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Generalized Drazin-g-Meromorphic Invertible Operators and Generalized Kato-g-Meromorphic Decomposition

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Abstract. In this paper we generalize the concept of Koliha-Drazin invertible operators by introducing generalized Drazin-g-meromorphic invertible operators. A bounded linear operator T on a Banach space X is said to be g-meromorphic if every non-zero point of its spectrum is an isolated point. For T we say that it is generalized Drazin-g-meromorphic invertible if there exists a bounded linear operator S acting on X such that $TS = ST$, $STS = S$, $TST - T$ is g-meromorphic, while T admits a generalized Kato-g-meromorphic decomposition if there exists a pair of T-invariant closed subspaces (M, N) such that $X = M \oplus N$, the reduction T_M is Kato and T_N is g-meromorphic.

1. Introduction

Let X be an infinite dimensional Banach space and let $L(X)$ be the Banach algebra of all bounded linear operators acting on X. The group of all invertible operators is denoted by $L(X)^{-1}$, and the set of all bounded below (resp., surjective) operators is denoted by $\mathcal{J}(X)$ (resp., $\mathcal{S}(X)$). Given $T \in L(X)$, we denote by $\sigma(T)$, $\sigma_{ap}(T)$ and $\sigma_{su}(T)$ its spectrum, approximate point spectrum and surjective spectrum, respectively. The space of bounded linear functionals on X is denoted by X'. For $T \in L(X)$ we shall write $\alpha(T)$ for the dimension of the kernel $N(T)$ and $\beta(T)$ for the codimension of the range $R(T)$. We call $T \in L(X)$ an *upper semi-Fredholm* operator if $\alpha(T) < \infty$ and $R(T)$ is closed, and we say that T is a *lower semi-Fredholm* operator if $\beta(T) < \infty$. We use $\Phi_+(X)$ (resp. $\Phi_-(X)$) to denote the set of upper (resp. lower) semi-Fredholm operators. The set of semi-Fredholm operators is defined by $\Phi_\pm(X) = \Phi_+(X) \cup \Phi_-(X)$, while the set of Fredholm operators is defined by $\Phi(X) = \Phi_+(X) \cap \Phi_-(X)$. If $T \in \Phi_\pm(X)$, the index is defined by $i(T) = \alpha(T) - \beta(T)$. For $T \in L(X)$ the semi-Fredholm spectrum of T and the Fredholm spectrum of T are defined, respectively, by:

$$\begin{aligned}\sigma_{\Phi_\pm}(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \notin \Phi_\pm(X)\},\\ \sigma_{\Phi}(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \notin \Phi(X)\}.\end{aligned}$$

The sets of upper semi-Weyl, lower semi-Weyl and Weyl operators are defined by $\mathcal{W}_+(X) = \{T \in \Phi_+(X) : \text{ind}(T) \leq 0\}$, $\mathcal{W}_-(X) = \{T \in \Phi_-(X) : \text{ind}(T) \geq 0\}$ and $\mathcal{W}(X) = \{T \in \Phi(X) : \text{ind}(T) = 0\}$, respectively. An

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operator $T \in L(X)$ is said to be a *Riesz operator*, if $T - \lambda \in \Phi(X)$ for every non-zero $\lambda \in \mathbb{C}$, and this is equivalent to the fact that its non-zero spectral points are poles of its resolvent of the finite algebraic multiplicity. An operator $T \in L(X)$ is *meromorphic* if its non-zero spectral points are poles of its resolvent, and in that case we shall write $T \in (\mathcal{M})$. Therefore, every Riesz operator is meromorphic.

If $K \subset \mathbb{C}$, then ∂K is the boundary of K, acc K is the set of accumulation points of K, iso $K = K \setminus \operatorname{acc} K$ and $\operatorname{int} K$ is the set of interior points of K. For $\lambda_0 \in \mathbb{C}$, the open disc, centered at λ_0 with radius ϵ in $\mathbb{C}$, is denoted by $D(\lambda_0, \epsilon)$.

