_{\beta\gamma}^{*}$.

The self-adjoint operator $\mathcal{S}_{\beta\gamma}$ generates in $\mathcal{H}$ a unitary group $\mathcal{X}(s)=\exp[i\mathcal{S}_{\beta\gamma}s]$ $(s\in\mathbb{R})$. Denote by $\mathcal{P}:\mathcal{H}\to H$ and $\mathcal{P}_{1}:H\to\mathcal{H}$ the mappings acting according to the formulas $\mathcal{P}:\langle\sigma_{-},y,\sigma_{+}\rangle\to y$ and $\mathcal{P}_{1}:y\to\langle 0,y,0\rangle$, respectively. Let $\mathcal{Z}(s)=\mathcal{P}\mathcal{X}(s)\mathcal{P}_{1}$ $(s\geq 0)$. The family $\{\mathcal{Z}(s)\}$ $(s\geq 0)$ of operators is a strongly continuous semigroup of non-unitary contraction on H. Denote by $B_{\beta\gamma}$ the generator of this semigroup, $B_{\beta\gamma}y=\lim_{s\to+0}(is)^{-1}(\mathcal{Z}(s)y-y)$. The domain of $B_{\beta\gamma}$ consists of all the vectors for which the limit exists. The operator $B_{\beta\gamma}$ is a dissipative. The operator $\mathcal{S}_{\beta\gamma}$ is called the *self-adjoint dilation* of $B_{\beta\gamma}$ ([17, 19]). We show that $B_{\beta\gamma}=T_{\beta\gamma}$, and hence, $\mathcal{S}_{\beta\gamma}$ is a self-adjoint dilation of $T_{\beta\gamma}$. To do this, we first verify the equality

$$\mathcal{P}(\mathcal{S}_{\beta\gamma}-\lambda I)^{-1}\mathcal{P}_{1}y=(T_{\beta\gamma}-\lambda I)^{-1}y,\,y\in H,\ \Im\lambda<0. \tag{2.19}$$

With this goal, we set $(\mathcal{S}_{\beta\gamma}-\lambda I)^{-1}\mathcal{P}_{1}y=g=\langle\rho_{-},z,\rho_{+}\rangle$. Then $(\mathcal{S}_{\beta\gamma}-\lambda I)g=\mathcal{P}_{1}y$, and hence, $\tau z-\lambda z=y$, $\rho_{-}(\xi)=\rho_{-}(0)e^{-i\lambda\xi}$, and $\rho_{+}(\varsigma)=\rho_{+}(0)e^{-i\lambda\varsigma}$. Since $g\in\mathfrak{D}(\mathcal{S}_{\beta\gamma})$, and hence $\rho_{-}\in\mathcal{L}^{2}(\mathbb{R}_{-})$; it follows that $\rho_{-}(0)=0$, and consequently, z satisfies the boundary conditions $(rD_{q^{-1}}y)(0)-\beta y(0)=0$, $[y,\varphi]_{\infty}-[y,\psi]_{0}=0$. Therefore,

$z \in \mathfrak{D}(T_{\beta\gamma})$, and since a point λ with $\Im\lambda < 0$ cannot be an eigenvalue of a dissipative operator, it follows that $z = (T_{\beta\gamma} - \lambda I)^{-1} y$. We remark that $\rho_+(0)$ is found from the formula $\rho_+(0) = \alpha^{-1}((rD_{q^{-1}})z(0) - \overline{\beta} z(0))$. Thus,

$$(\mathcal{S}_{\beta\gamma} - \lambda I)^{-1}\mathcal{P}_1 y = \langle 0, (T_{\beta\gamma} - \lambda I)^{-1} y, \alpha^{-1}((rD_{q^{-1}})z(0) - \overline{\beta} z_1(0))e^{-i\lambda\varsigma}\rangle,$$

for $y \in H$ and $\Im\lambda < 0$. By applying $\mathcal{P}$, one obtains (2.19).

