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A New Result for Fractional Differential Equation With Nonlocal Initial Value Using Caputo-Fabrizio Derivative

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Abstract. In this paper, using Caputo-Fabrizio fractional derivative, we obtain some new results for the existence and uniqueness of solutions of differential equation with nonlocal initial value about fractional order $0 < \alpha < 1$. These results are applied with the help of Arzela-Ascoli theorem and Schauder fixed point. Also based on some α-contractive maps for such problems, some new unique theorem has been introduced and proved. Finally, some illustrative example is considered to show the effectiveness of the results.

1. Introduction

The problem of fractional calculation was directly and indirectly attracted by large mathematicians such as Euler, Lagrange, Laplace, Liouville, Riemann, and so on. Comparative studies of ordinary differential equations (ODE) with fractional differential equations (FDE) has been presented. Therefore, differential equations have been the focus of many changes [18],[27].

In recent years, due to the frequent occurrence of the differential fractional equations in engineering, economics, electrochemical, biology and image processing, aerodynamics, economics, polymorphism, electrodynamics, economics, control theory, biophysics, ... more attention has been given to the solution for the Linear and nonlinear differential fractional equation [13], [33], [16], [24], [35], [41], [3], [39], [40], [25], [45], [42], [22].

Recent applications of fractional calculations in dynamic systems in control theory, compression electric circuits, total voltage divider, viscosity, electromagnetism, electrochemical, fluid flow detector, and neuron model in biology were investigated [17]. The approximation of the solution of the nonlinear differential equation was derived from the fractional derivative, and the existence and uniqueness of mild solution was investigated for the problem of the initial value of the semi-linear derivative of the fractional order in [1]. Adomian decomposition method was used to obtain the solutions of linear and nonlinear equations of propagation and wave with fractional derivative. Homotopy analysis method, the solution for the partial differential equations was shown by a fractional derivative [15]. The solution for the wave-propagation

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fraction equation was proved by a finite difference method (FDM) [20]. Using the compact discrepancy method, the existence of the solution for the sub-diffusion fractional equation has been investigated and the stability and convergence of the obtained result have been proved [21]. Some existence of mild solutions for a class of non-linear nonlinear delay-reaction-diffusion equations was investigated using fractional derivatives [58]. Also, the solutions of the fractional differential-integral equation were proved by a numerical method using the Legendre and Chebyshev polynomials [8].

Nagumo equations of Heat transfer and nonlinear propagation used in physiology, biochemical reactions, and other fields were also considered. Using the fractional derivative and fixed point theorems, the solution of these equations was investigated [36], [37], [7]. Existence, uniqueness and stability of the solutions of linear fractional differential equations were investigated using the Banach fixed point theorem [18]. Using Schauder's fixed point theorem, the analysis of the solutions of the differential fractional equations for square matrices with real values was performed [16]. Moreover, existence and uniqueness of the solutions of the linear fractional differential equations of the $C^{n-1}[a,b]$ space were investigated using the Banach fixed point theorem [26]. Banach fixed point theorem and the Leray-Schauder nonlinear method are used to find the solution of the finite-delayed fractional differential equations [11]. The existence of solutions of differential fractional inclusion with boundary conditions has been investigated. A sufficient condition for solving these problems is to use the Bohnenblast-Karlinfixed point theorem [15]. The solution for integral equations in relation to boundary value problems for a nonlinear differential fractional equation of along with the existence of at least one solution for integral equations using a nonlinear Leray-Schauder method for several types of initial value problems has been investigated.

Using the Banach contraction principle, sufficient conditions for unique solutions have been proposed [30]. Uniqueness and convergence of successive approximations of the Nagumo differential fractional equation have been extended. For this purpose, the initial value problem is changed to the form of equivalent fractional integral Volterra equation. It is assumed that the initial value problem satisfies to the Nagumo type condition. Then, using the successive approximation of existence and uniqueness, the answer is determined [29].

Two general existential results for an initial boundary value problem with a large class of fractional differential equations were investigated with two fixed points of Arzela-Ascoli and Knaster-Tarski [9]. By transforming the differential fraction equation into its integral equivalent form in the $L^1[0,\infty)$ space and applying the Schauder theorem and the Banach contraction principle, the solution to the differential fraction equation was investigated [12]. An equivalent fractional integral equation is constructed with an appropriate replacement. Then, using a fractional inequality of a nonlinear Leray-schauder method, the answer to the fractional differential equation is proved. Under the additional condition of singularity, the solutions to the proposed Banach contraction principle have been proved [2]]. The existence and uniqueness of the solutions of the nonlinear differential equations of general fraction order has been concluded by introducing a new norm and using the Banach contraction principle [34]. Using the Leray-schauder fixed point theorem and applying Banach contraction principle, various theorems for the existence and uniqueness of generalized solutions for the problem of nonlinear quantitative reaction equations have been extended by delay [58]. The existence and uniqueness of the solution for a class of boundary value problems of nonlinear differential fractional equations were investigated using the Banach contraction principle and the Schauder fixed point theorem [8].