If $K \subset \mathbb{C}$ is a compact set, we write $f \in \text{Holo}(K)$ if f is a holomorphic function in a neighborhood of K, and $\text{Holo}_1(K) \subseteq \text{Holo}(K)$ for those holomorphic functions $g : U \to \mathbb{C}$ which are non constant on each connected component of open $U \supseteq K$.

For $T \in L(X)$, a subset σ of $\sigma(T)$ is called a *spectral set* of T if it is both open and closed in the relative topology of $\sigma(T)$.

If M is a subspace of X such that $T(M) \subset M$, $T \in L(X)$, it is said that M is *T-invariant*. We define $T_M : M \to M$ as $T_M x = Tx$, $x \in M$. If M and N are two closed T-invariant subspaces of X such that $X = M \oplus N$, we say that T is *completely reduced* by the pair (M, N) and it is denoted by $(M, N) \in Red(T)$. In this case we write $T = T_M \oplus T_N$ and say that T is the *direct sum* of T_M and T_N.

For $T \in L(X)$ we say that it is *Kato* if $R(T)$ is closed and $N(T) \subset R(T^n)$ for every $n \in \mathbb{N}$. It is said that $T \in L(X)$ admits a Kato decomposition or T is of Kato type if there exist two closed T-invariant subspaces M and N such that $X = M \oplus N$, T_M is Kato and T_N is nilpotent. If we require that T_N is quasinilpotent instead of nilpotent in the definition of the Kato decomposition, then it leads us to the generalized Kato decomposition, abbreviated as GKD. An operator $T \in L(X)$ is said to admit a generalized Kato-Riesz decomposition (a generalized Kato-meromorphic decomposition) if there exists a pair $(M, N) \in Red(T)$ such that T_M is Kato and T_N is Riesz (meromorphic), abbreviated as *GKRD* (*GK*($\mathcal{M}$)*D*) [15], [16].

For $T \in L(X)$, the Kato type spectrum, the generalized Kato spectrum, the generalized Kato-Riesz spectrum and the generalized Kato-meromorphic spectrum are defined, respectively, by:

$$\begin{aligned}
\sigma_{Kt}(T) &= \{\lambda \in \mathbb{C} : T - \lambda \text{ is not of Kato type}\},\\
\sigma_{gK}(T) &= \{\lambda \in \mathbb{C} : T - \lambda \text{ does not admit a generalized Kato decomposition}\},\\
\sigma_{gKR}(T) &= \{\lambda \in \mathbb{C} : T - \lambda \text{ does not admit a } GKRD\},\\
\sigma_{gK(\mathcal{M})}(T) &= \{\lambda \in \mathbb{C} : T - \lambda \text{ does not admit a } GK(\mathcal{M})D\}.
\end{aligned}$$

An operator $T \in L(X)$ is said to be Drazin invertible if there exists $S \in L(X)$ such that $TS = ST$, $STS = S$ and $TST - T$ is nilpotent. This concept has been generalized by Koliha [9]: an operator $T \in L(X)$ is generalized Drazin invertible (Koliha-Drazin invertible) if there is $S \in L(X)$ such that

$$TS = ST,\ STS = S,\ TST - T \text{ is quasinilpotent.} \tag{1.1}$$

Recall that T is generalized Drazin invertible if and only if $0 \notin \operatorname{acc} \sigma(T)$, and this is also equivalent to the fact that $T = T_1 \oplus T_2$ where T_1 is invertible and T_2 is quasinilpotent. In [15] this concept is further generalized by replacing the third condition in the previous definitions by the condition that $TST - T$ is Riesz, and so it is introduced the concept of generalized Drazin-Riesz invertible operators. Further generalization is done in [16] by replacing the third condition in (1.1) by the condition that $TST - T$ is meromorphic, and so it is introduced the concept of generalized Drazin-meromorphic invertible operators. Recall that T is generalized Drazin-Riesz invertible (generalized Drazin-meromorphic invertible) if and only if $T = T_1 \oplus T_2$ where T_1 is invertible and T_2 is Riesz (meromorphic) [15], [16]. In [1] it is proved that T is generalized Drazin-Riesz invertible if and only if 0 is not an accumulation point of its Browder spectrum.