It is now easy to show that $B_{\beta\gamma} = T_{\beta\gamma}$. Indeed, by (2.19),

$$(T_{\beta\gamma} - \lambda I)^{-1} = \mathcal{P}(\mathcal{S}_{\beta\gamma} - \lambda I)^{-1}\mathcal{P}_1 = -i\mathcal{P}\int_0^\infty \mathcal{X}(s)e^{-i\lambda s} ds \mathcal{P}_1 =$$

$$= -i\int_0^\infty \mathcal{Z}(s)e^{-i\lambda s} ds = (B_{\beta\gamma} - \lambda I)^{-1} \ (\Im\lambda < 0),$$

and therefore $T_{\beta\gamma} = B_{\beta\gamma}$. Theorem 2.5 is proved. □

3. Scattering theory of the dilation, functional model of the dissipative operator and completeness theorems of the dissipative and accumulative operators

The unitary group $\{\mathcal{X}(s)\}$ $(s \in \mathbb{R})$ has an important property which makes it possible to apply to it the Lax–Phillips scheme [16]. Namely, it has 'incoming' and 'outgoing' subspaces $\mathcal{D}^- := \left\langle \mathcal{L}^2(\mathbb{R}_-), 0, 0 \right\rangle$ and $\mathcal{D}^+ := \left\langle 0, 0, \mathcal{L}^2(\mathbb{R}_+) \right\rangle$ possessing the following properties:

(**i**) $\mathcal{X}(s)\mathcal{D}^- \subset \mathcal{D}^-, s \leq 0$, and $\mathcal{X}(s)\mathcal{D}^+ \subset \mathcal{D}^+, s \geq 0$;

(**ii**) $\bigcap_{s\leq 0} \mathcal{X}(s)\mathcal{D}^- = \bigcap_{s\geq 0} \mathcal{X}(s)\mathcal{D}^+ = \{0\}$;

(**iii**) $\overline{\bigcup_{s\geq 0} \mathcal{X}(s)\mathcal{D}^+} = \overline{\bigcup_{s\leq 0} \mathcal{X}(s)\mathcal{D}^+} = \mathcal{H}$;

(**iv**) $\mathcal{D}^- \perp \mathcal{D}^+$.

Property (**iv**) is obvious. To prove property (**i**) for $\mathcal{D}^+$ (the proof for $\mathcal{D}^-$ is similar), we set $\mathcal{R}_\lambda = (\mathcal{S}_{\beta\gamma} - \lambda I)^{-1}$, for all λ with $\Im\lambda < 0$. Then, for any $f = \langle 0, 0, \sigma_+\rangle \in \mathcal{D}^+$, we have

$$\mathcal{R}_\lambda f = \langle 0, 0, -ie^{-i\lambda\xi}\int_0^\xi e^{i\lambda s}\sigma_+(s) ds\rangle.$$

So we have $\mathcal{R}_\lambda f \in \mathcal{D}^+$. Therefore, if $g \perp \mathcal{D}^+$, then

$$0 = (\mathcal{R}_\lambda f, g)_{\mathcal{H}} = -i\int_0^\infty e^{-i\lambda s}(\mathcal{X}(s)f, g)_{\mathcal{H}} ds, \Im\lambda < 0.$$

From this it follows that $(\mathcal{X}(s)f, g)_{\mathcal{H}} = 0$ for all $s \geq 0$. Hence, $\mathcal{X}(s)\mathcal{D}^+ \subset \mathcal{D}^+$ for $s \geq 0$, and property (**i**) has thus been proved.

To prove property (**ii**), we denote by $\mathcal{P}^+ : \mathcal{H} \to \mathcal{L}^2(\mathbb{R}_+)$ and $\mathcal{P}_1^+ : \mathcal{L}^2(\mathbb{R}_+) \to \mathcal{D}^+$ the mappings acting according to the formulae $\mathcal{P}^+ : \langle \sigma_-, u, \sigma_+\rangle \to \sigma_+$ and $\mathcal{P}_1^+ : \sigma \to \langle 0, 0, \sigma\rangle$, respectively. We note that the semigroup of isometries $\mathcal{X}^+(s) = \mathcal{P}^+\mathcal{X}(s)\mathcal{P}_1^+, s \geq 0$, is a one-sided shift in $\mathcal{L}^2(\mathbb{R}_+)$. Indeed, the generator of the semigroup of the one-sided shift $\mathcal{Y}(s)$ in $\mathcal{L}^2(\mathbb{R}_+)$ is the differential operator $i\frac{d}{d\xi}$ with the boundary condition $\sigma(0) = 0$. On the other hand, the generator S of the semigroup of isometries $\mathcal{X}^+(s), s \geq 0$, is the operator $S\sigma = \mathcal{P}^{+}\mathcal{S}_{\beta\gamma}\mathcal{P}_{1}^{+}\sigma = \mathcal{P}^{+}\mathcal{S}_{\beta\gamma}\langle 0,0,\sigma\rangle = \mathcal{P}^{+}\langle 0,0,i\frac{d\sigma}{d\xi}\rangle = i\frac{d\sigma}{d\xi}$, where $\sigma \in \mathcal{W}_{2}^{1}(\mathbb{R}_{+})$ and $\sigma(0) = 0$. Since a semigroup is determined by its generator, it follows that $\mathcal{X}^{+}(s) = \mathcal{Y}(s)$, and hence,