Srivastava et al. in [48], [53], [54] and [43] have recently proposed some new approximate solution of the time-fractional Nagumo equation involving fractional integrals without singular kernel, investigated the properties of spiral-like close-to-convex functions associated with conic domains, studied the fractional-order mathematical model of diabetes and its resulting complications, and applied Jacobi collocation method for obtaining the approximate solution of some fractional-order Riccati differential equations with variable coefficients, respectively. Ali et al. [5] discussed the solution of fractional Volterra–Fredholm integro-differential equations under mixed boundary conditions by using the HOBW method. Some application of the Gegenbauer Wavelet Method for the numerical solution of the fractional Bagley-Torvik Equation has considered in [50]. Also, the analytical and approximate solutions of fractional-order susceptible-infected-recovered epidemic model of childhood disease is studied in [46]. Moreover, in [49], the authors have analyzed the time-fractional and space-time fractional-order Nagumo equation. A Class of Nonlinear

Boundary Value Problems for an arbitrary fractional-Order differential equation with the Riemann-Stieltjes functional integral and infinite-point boundary conditions is verified in [52]. Some related and interesting results can be found in [10], [47], [55], [51], [56], [44] and [57].

In 2015, Caputo and Fabrizio introduced a new fractional derivative [14]. In other articles, new derivative properties have been reviewed and extended [8], [7], [31], [23] and [28].

In this paper, using Caputo-Fabrizio derivative, some existence and uniqueness results for the solutions of the differential fractional equations of order $0 < \alpha < 1$ is considered.

$$\begin{aligned} &{}^{CF}D^{\alpha}x(t) = f(t, x(t)), \quad 0 \le t \le 1, \\ &x(0) = \int_0^1 g(s)x(s)ds \end{aligned} \tag{1}$$

In the above equation, ${}^{CF}D^{\alpha}$ is the fractional derivative of Caputo-Fabrizio sense, $0 < \alpha < 1$, $g \in L^1([0,1], R_+)$, $g(t) \in [0,1)$ and f is a function of E-value.

The organization of paper is as follows: In the first section we recall and introduce some spaces, definitions and results for further applications. In section 2, we propose the new results in two fold. First, we prove the existence and uniqueness results using Arzela-Ascoli theorem and Schauder fixed point included some illustrative example. After that, in the second part, we suggest some approximation of the solution using some $\alpha-$contractive maps.

2. Preliminaries

We define E as Banach space, let $C([0,1], E)$ as the Banach space of all of the continuous functions where $x : [0,1] \to E$ and the norm $\|x\|_c = \sup_{t\in[0,1]} \|x(t)\|$ are defined on it. Set $L^1([0,1], E)$ as the Banach space of measurable functions $x : [0,1] \to E$ as integral and is equipped with $\|x\|_{L^1} = \int_0^1 \|x(s)\|ds$ norm. The function of $x \in C([0,1], E)$ is the solution of the equation 1 if it satisfies to the equation 1.

Definition 2.1. *[27]. The fractional derivative Caputo of order α , $0 < \alpha < 1$, is defined as follows:*

$$ {}^{C}D^{\alpha}x(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha}x'(s)ds, \qquad t > 0. \tag{2}$$

It is achieved by replacing the core $(t-s)^{-\alpha}$ with the function $\exp(-\alpha(t-s)/(1-\alpha))$ and also by changing $\frac{1}{\Gamma(1-\alpha)}$ to $\frac{M(\alpha)}{(1-\alpha)}$ new fractional derivative of Caputo-Fabrizio.

Definition 2.2. *[14]. The fractional derivative Caputo-Fabrizio of order $0 < \alpha < 1$ is defined as follows:*

$$ {}^{CF}D^{\alpha}x(t) = \frac{(2-\alpha)M(\alpha)}{2(1-\alpha)} \int_0^t \exp(-\frac{\alpha}{1-\alpha}(t-s))x'(s)ds, \qquad t \ge 0 \tag{3}$$

Definition 2.3. *[14] The fractional integral of the order $0 < \alpha < 1$ as follows:*

$$ {}^{CF}I^{\alpha}x(t) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} f(t, x(t)) + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t f(s, x(s))ds, \qquad t \ge 0 \tag{4}$$

Remark 2.4. *[14] Based on the above definition, it is clearly seen that if x is a constant function, then we have the same derivative as normal Caputo ${}^{CF}D^{\alpha}x = 0$.*

Lemma 2.5. *Suppose that $0 < \alpha < 1$, then the unique solution of the problem ${}^{CF}D^{\alpha}x(t) = f(t, x(t))$ with the boundary condition $x(0) = c$ is as follows [31]:*

$$ x(t) = c + a_{\alpha} f(t, x(t)) + b_{\alpha} \int_0^t f(s, x(s))ds \tag{5}$$

where

$$ a_{\alpha} = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} = 1-\alpha, b_{\alpha} = \frac{2\alpha}{(2-\alpha)M(\alpha)} = \alpha \tag{6}$$

3. Main Results

This section included two parts. Firstly, we discuss about the existence and uniqueness results using well-known fixed point theorems. Second, we propose some approximation of the solution using α-contractive maps.

3.1. Existence and uniqueness results

In this subsection, we used some theorems of fixed point for existence and uniqueness of solutions of the fractional differential equation 1 in Banach space E. These results are proved in [32] for fractional derivative of Caputo type. In fact, we proved using Caputo-Fabrizio derivative.

Suppose that the following conditions are satisfied:

(H_1), Let $f \in C([0,1] \times E, E)$ such that there is a constant $M > 0$, $p_f(t) \leq M$ for $t \in [0,1]$ and every $x \in E$, there is $p_f(t) \in L^1([0,1], R_+)$ such that $\|.f(t,x)\|. \leq p_f(t)\|.x\|.$.

(H_2) For each $t \in [0,1]$ and $R > 0$, $f(t, B_R) = \{f(t,x) : x \in B_R\}$ is a relatively compact in E , where $B_R = \{x \in C([0,1], E), \|.x\|_C \leq R.\}$, $\Lambda_1 = \frac{2-\mu}{1-\mu}M < 1$ and $\mu = \int_0^1 g(s)ds$.

Lemma 3.1. *If the condition (H_1) holds, then the problem 1 is equivalent to the equation:*

$$x(t) = \frac{1-\alpha}{1-\mu}\int_0^1 g(s)f(s,x(s))ds + \frac{\alpha}{1-\mu}\int_0^1 Q(\tau)f(\tau,x(\tau))d\tau + (1-\alpha)f(t,x(t)) + \alpha\int_0^t f(s,x(s))ds$$

Proof. Using Lemma 2.5, we have:

$$x(t) = x(0) + (1-\alpha)f(t,x(t)) + \alpha\int_0^t f(s,x(s))ds \tag{7}$$

Then, we obtain:

$$\begin{aligned} x(0) &= \int_0^1 g(s)x(s)ds \\ &= \int_0^1 g(s)[x(0) + (1-\alpha)f(s,x(s)) + \alpha\int_0^s f(\tau,x(\tau))d\tau] \\ &= \int_0^1 g(s)dsx(0) + (1-\alpha)\int_0^1 g(s)f(s,x(s))ds + \int_0^1 g(s)\int_0^s f(\tau,x(\tau))d\tau ds \end{aligned}$$

Therefore, if $Q(\tau) = \int_\tau^1 g(s)ds$ then, we have:

$$\begin{aligned} x(0) &= \tfrac{1-\alpha}{(1-\int_0^1 g(s)ds)}\int_0^1 g(s)f(s,x(s))ds + \tfrac{\alpha}{(1-\int_0^1 g(s)ds)}\int_0^1 Q(\tau)f(\tau,x(\tau))d\tau \\ &= \tfrac{1-\alpha}{1-\mu}\int_0^1 g(s)f(s,x(s))ds + \tfrac{\alpha}{1-\mu}\int_0^1 Q(\tau)f(\tau,x(\tau))d\tau. \end{aligned} \tag{8}$$

So:

$$x(t) = \frac{1-\alpha}{1-\mu}\int_0^1 g(s)f(s,x(s))ds + \frac{\alpha}{1-\mu}\int_0^1 Q(\tau)f(\tau,x(\tau))d\tau + (1-\alpha)f(t,x(t)) + \alpha\int_0^t f(s,x(s))ds \tag{9}$$

Conversely, if x be a solution, then for each $t \in [0,1]$, according to the derivative definition, we have:

$$\begin{aligned} {}^{CF}D^\alpha x(t) &= {}^{CF}D^\alpha(\tfrac{1-\alpha}{1-\mu}\int_0^1 g(s)f(s,x(s))ds + \tfrac{\alpha}{1-\mu}\int_0^1 Q(\tau)f(\tau,x(\tau))d\tau + (1-\alpha)f(t,x(t)) + \alpha\int_0^t f(s,x(s))ds) \\ &= {}^{CF}D^\alpha((1-\alpha)f(t,x(t)) + \alpha\int_0^t f(s,x(s))) = {}^{CF}D^\alpha({}^{CF}I^\alpha(f(t,x(t)))) \\ &= f(t,x(t)) \end{aligned} \tag{10}$$

which completes the proof.

Theorem 3.2. *Suppose that conditions(H_1) and (H_2) hold and function f satisfied in Lipschitz condition, then the initial value problem 1 has at least one solution.*

Proof.
Consider the operator $A : C([0,1],E) \to C([0,1],E)$ as follows:

$$(Ax)(t) = \frac{1-\alpha}{1-\mu}\int_0^1 g(s)f(s,x(s))ds + \frac{\alpha}{1-\mu}\int_0^1 Q(\tau)f(\tau,x(\tau))d\tau + (1-\alpha)f(t,x(t)) + \alpha\int_0^t f(s,x(s))ds \quad (11)$$

Obviously, the fixed point of the operator A is the solution to the problem 1. It is easy to verify that $B_R = \{x \in C([0,1],E), \|.x\|_C \leq R\}$ is convex, closed and bounded.
We follow the proof in several steps:
Step 1. We show that the operator A is continuous. We set:

$$x_n, \overline{x} \in C([0,1],E), \qquad \|x_n - \overline{x}\|_c \to 0. \quad (12)$$