$$\bigcap_{s\geq 0}\mathcal{X}(s)\mathcal{D}^{+} = \langle 0,0,\bigcap_{s\geq 0}\mathcal{Y}(s)\mathcal{L}^{2}(\mathbb{R}_{+}) = \{0\},$$

i.e., property (**ii**) is proved.

In this scheme of the Lax–Phillips scattering theory, the scattering matrix is defined in terms of the theory of spectral representations. We proceed to their construction. Along the way, we also prove property (**iii**) of the incoming and outgoing subspaces.

We recall that the linear operator $\mathbf{A}$ (with domain $\mathfrak{D}(\mathbf{A})$) acting in the Hilbert space $\mathbf{H}$ is called *completely non-self-adjoint* (or *simple*) if there is no invariant subspace $M \subseteq \mathfrak{D}(\mathbf{A})$ ($M \neq \{0\}$) of the operator $\mathbf{A}$ on which the restriction of $\mathbf{A}$ to M is self-adjoint.

We first prove the following lemma.

Lemma 3.1. *The dissipative operator $T_{\beta\gamma}$is completely non-self-adjoint (simple).*

Proof. Let $H' \subset H$ be a non-trivial subspace in which $T_{\beta\gamma}$ induces a self-adjoint operator $T'_{\beta\gamma}$ with domain $\mathfrak{D}(T'_{\beta\gamma}) = H' \cap \mathfrak{D}(T_{\beta\gamma})$. If $f \in \mathfrak{D}(T'_{\beta\gamma})$, then $f \in \mathfrak{D}(T'^{*}_{\beta\gamma})$, and $(rD_{q^{-1}}y)(0) - \beta y(0) = 0$, $(rD_{q^{-1}}y)(0) - \overline{\beta}y(0) = 0$, $[y,\varphi](\infty) - \gamma[y,\psi](\infty) = 0$. From this for the eigenfunctions $y(t,\lambda)$ of the operator $T_{\beta\gamma}$ that lie in H' and are eigenvectors of $T'_{\beta\gamma}$ we have $y(0,\lambda) = 0$, $(rD_{q^{-1}}y)(0,\lambda) = 0$, and then by the uniqueness theorem of the Cauchy problem for the equation $\tau y = \lambda y$, $t \in \mathbb{R}_{+}$, we have $y(t,\lambda) \equiv 0$. Since all solutions of $\tau y = \lambda y$ ($t \in \mathbb{R}_{+}$) belong to $\mathcal{L}^{2}_{w,q}(\mathbb{R}_{+})$, it can be concluded that the resolvent $\mathcal{R}_{\lambda}(T_{\beta\gamma})$ of the operator $T_{\beta\gamma}$ is a Hilbert-Schmidt operator, and hence the spectrum of $T_{\beta\gamma}$ is purely discrete. Hence by the theorem on expansion in eigenvectors of the self-adjoint operator $T'_{\beta\gamma}$, we have $H' = \{0\}$, i.e., the operator $T_{\beta\gamma}$ is simple. The lemma is proved. □

We set

$$\mathcal{H}^{-} = \overline{\bigcup_{s\geq 0}\mathcal{X}(s)\mathcal{D}^{-}},\ \mathcal{H}^{+} = \overline{\bigcup_{s\leq 0}\mathcal{X}(s)\mathcal{D}^{+}}.$$

Lemma 3.2. *The equality $\mathcal{H}^{-} + \mathcal{H}^{+} = \mathcal{H}$ holds.*