Then, we have:

$$r = \sup_n \|x_n\|_C < \infty \qquad , \qquad \|\overline{x}\|_C \leq r, \quad (13)$$

for each $t \in [0,1]$, and

$$\begin{aligned}\|(Ax_n)(t) - (A\overline{x})(t)\| \leq & \tfrac{1-\alpha}{1-\mu}\int_0^1 g(s)\|f(s,x_n(s)) - f(s,\overline{x}(s))\|ds + \tfrac{\alpha}{1-\mu}\int_0^1 Q(\tau)\|f(\tau,x_n(\tau)) - f(\tau,\overline{x}(\tau))\|d\tau \\ & +(1-\alpha)\|f(t,x_n(t)) - f(t,\overline{x}(t))\| + \alpha\int_0^t \|f(s,x_n(s)) - f(s,\overline{x}(s))\|ds.\end{aligned} \quad (14)$$

It is easy to verify that:

$$\begin{aligned}& f(t,x_n(t)) \to f(t,\overline{x}(t)), \qquad as \quad n \to \infty, \quad t \in [0,1], \\ & \|f(t,x_n(t)) - f(t,\overline{x}(t))\| \leq 2Mr.\end{aligned} \quad (15)$$

Therefore:

$$\|(Ax_n) - (A\overline{x})\|_c \to 0, \quad n \to \infty \quad (16)$$

Step 2. We show that $A(B_R)$ is equicontinuous.
Let $t_1, t_2 \in [0,1]$, $t_1 < t_2$, and $x \in B_R$. then:

$$\begin{aligned}& \|(Ax)(t_2) - (Ax)(t_1)\| \\ & \quad \leq (1-\alpha)\|f(t_2,x(t_2)) - f(t_1,x(t_1))\| + \alpha\int_{t_1}^{t_2} \|f(s,x(s))\|ds.\end{aligned} \quad (17)$$

If $t_1 \to t_2$, the right side of inequality goes to zero, and the continuous property is proved.
Step 3. Let us consider$x \in B_R$. For each $t \in [0,1]$ we have:

$$\begin{aligned}\|(Ax)(t)\| \leq & \tfrac{1-\alpha}{1-\mu}\int_0^1 g(s)\|f(s,x(s))\|ds + \tfrac{\alpha}{1-\mu}\int_0^1 Q(\tau)\|f(\tau,x(\tau))\|d\tau + (1-\alpha)\|f(t,x(t))\| + \alpha\int_0^t \|f(s,x(s))\|ds \\ \leq & M_1 + M_2 + (1-\alpha)MR + \alpha MR < R.\end{aligned} \quad (18)$$

So the uniformly bounded condition is proven. It follows from Arzela-Ascoli theorem we deduce that it is relatively compact, then, based on Schauder theorem, the existence of the fixed point of the mapping in B_R is proved.

Theorem 3.3. *Under assumptions of Theorem 3.2, and considering the following assumptions:*

$$\begin{aligned}& 0 < L < \Lambda_2, \\ & \|f(t,u) - f(t,w)\| \leq L\|u - w\|,\end{aligned} \quad (19)$$

where $u, w \in B_R$ and $\Lambda_2 = \frac{1-\mu}{2-\mu}$, then $x(t)$ is the solution of problem 1 . More precisely, the solution will be unique in B_R.

Proof.
Using Theorem 3.2, we know that there is at least one solution $x(t)$ in B_R . Now, let us consider there are two solutions $w(t), u(t)$ belong to B_R. Then we have:

$$\begin{aligned}\|u(t)-w(t)\| &\leq \tfrac{1-\alpha}{1-\mu}\int_0^1 \|f(s,u(s))-f(s,w(s))\|ds + \tfrac{\alpha}{1-\mu}\int_0^1 \|f(\tau,u(\tau))-f(\tau,w(\tau))\|d\tau \\ &\quad +(1-\alpha)\|f(t,u(t))-f(t,w(t))\| + \alpha\int_0^t \|f(s,u(s))-f(s,w(s))\|ds \\ &\leq (\tfrac{1-\alpha}{1-\mu}+\tfrac{\alpha}{1-\mu}+(1-\alpha)+\alpha)L\|u-w\|_C \\ &= \tfrac{2-\mu}{1-\mu}L\|u-w\|_C.\end{aligned} \tag{20}$$

According to the assumption, it is concluded that,

$$\|u-w\|_C = 0, \tag{21}$$

which completes the proof.

Now, we recall a similar example given in [32] to show the ability of proposed approach under Caputo-Fabrizio derivative.

Example 3.4. *Suppose* $E = c_0 = \{x = (x_1, x_2, ..., x_n, ...) : x_n \to 0\}$ *and* $\|x\| = \sup|x_n|$ *.Consider the following differential fractional equation of order* $0 < \alpha < 1$ *on E:*

$$\begin{aligned}&{}^{CF}D^{\alpha}\, x_n(t) = \tfrac{3t+2}{200n^3}x_n(t), \qquad t\in[0,1]\\ &x_n(0) = \int_0^1 \tfrac{1}{5}\; x_n(s)ds\end{aligned} \tag{22}$$

Then the above equation in $[0,1]$ *has an unique solution.*

Solution 3.5. *Let*

$$\begin{aligned}&f_n(t,x) = \tfrac{3t+2}{200n^3}x_n, \qquad f = (f_1, f_2, ..., f_n),\\ &g(s) = \tfrac{1}{5}, \qquad p_f(t) = \tfrac{3t+2}{200n^3}.\end{aligned} \tag{23}$$