Proof. Considering property (**i**) of the subspace $\mathcal{D}^{+}$, it is easy to show that the subspace $\mathcal{H}' = \mathcal{H} \ominus (H + \mathcal{H}^{+})$ is invariant relative to group $\{\mathcal{X}(s)\}$ and has the form $\mathcal{H}' = \langle 0, H', 0\rangle$, where H' is a subspace in H. Therefore, if the subspace $\mathcal{H}'$ (and hence, also H') were non-trivial, then the unitary group $\{\mathcal{X}'(s)\}$, restricted to this subspace, would be a unitary part of the group $\{\mathcal{X}(s)\}$, and hence the restriction $T'_{\beta\gamma}$ of $T_{\beta\gamma}$ to H' would be a self-adjoint operator in H'. From the simplicity of the operator $T_{\beta\gamma}$, it follows that $H' = \{0\}$, i.e. $\mathcal{H}' = \{0\}$. The lemma is proved. □

Let denote by $T_{\infty\gamma}$ the self-adjoint operator generated by the expression τ and the boundary conditions $y(0) = 0$, $[y,\varphi](\infty) - \gamma[y,\psi](\infty) = 0$ ($y \in \mathfrak{D}_{\max}$).

Let $\phi(t,\lambda)$ and $\omega(t,\lambda)$ be the solution of the equation $\tau(y) = \lambda y$ ($t \in \mathbb{R}_{+}$) satisfying the conditions $\phi(0,\lambda) = 0$, $(pD_{q^{-1}}\phi)(0,\lambda) = 1$, $\omega(0,\lambda) = 1$, $(pD_{q^{-1}}\omega)(0,\lambda) = 0$. The *Weyl–Titchmarch function* $m_{\infty\gamma}(\lambda)$ of the self-adjoint operator $T_{\infty\gamma}$ is determined by the condition $[\omega + m_{\infty\gamma}\phi,\varphi](\infty) - \gamma[\omega + m_{\infty\gamma}\phi,\psi](\infty) = 0$. From this we have

$$m_{\infty\gamma}(\lambda) = -\frac{[\omega,\varphi](\infty) - \gamma[\omega,\psi](\infty)}{[\phi,\varphi](\infty) - \gamma[\phi,\psi](\infty)}. \tag{3.1}$$

From (3.1), it follows that $m_{\infty\gamma}(\lambda)$ is a meromorphic function on the complex plane $\mathbb{C}$ with a countable number of poles on the real axis and these poles coincide with the eigenvalues of the operator $T_{\infty\gamma}$ ([5, 6]). Further, it is possible to show that the function $m_{\infty\gamma}$ possesses the following properties: $\Im\lambda\Im m_{\infty\gamma}(\lambda) > 0$ for $\Im\lambda \neq 0$ and $\overline{m_{\infty\gamma}(\lambda)} = m_{\infty\gamma}(\bar{\lambda})$ for $\lambda \in \mathbb{C}$, except the reel poles of $m_{\infty\gamma}(\lambda)$.

Let us adopt the following notations: $\theta(t,\lambda) := \omega(t,\lambda) + m_{\infty\gamma}(\lambda)\phi(t,\lambda)$,

$$\Theta_{\beta\gamma}(\lambda) := \frac{m_{\infty\gamma}(\lambda) - \beta}{m_{\infty\gamma}(\lambda) - \overline{\beta}}. \tag{3.2}$$

Let

$$\Upsilon_\lambda^-(t,\xi,\varsigma) = \langle e^{-i\lambda\xi}, (m_{\infty\gamma}(\lambda)-\beta)^{-1}\alpha\theta(t,\lambda), \overline{\times}_{\beta\gamma}(\lambda)e^{-i\lambda\varsigma}\rangle.$$

Note that the vectors $\Upsilon_\lambda^-(t,\xi,\varsigma)$ for real λ do not belong to the space $\mathcal{H}$. However, $\Upsilon_\lambda^-(t,\xi,\varsigma)$ satisfies the equation $\mathcal{L}\Upsilon = \lambda\Upsilon$ and the corresponding boundary conditions for the operator $\mathcal{S}_{\beta\gamma}$. Using $\Upsilon_\lambda^-(t,\xi,\varsigma)$, we define the transformation $\Phi_- : f \to \tilde{f}_-(\lambda)$ by $(\Phi_- f)(\lambda) := \tilde{f}_-(\lambda) := \frac{1}{\sqrt{2\pi}}(f,\Upsilon_\lambda^-)_{\mathcal{H}}$ on the vector $f = \langle\sigma_-, y, \sigma_+\rangle$ in which σ_-, σ_+, and y are smooth, compactly supported functions.