Clearly,

$$\begin{aligned}&f \in C([0,1]\times E, E), \qquad p_f(t) \leq \tfrac{1}{40} = M,\\ &p_f \in L([0,1], R^+), \qquad \|f(t,x)\| \leq p_f\|x\|.\end{aligned} \tag{24}$$

So the assumption(H_1) *is valid. Based on the Arzela-Ascoli theorem, we conclude that the* $f(t, B_R)$ *in* c_0 *has a relative compression. With a simple discussion, we have:*

$$\Lambda_1 = \frac{2-\mu}{1-\mu}M \leq \frac{2-\mu}{1-\mu}\times\frac{1}{40} < 1. \tag{25}$$

Therefore, hypothesis and consequently (H_2)*, Theorem 3.2 is established for this example. To prove uniqueness, we must use Theorem 3.3.*

Indeed,

$$|f_n(t,u) - f_n(t,w)| = |\frac{3t+2}{200n^3}u_n - \frac{3t+2}{200n^3}w_n| \leq \frac{1}{40}|u_n - w_n|, \tag{26}$$

Therefore:

$$\|f(t,u) - f(t,w)\| \leq \frac{1}{40}\|u-w\|. \tag{27}$$

So,

$$\Lambda_2 = \frac{1-\mu}{2-\mu} = \frac{1-1/5}{2-1/5} = \frac{4}{9} \tag{28}$$

Hence:

$$L = \frac{1}{40} < \frac{4}{9} \tag{29}$$

Now, using Theorem 3.3, the equation 22 on the interval $[0,1]$ *has a unique solution.*

3.2. Approximation of the solution

In this section, an approximate solution for the initial value problem 1 is proved based on α-contractive maps derived from Caputo-Fabrizio of order $0 < \alpha < 1$. Indeed, in [8], the authors have applied similar derivative to obtain some existence and uniqueness results using initial value equal to zero. Now, we extend this results to apply the problem under nonlocal initial value.

Lemma 3.6. *Consider the problem 1, then we have:*

$$f(t,x(t)) = \frac{1}{1-\alpha}x(t) - \frac{1}{1-\alpha}\exp(-\frac{\alpha}{1-\alpha}t)x(0) - \frac{\alpha}{(1-\alpha)^2}\int_0^t \exp(-\frac{\alpha}{1-\alpha}(t-s))x(s)ds \tag{30}$$

Proof. Using definition of Caputo-Fabrizio derivative, we have:

$$\begin{aligned} f(t,x(t)) &= {}^{CF}D^{\alpha}x(t) = \tfrac{1}{1-\alpha}\int_0^t \exp(-\tfrac{\alpha}{1-\alpha}(t-s))x'(s)ds \\ &= \tfrac{1}{1-\alpha}\exp(-\tfrac{\alpha}{1-\alpha}(t-s))x(s)|_0^t - \tfrac{1}{1-\alpha}\int_0^t \tfrac{\alpha}{1-\alpha}\exp(-\tfrac{\alpha}{1-\alpha}(t-s))x(s)ds \\ &= \tfrac{1}{1-\alpha}x(t) - \tfrac{1}{1-\alpha}\exp(-\tfrac{\alpha}{1-\alpha}t)x(0) - \tfrac{\alpha}{(1-\alpha)^2}\int_0^t \exp(-\tfrac{\alpha}{1-\alpha}(t-s))x(s)ds, \end{aligned} \tag{31}$$

which completes the proof.

Suppose that continuous maps γ and $\lambda : [0,1]\times[0,1] \to [0,\infty)$ are defined in such a way that:

$$\sup_{t\in I} |\int_0^t \lambda(t,s)ds| < \infty, \tag{32}$$

and

$$\sup_{t\in I} |\int_0^t \gamma(t,s)ds| < \infty. \tag{33}$$

If the mapping ϕ and φ are defined as follows:

$$(\varphi x)(t) = \int_0^t \lambda(t,s)x(s)ds, \tag{34}$$

and

$$(\phi x)(t) = \int_0^t \gamma(t,s)x(s)ds. \tag{35}$$

Also consider the following definitions, $\lambda_0 = \sup_{t\in I} |\int_0^t \lambda(t,s)ds|, \gamma_0 = \sup_{t\in I} |\int_0^t \gamma(t,s)ds|$, where

$$\eta(t) \in L^{\infty}(I) \quad , \quad \eta^* = \sup_{t\in I} |\eta(t)|. \tag{36}$$

Theorem 3.7. *Let $\eta(t) \in L^{\infty}(I)$ and $f \in I \times R^3 \to R$ be a continuous function such that for each $t \in I$ and$x, y, x', y' \in R$,*

$$|f(t, x, y, w) - f(t, x', y', w')| \leq \eta(t)(|x - x'| + |y - y'| + |w - w'|). \tag{37}$$

Then the problem ${}^{CF}D^{\alpha}x(t) = f(t, x(t), (\phi x)(t))$ with a boundary condition $x(0) = 0$ has an approximate solution whenever

$$\Delta_1 = \eta^*(1 + \gamma_0 + \lambda_0) < 1. \tag{38}$$

Proof. See [8].