Lemma 3.3. *The transformation Φ_- isometrically maps $\mathcal{H}^-$ onto $\mathcal{L}^2(\mathbb{R})$. For all vectors $f, g \in \mathcal{H}^-$, the Parseval equality and the inversion formula hold:*

$$(f,g)_{\mathcal{H}} = (\tilde{f}_-, \tilde{g}_-)_{\mathcal{L}^2} = \int_{-\infty}^{\infty} \tilde{f}_-(\lambda)\overline{\tilde{g}_-(\lambda)}d\lambda,\ f = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{f}_-(\lambda)\Upsilon_\lambda^- d\lambda,$$

where $\tilde{f}_-(\lambda) = (\Phi_- f)(\lambda)$ and $\tilde{g}_-(\lambda) = (\Phi_- g)(\lambda)$.

Proof. For $f, g \in \mathcal{D}^-, f = \langle\sigma_-, 0, 0\rangle, g = \langle\rho_-, 0, 0\rangle$, we have that

$$\tilde{f}_-(\lambda) := \frac{1}{\sqrt{2\pi}}(f,\Upsilon_\lambda^-)_{\mathcal{H}} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0}\sigma_-(\xi)e^{i\lambda\xi}d\xi \in H_-^2,$$

and, in view of the usual Parseval equality for Fourier integrals,

$$(f,g)_{\mathcal{H}} = \int_{-\infty}^{0}\sigma_-(\xi)\overline{\rho_-}(\xi)d\xi = \int_{-\infty}^{\infty}\tilde{f}_-(\lambda)\overline{\tilde{g}_-(\lambda)}d\lambda = (\Phi_- f, \Phi_- g)_{\mathcal{L}^2}.$$

Here and below, $\mathcal{H}_\pm^2$ denote the Hardy classes in $\mathcal{L}^2(\mathbb{R})$ consisting of the functions analytically extendable to the upper and lower half-planes, respectively.

We now extend the Parseval equality to the whole of $\mathcal{H}^-$. For this purpose, we consider in $\mathcal{H}^-$ the dense set $\mathcal{H}'_-$ of vectors obtained from the smooth, compactly supported functions in $\mathcal{D}^- : f \in \mathcal{H}'_-$ if $f = \mathcal{X}(l)f_0$, $f_0 = \langle\sigma_-, 0, 0\rangle$, $\sigma_- \in C_0^\infty(\mathbb{R}_-)$, where $l = l_f$ is a non-negative number (depending on f). In this case, if $f, g \in \mathcal{H}^-$, then for $l > l_f$ and $l > l_g$ we have that $\mathcal{X}(-l)f, \mathcal{X}(-l)g \in \mathcal{D}^-$ and moreover, the first components of these vectors belong to $C_0^\infty(\mathbb{R}_-)$. Therefore, since the operators $\mathcal{X}(s)$ $(s \in \mathbb{R})$ are unitary, the equality $\Phi_-\mathcal{X}(-l)f = (\mathcal{X}(-l)f, U_\lambda^-)_{\mathcal{H}} = e^{-i\lambda l}(f, U_\lambda^-)_{\mathcal{H}} = e^{-i\lambda l}\Phi_- f$, implies that

$$\begin{aligned}(f,g)_{\mathcal{H}} &= (\mathcal{X}(-l)f, \mathcal{X}(-l)g)_{\mathcal{H}} = (\Phi_-\mathcal{X}(-l)f, \Phi_-\mathcal{X}(-l)g)_{L^2}\\ &= (e^{-i\lambda l}\Phi_- f, e^{-i\lambda l}\Phi_- g)_{\mathcal{L}^2} = (\Phi_- f, \Phi_- g)_{\mathcal{L}^2}.\end{aligned} \tag{3.3}$$

By taking the closure in (3.3), we obtain the Parseval equality for the whole space $\mathcal{H}^-$. The inversion formula follows from the Parseval equality if all integrals in it are understood as limits in the mean of integrals over finite intervals. Finally,

$$\Phi_-\mathcal{H}^- = \overline{\bigcup_{s\geq 0}\Phi_-\mathcal{X}(s)\mathcal{D}^-} = \overline{\bigcup_{s\geq 0}e^{-i\lambda s}\mathcal{H}_-^2} = \mathcal{L}^2(\mathbb{R}),$$

i.e. Φ_- maps $\mathcal{H}^-$ onto the whole of $\mathcal{L}^2(\mathbb{R})$. The lemma is proved. □

We set

$$\Upsilon_\lambda^+(t,\xi,\varsigma) = \langle\Theta_{\beta\gamma}(\lambda)e^{-i\lambda\xi}, (m_{\infty\gamma}(\lambda)-\overline{\beta})^{-1}\alpha\theta(t,\lambda), e^{-i\lambda\varsigma}\rangle.$$

Note that the vectors $\Upsilon_\lambda^+(t,\xi,\varsigma)$ for real λ do not belong to the space $\mathcal{H}$. However, $\Upsilon_\lambda^+(t,\xi,\varsigma)$ satisfies the equation $\mathcal{S}\Upsilon = \lambda\Upsilon$ $(\lambda \in \mathbb{R})$ and the boundary conditions (2.14) and (2.15). Using $\Upsilon_\lambda^+(t,\xi,\varsigma)$, we define the transformation $\Phi_+ : f \to \tilde{f}_+(\lambda)$ on vectors $f = \langle \sigma_-, y, \sigma_+ \rangle$, in which σ_-, σ_+, and y are smooth, compactly supported functions by setting $(\Phi_+ f)(\lambda) := \tilde{f}_+(\lambda) := \frac{1}{\sqrt{2\pi}}(f, \Upsilon_\lambda^+)_{\mathcal{H}}$.

The proof of the next result is analogous to that of Lemma 3.3.

Lemma 3.4. *The transformation Φ_+ isometrically maps $\mathcal{H}^+$ onto $\mathcal{L}^2(\mathbb{R})$. For all vectors $f, g \in \mathcal{H}^+$, the Parseval equality and the inversion formula hold:*

$$(f, g)_{\mathcal{H}} = (\tilde{f}_+, \tilde{g}_+)_{\mathcal{L}^2} = \int_{-\infty}^{\infty} \tilde{f}_+(\lambda)\overline{\tilde{g}_+(\lambda)} d\lambda, \ f = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}_+(\lambda) U_\lambda^+ d\lambda,$$

where $\tilde{f}_+(\lambda) = (\Phi_+ f)(\lambda)$ and $\tilde{g}_+(\lambda) = (\Phi_+ g)(\lambda)$.

According to (3.2), the function $\Theta_{\beta\gamma}(\lambda)$ satisfies $\left|\Theta_{\beta\gamma}(\lambda)\right| = 1$ for $\lambda \in \mathbb{R}$; therefore, it follows from the explicit formula for the vectors Υ_λ^+ and Υ_λ^- that

$$\Upsilon_\lambda^- = \overline{\Theta}_{\beta\gamma}(\lambda)\Upsilon_\lambda^+ \ (\lambda \in \mathbb{R}). \tag{3.4}$$

Therefore, it follows from Lemmas 3.3 and 3.4 that $\mathcal{H}^- = \mathcal{H}^+$. Together with Lemma 3.2, this shows that $\mathcal{H} = \mathcal{H}^- = \mathcal{H}^+$, and property (**iii**) above has been established for the incoming and outgoing subspaces.