Theorem 3.8. *Consider the following initial value problem:*

$$\begin{array}{l} {}^{CF}D^{\alpha}x(t) = f(t, x(t)) \\ x(0) = \int_0^1 g(s)x(s)ds \end{array} . \tag{39}$$

Where ${}^{CF}D^{\alpha}$ is the Caputo-Fabrizio derivative, and $0 < \alpha < 1$, $g \in L^1([0, 1], R_+)$, $g(t) \in [0, 1)$. If $(1 + m\lambda_0 + k\gamma_0) < 1 - \alpha$, then the problem has an approximate solution.

Proof. Set $d(x, y) = \|x - y\|$, and $\|x\| = \sup_{t\in[0,1]} |x|$. Using Lemma 3.6 we have:

$$\begin{array}{l} |f(t, x(t)) - f(t, y(t))| \\ \leq \frac{1}{1-\alpha}|x(t) - y(t)| + \frac{1}{1-\alpha} \exp(-\frac{\alpha}{1-\alpha}t)|x(0) - y(0)| + \frac{\alpha}{(1-\alpha)^2} \int_0^t \exp(-\frac{\alpha}{1-\alpha}(t - s))|x(s) - y(s)|ds. \end{array} \tag{40}$$

Set:

$$(\phi x)(t) = \int_0^t \exp(-\frac{\alpha}{1 - \alpha}(t - s))x(s)ds, \tag{41}$$

using the above result, we obtain:

$$\begin{aligned} &|f(t, x(t), (\phi x)(t)) - f(t, y(t), (\phi y)(t))| \\ &\qquad \leq \frac{1}{1-\alpha}|x(t) - y(t)| + \frac{1}{1-\alpha} \exp(-\frac{\alpha}{1-\alpha}t)(\int_0^1 g(s)|x(s) - y(s)|ds) + \frac{\alpha}{(1-\alpha)^2}|(\phi x)(t) - (\phi y)(t)| \\ &\qquad \leq \frac{1}{1-\alpha}(|x(t) - y(t)| + m\lambda_0 \sup |x(t) - y(t)| + k\gamma_0|x(t) - y(t)|) \\ &\qquad \leq \frac{1}{1-\alpha}(1 + m\lambda_0 + k\gamma_0)\|x - y\| \end{aligned} \tag{42}$$

Now, using the hypothesis and in accordance with Theorem 3.7, we conclude that the problem 1 has an approximate solution.

4. Conclusion

In this paper, some new results for the existence and uniqueness of the fractional differential equations involving Caputo-Fabrizio derivative are proposed with nonlocal condition. In fact, two well-known theorems, Arzela-Ascoli and Schauder fixed point results have been applied to prove some new existence and uniqueness theorems for the problem 39 Also, one illustrative examples is solved in detail. After that, using the non-singular type of fractional differentiability, called Caputo-Fabrizio, some new approximation of the solution is derived using α-contractive maps under nonlocal initial value problems. For future work, we will apply similar procedure to consider higher fractional order between 1< α < 2. Also, such problem under deterministic and uncertain frameworks can be considered for further investigations [28], [4], [6], [38] and [19].

References

[1] Abbasbandy, S. An approximation solution of a nonlinear equation with riemann–liouville's fractional derivatives by he's variational iteration method. *Journal of Computational and Applied Mathematics 207*, 1 (2007), 53–58.

[2] Aghajani, A., Jalilian, Y., and Trujillo, J. J. On the existence of solutions of fractional integro-differential equations. *Fractional Calculus and Applied Analysis 15*, 1 (2012), 44–69.

[3] Ahmadian, A., Chan, C. S., Salahshour, S., and Vaitheeswaran, V. FTFBE: a numerical approximation for fuzzy time-fractional bloch equation. In *2014 IEEE international conference on fuzzy systems (FUZZ-IEEE)* (2014), IEEE, pp. 418–423.

[4] Ahmadian, A., Salahshour, S., and Chan, C. S. Fractional differential systems: a fuzzy solution based on operational matrix of shifted chebyshev polynomials and its applications. *IEEE Transactions on Fuzzy Systems 25*, 1 (2016), 218–236.

[5] Ali, M. R., Hadhoud, A. R., and Srivastava, H. Solution of fractional volterra–fredholm integro-differential equations under mixed boundary conditions by using the hobw method. *Advances in Difference Equations 2019*, 1 (2019), 115.

[6] Allahviranloo, T., and Ezadi, S. Z-advanced numbers processes. *Information Sciences 480* (2019), 130–143.

[7] Alqahtani, R. T. Fixed-point theorem for caputo-fabrizio fractional nagumo equation with nonlinear diffusion and convection. *J. Nonlinear Sci. Appl 9*, 5 (2016), 1991–1999.

[8] Baleanu, D., Mousalou, A., and Rezapour, S. A new method for investigating approximate solutions of some fractional integro-differential equations involving the caputo-fabrizio derivative. *Advances in Difference Equations 2017*, 1 (2017), 1–12.

[9] Băleanu, D., and Mustafa, O. G. On the global existence of solutions to a class of fractional differential equations. *Computers & Mathematics with Applications 59*, 5 (2010), 1835–1841.

[10] Baleanu, D., Shiri, B., Srivastava, H., and Al Qurashi, M. A chebyshev spectral method based on operational matrix for fractional differential equations involving non-singular mittag-leffler kernel. *Advances in Difference Equations 2018*, 1 (2018), 353.

[11] Benchohra, M., Henderson, J., Ntouyas, S., and Ouahab, A. Existence results for fractional order functional differential equations with infinite delay. *Journal of Mathematical Analysis and Applications 338*, 2 (2008), 1340–1350.

[12] Burton, T., and Zhang, B. Fractional equations and generalizations of schaefer's and krasnoselskii's fixed point theorems. *Nonlinear Analysis: Theory, Methods & Applications 75*, 18 (2012), 6485–6495.

[13] Caputo, M. Linear models of dissipation whose q is almost frequency independent—ii. *Geophysical Journal International 13*, 5 (1967), 529–539.

[14] Caputo, M., and Fabrizio, M. A new definition of fractional derivative without singular kernel. *Progr. Fract. Differ. Appl 1*, 2 (2015), 1–13.

[15] Chang, Y.-K., and Nieto, J. J. Some new existence results for fractional differential inclusions with boundary conditions. *Mathematical and Computer Modelling 49*, 3-4 (2009), 605–609.

[16] Daftardar-Gejji, V., and Babakhani, A. Analysis of a system of fractional differential equations. *Journal of Mathematical Analysis and Applications 293*, 2 (2004), 511–522.

[17] Debnath, L. Recent applications of fractional calculus to science and engineering. *International Journal of Mathematics and Mathematical Sciences 2003*, 54 (2003), 3413–3442.

[18] Diethelm, K., and Ford, N. J. Analysis of fractional differential equations. *Journal of Mathematical Analysis and Applications 265*, 2 (2002), 229–248.

[19] Dizicheh, A. K., Salahshour, S., Ahmadian, A., and Baleanu, D. A novel algorithm based on the legendre wavelets spectral technique for solving the lane–emden equations. *Applied Numerical Mathematics* (2020).

[20] Du, R., Cao, W., and Sun, Z. A compact difference scheme for the fractional diffusion-wave equation. *Applied Mathematical Modelling 34*, 10 (2010), 2998–3007.

[21] Gao, G.-h., and Sun, Z.-z. A compact finite difference scheme for the fractional sub-diffusion equations. *Journal of Computational Physics 230*, 3 (2011), 586–595.

[22] Ghaemi, F., Yunus, R., Ahmadian, A., Salahshour, S., Suleiman, M., and Saleh, S. F. Application of fuzzy fractional kinetic equations to modelling of the acid hydrolysis reaction. In *Abstract and Applied Analysis* (2013), vol. 2013, Hindawi.

[23] Goufo, E. F. D. Application of the caputo-fabrizio fractional derivative without singular kernel to korteweg-de vries-burgers equation. *Mathematical Modelling and Analysis 21*, 2 (2016), 188–198.

[24] Hayat, T., Nadeem, S., and Asghar, S. Periodic unidirectional flows of a viscoelastic fluid with the fractional maxwell model. *Applied Mathematics and Computation 151*, 1 (2004), 153–161.

[25] Hosseini, K., Salahshour, S., and Mirzazadeh, M. Bright and dark solitons of a weakly nonlocal schrödinger equation involving the parabolic law nonlinearity. *Optik 227* (2021), 166042.

[26] Kilbas, A. A., and Marzan, S. A. Nonlinear differential equations with the caputo fractional derivative in the space of continuously differentiable functions. *Differential Equations 41*, 1 (2005), 84–89.

[27] Kilbas, A. A., Srivastava, H. M., and Trujillo, J. J. *Theory and applications of fractional differential equations*, vol. 204. elsevier, 2006.

[28] Korpinar, Z. On numerical solutions for the caputo-fabrizio fractional heat-like equation. *Thermal Science 22*, Suppl. 1 (2018), 87–95.

[29] Kosmatov, N. Integral equations and initial value problems for nonlinear differential equations of fractional order. *Nonlinear Analysis: Theory, Methods & Applications 70*, 7 (2009), 2521–2529.

[30] Lakshmikantham, V., and Leela, S. Nagumo-type uniqueness result for fractional differential equations. *Nonlinear Analysis 71*, 7-8 (2009), 2886–2889.

[31] Losada, J., and Nieto, J. J. Properties of a new fractional derivative without singular kernel. *Progr. Fract. Differ. Appl 1*, 2 (2015), 87–92.

[32] Lv, Z.-W., Liang, J., and Xiao, T.-J. Solutions to fractional differential equations with nonlocal initial condition in banach spaces. *Advances in Difference Equations 2010* (2010), 1–10.

[33] Mainardi, F. Fractional calculus, fractals and fractional calculus in continuum mechanics, 1997.

[34] Mei, Z.-D., Peng, J.-G., and Gao, J.-H. Existence and uniqueness of solutions for nonlinear general fractional differential equations in banach spaces. *Indagationes Mathematicae 26*, 4 (2015), 669–678.

[35] Momani, S., and Shawagfeh, N. Decomposition method for solving fractional riccati differential equations. *Applied Mathematics and Computation 182*, 2 (2006), 1083–1092.

[36] Nagumo, J., Arimoto, S., and Yoshizawa, S. An active pulse transmission line simulating nerve axon. *Proceedings of the IRE 50*, 10 (1962), 2061–2070.