Thus, the transformation Φ_- isometrically maps onto $\mathcal{L}^2(\mathbb{R})$ with the subspace $\mathcal{D}^-$ mapped onto $\mathcal{H}_-^2$ and the operators $\mathcal{X}(s)$ are transformed into the operators of multiplication by $e^{i\lambda s}$. In order words, Φ_- is the incoming spectral representation for the group $\{\mathcal{X}(s)\}$. Similarly, Φ_+ is the outgoing spectral representation for $\{\mathcal{X}(s)\}$. It follows from (3.4) that the passage from the Φ_+-representation of a vector $f \in \mathcal{H}$ to its Φ_--representation is realized by multiplication of the function $\Theta_{\beta\gamma}(\lambda) : \tilde{f}_-(\lambda) = \Theta_{\beta\gamma}(\lambda)\tilde{f}_+(\lambda)$. According to [16], the scattering function (matrix) of the group $\{\mathcal{X}(s)\}$ with respect to the subspaces $\mathcal{D}^-$ and $\mathcal{D}^+$, is the coefficient by which the Φ_--representation of a vector $f \in \mathcal{H}$ must be multiplied in order to get the corresponding Φ_+-representation: $\tilde{f}_+(\lambda) = \overline{\Theta}_{\beta\gamma}(\lambda)\tilde{f}_-(\lambda)$. According to, we have now proved the following theorem.

Theorem 3.5. *The function $\overline{\Theta}_{\beta\gamma}(\lambda)$ is the scattering matrix of the group $\{\mathcal{X}(s)\}$ (of the self-adjoint operator $\mathcal{S}_{\beta\gamma}$).*

Let $\Theta(\lambda)$ be an arbitrary non-constant inner function on the upper half-plane (we recall that a function $\times(\lambda)$ analytic in the upper half-plane $\mathbb{C}_+$ is called *inner function* on $\mathbb{C}_+$ if $|\Theta(\lambda)| \leq 1$ for $\lambda \in \mathbb{C}_+$, and $|\Theta(\lambda)| = 1$ for almost all $\lambda \in \mathbb{R}$). Define $\mathcal{K} = \mathcal{H}_+^2 \ominus \Theta\mathcal{H}_+^2$. Then $\mathcal{K} \neq \{0\}$ is a subspace of the Hilbert space $\mathcal{H}_+^2$. We consider the semigroup of the operators $\mathcal{Z}(s)$ $(s \geq 0)$ acting in $\mathcal{K}$ according to the formula $\mathcal{Z}(s)\varphi = \mathcal{P}\left[e^{i\lambda s}\varphi\right]$, $\varphi := \varphi(\lambda) \in \mathcal{K}$, where $\mathcal{P}$ is the orthogonal projection from $\mathcal{H}_+^2$ onto $\mathcal{K}$. The generator of the semigroup $\{\mathcal{Z}(s)\}$ $(s \geq 0)$ is denoted by $T : T\varphi = \lim_{s\to+0}(is)^{-1}(\mathcal{Z}(s)\varphi - \varphi)$, which is a dissipative operator acting in $\mathcal{K}$ and with the domain $\mathfrak{D}(T)$ consisting of all functions $\varphi \in \mathcal{K}$, such that the limit exists. The operator T is called a *model dissipative operator* (we remark that this model dissipative operator, which is associated with the names of Lax and Phillips [16], is a special case of a more general model dissipative operator constructed by Sz.-Nagy and Foiaş [20]). The basic assertion is that $\Theta(\lambda)$ is the *characteristic function* of the operator T.

Let $\mathbf{K} = \langle 0, H, 0 \rangle$, so that $\mathcal{H} = \mathcal{D}^- \oplus \mathbf{K} \oplus \mathcal{D}^+$. From the explicit form of the unitary transformation Φ_- it follows that

$$\mathcal{H} \to \mathcal{L}^2(\mathbb{R}), \ f \to \tilde{f}_-(\lambda) = (\Phi_- f)(\lambda), \ \mathcal{D}^- \to \mathcal{H}_-^2, \ \mathcal{D}^+ \to \Theta_{\beta\gamma}\mathcal{H}_+^2, \tag{3.5}$$

$$\mathbf{K} \to \mathcal{H}_+^2 \ominus \Theta_{\beta\gamma}\mathcal{H}_+^2, \ \mathcal{X}(s)f \to (\Phi_- \mathcal{X}(s)\Phi_-^{-1}\tilde{f}_-)(\lambda) = e^{i\lambda s}\tilde{f}_-(\lambda). \tag{3.6}$$

The formulae (3.5) and (3.6) show that our operator $T_{\beta\gamma}$ is a unitary equivalent to the model dissipative operator with the characteristic function $\Theta_{\beta\gamma}(\lambda)$. Since the characteristic functions of unitary equivalent dissipative operators coincide (see [18-20]), we have proved

Theorem 3.6. *The characteristic function of the dissipative operator $T_{\beta\gamma}$ coincides with the function $\Theta_{\beta\gamma}(\lambda)$ defined in* (3.2).