[37] Nucci, M., and Clarkson, P. The nonclassical method is more general than the direct method for symmetry reductions. an example of the fitzhugh-nagumo equation. *Physics Letters A 164*, 1 (1992), 49–56.

[38] Rahaman, M., Mondal, S. P., Shaikh, A. A., Ahmadian, A., Senu, N., and Salahshour, S. Arbitrary-order economic production quantity model with and without deterioration: generalized point of view. *Advances in Difference Equations 2020*, 1 (2020), 16.

[39] Salahshour, S., Ahmadian, A., Abbasbandy, S., and Baleanu, D. M-fractional derivative under interval uncertainty: Theory, properties and applications. *Chaos, Solitons & Fractals 117* (2018), 84–93.

[40] Salahshour, S., Ahmadian, A., and Chan, C. S. Successive approximation method for caputo q-fractional IVPs. *Communications in Nonlinear Science and Numerical Simulation 24*, 1-3 (2015), 153–158.

[41] Salahshour, S., Ahmadian, A., Ismail, F., Baleanu, D., and Senu, N. A new fractional derivative for differential equation of fractional order under interval uncertainty. *Advances in Mechanical Engineering 7*, 12 (2015), 1687814015619138.

[42] Senol, M., Atpinar, S., Zararsiz, Z., Salahshour, S., and Ahmadian, A. Approximate solution of time-fractional fuzzy partial differential equations. *Computational and Applied Mathematics 38*, 1 (2019), 1–18.

[43] Singh, H., and Srivastava, H. Jacobi collocation method for the approximate solution of some fractional-order riccati differential equations with variable coefficients. *Physica A: Statistical Mechanics and its Applications 523* (2019), 1130–1149.

[44] Singh, H., Srivastava, H., and Kumar, D. A reliable numerical algorithm for the fractional vibration equation. *Chaos, Solitons & Fractals 103* (2017), 131–138.

[45] Singh, J., Ahmadian, A., Rathore, S., Kumar, D., Baleanu, D., Salimi, M., and Salahshour, S. An efficient computational approach for local fractional poisson equation in fractal media. *Numerical Methods for Partial Differential Equations 37*, 2 (2021), 1439–1448.

[46] Srivastava, H., and Günerhan, H. Analytical and approximate solutions of fractional-order susceptible-infected-recovered epidemic model of childhood disease. *Mathematical Methods in the Applied Sciences 42*, 3 (2019), 935–941.

[47] Srivastava, H., and Saad, K. Some new models of the time-fractional gas dynamics equation. *Advanced Mathematical Models and Applications 3*, 1 (2018), 5–17.

[48] Srivastava, H., and Saad, K. M. New approximate solution of the time-fractional nagumo equation involving fractional integrals without singular kernel. *Appl. Math. Inf. Sci. 14*, 1 (2020), 1–8.

[49] Srivastava, H., Saad, K. M., and Al-Sharif, E. H. A new analysis of the time-fractional and space-time fractional-order nagumo equation. *Journal of Informatics and Mathematical Sciences 10*, 4 (2018), 545–561.

[50] Srivastava, H., Shah, F., and Abass, R. An application of the gegenbauer wavelet method for the numerical solution of the fractional bagley-torvik equation. *Russian Journal of Mathematical Physics 26*, 1 (2019), 77–93.

[51] Srivastava, H. M., Abbas, S., Tyagi, S., and Lassoued, D. Global exponential stability of fractional-order impulsive neural network with time-varying and distributed delay. *Mathematical Methods in the Applied Sciences 41*, 5 (2018), 2095–2104.

[52] Srivastava, H. M., El-Sayed, A., and Gaafar, F. M. A class of nonlinear boundary value problems for an arbitrary fractional-order differential equation with the riemann-stieltjes functional integral and infinite-point boundary conditions. *Symmetry 10*, 10 (2018), 508.

[53] Srivastava, H. M., Khan, N., Darus, M., Rahim, M. T., Ahmad, Q. Z., and Zeb, Y. Properties of spiral-like close-to-convex functions associated with conic domains. *Mathematics 7*, 8 (2019), 706.

[54] Srivastava, H. M., Shanker Dubey, R., and Jain, M. A study of the fractional-order mathematical model of diabetes and its resulting complications. *Mathematical Methods in the Applied Sciences 42*, 13 (2019), 4570–4583.

[55] Yang, X.-J., Gao, F., and Srivastava, H. A new computational approach for solving nonlinear local fractional pdes. *Journal of Computational and Applied Mathematics 339* (2018), 285–296.

[56] Yang, X.-J., Gao, F., and Srivastava, H. M. Exact travelling wave solutions for the local fractional two-dimensional burgers-type equations. *Computers & Mathematics with Applications 73*, 2 (2017), 203–210.

[57] Yang, X.-J., Hristov, J., Srivastava, H. M., and Ahmad, B. Modelling fractal waves on shallow water surfaces via local fractional korteweg-de vries equation. In *Abstract and Applied Analysis* (2014), vol. 2014, Hindawi.

[58] Zhu, B., Liu, L., and Wu, Y. Existence and uniqueness of global mild solutions for a class of nonlinear fractional reaction–diffusion equations with delay. *Computers & Mathematics with Applications 78*, 6 (2019), 1811–1818.