Characteristic function is very useful to answer the question that whether all eigenfunctions and associated functions of a dissipative operator $T_{\beta\gamma}$ span the whole space or not. We can perform this analysis by ensuring that the singular factor $s(\lambda)$ in the factorization $\Theta_{\beta\gamma}(\lambda) = s(\lambda)B(\lambda)$ ($B(\lambda)$ is a Blaschke product) is absent ([1-3, 18-20]).

Theorem 3.7. *For all values of β with $\Im\beta > 0$, except possibly for a single value $\gamma = \gamma_0$, and for fixed γ* with $\Im\gamma = 0$ or $\gamma = 0$, *the characteristic function $\Theta_{\beta\gamma}$ of the dissipative operator $T_{\beta\gamma}$ is a Blaschke product. The spectrum of $T_{\beta\gamma}$ is purely discrete and lies in the open upper half-plane. The operator $T_{\beta\gamma}$ ($\beta \neq \beta_0$) has a countable number of isolated eigenvalues with finite multiplicity and limit points at infinity. The system of all eigenfunctions and associated functions (or root functions) of the dissipative operator $T_{\beta\gamma}$ ($\beta \neq \beta_0$) is complete in the space H.*

Proof. Using (3.2), it is easy to see that $\Theta_{\beta\gamma}$ is an inner function in the upper half-plane and it is meromorphic in the whole λ-plane. We have the factorization

$$\Theta_{\beta\gamma}(\lambda) = e^{i\lambda b} B_{\beta\gamma}(\lambda), \tag{3.7}$$

where $B_{\beta\gamma}(\lambda)$ is the Blaschke product and $b = b(\beta) \geq 0$. Therefore we obtain from (3.7) that

$$\left|\Theta_{\beta\gamma}(\lambda)\right| = \left|e^{i\lambda b}\right| \left|B_{\beta\gamma}(\lambda)\right| \leq e^{-b(\beta)\Im\lambda}, \Im\lambda \geq 0. \tag{3.8}$$

On the other hand, if we express $m_{\infty\gamma}(\lambda)$ in terms of $\Theta_{\beta\gamma}(\lambda)$ we get from (3.2) that

$$m_{\infty\gamma}(\lambda) = \frac{\overline{\beta}\Theta_{\beta\gamma}(\lambda) - \beta}{\Theta_{\beta\gamma}(\lambda) - 1}. \tag{3.9}$$

If $b(\beta) > 0$ for a given value β ($\Im\beta > 0$), then (3.8) gives us that $\lim_{s\to+\infty} \Theta_{\beta\gamma}(is) = 0$, and then (3.9) leads to $\lim_{s\to+\infty} m_{\infty\gamma}(is) = -\beta$. Since $m_{\infty\gamma}(\lambda)$ is independent of β, $b(\beta)$ can be non-zero at not more than a single point $\beta = \beta_0$ and, further $b(\beta)$ can be non-zero at not more then a single point $\beta = \beta_0$ (and, further, $\beta_0 = -\lim_{s\to+\infty} m_{\infty\gamma}(is)$). Therefore the proof is completed. □

Since a linear operator **T** acting in the Hilbert space **H** is accumulative if and only if −**T** is dissipative, all results concerning dissipative operators can be immediately stated for accumulative operators. Then the Theorem 3.7 yields the following result.

Corollary 3.8. *For all values of β with $\Im\beta < 0$, except possibly for a single value $\beta = \beta_1$, and for fixed γ* with $\Im\gamma = 0$ or $\gamma = 0$, *the characteristic function $\Theta_{\beta\gamma}$ of the accumulative operator $T_{\beta\gamma}$ is a Blaschke product. The spectrum of $T_{\beta\gamma}$ is purely discrete and lies in the open lower half-plane. The operator $T_{\beta\gamma}$ ($\beta \neq \beta_1$) has a countable number of isolated eigenvalues with finite multiplicity and limit points at infinity. The system of all eigenvectors and associated functions (or root functions) of the accumulative operator $T_{\beta\gamma}$ ($\beta \neq \beta_1$) is complete in the space H.*

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