In this paper we further generalize this concept of Koliha-Drazin invertibles by replacing the third condition in (1.1) by the condition that $TST - T$ is g-meromorphic:

Definition 1.1. An operator $T \in L(X)$ is said to be *g-meromorphic* if every non-zero point of its spectrum is an isolated point, and in that case we shall write $T \in (g\mathcal{M})$.

Definition 1.2. An operator $T \in L(X)$ is *generalized Drazin-g-meromorphic invertible*, if there exists $S \in L(X)$ such that

$$TS = ST, \quad STS = S, \quad TST - T \text{ is } g-\text{meromorphic}.$$

The set of all generalized Drazin invertible (Koliha-Drazin invertible) operators of the algebra $L(X)$ is denoted by $L(X)^{gD}$, while the set of all generalized Drazin-g-meromorphic invertible operators of the algebra $L(X)$ is denoted by $L(X)^{gD(g\mathcal{M})}$.

Definition 1.3. An operator $T \in L(X)$ is said to admit a *generalized Kato-g-meromorphic decomposition*, abbreviated to $GK(g\mathcal{M})D$, if there exists a pair $(M, N) \in Red(T)$ such that T_M is Kato and T_N is g-meromorphic (i.e. $T_N \in (g\mathcal{M})$). In that case we shall say that T admits a $GK(g\mathcal{M})D(M, N)$.

We use the following notation:

$\mathbf{R_1}(X) = L(X)^{-1}$	$\mathbf{R_2}(X) = \mathcal{J}(X)$	$\mathbf{R_3}(X) = \mathcal{S}(X)$
$\mathbf{R_4}(X) = \Phi(X)$	$\mathbf{R_5}(X) = \Phi_+(X)$	$\mathbf{R_6}(X) = \Phi_-(X)$
$\mathbf{R_7}(X) = \mathcal{W}(X)$	$\mathbf{R_8}(X) = \mathcal{W}_+(X)$	$\mathbf{R_9}(X) = \mathcal{W}_-(X)$

Henceforth, in common with current practice ([12], [13]) we abbreviate $R_i(X)$ to R_i, the Banach space X being understood: for example, if $T \in L(X)$, $T \in R_i$ means T satisfies $R_i(X)$. If $T \in L(X)$ and $1 \le i \le 9$, let $\sigma_{\mathbf{R}_i}(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin \mathbf{R_i}\}$. Recall that $\sigma_{\mathbf{R}_i}(T)$ is closed, $1 \le i \le 9$.

For $T \in L(X)$ we write $T \in GD\mathbf{R_i}$ if there exist $(M, N) \in Red(T)$ such that $T_M \in \mathbf{R_i}$ and T_N is quasinilpotent, $1 \le i \le 9$. For $T \in L(X)$ the generalized Drazin spectrum is defined by:

$$\sigma_{gD}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not generalized Drazin invertible}\}.$$

If $T \in L(X)$ and $2 \le i \le 9$, let

$$\sigma_{gD\mathbf{R}_i}(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin GD\mathbf{R_i}\}.$$

Definition 1.4. An operator $T \in L(X)$ satisfies $T \in GD(g\mathcal{M})\mathbf{R_i}$ if there exist $(M, N) \in Red(T)$ such that $T_M \in \mathbf{R_i}$ and $T_N \in (g\mathcal{M})$, $1 \le i \le 9$.

This paper is divided into four sections. In the second section we give some preliminary results. In the third section we give some properties of g-meromorphic operators and show that T is generalized Drazin-g-meromorphic invertible if and only if 0 is not an accumulation point of its generalized Drazin spectrum and this is also equivalent to the fact that T is a direct sum of a g-meromorphic operator and an invertible operator, as well as to the fact that T admits a generalized Kato-meromorphic decomposition and 0 is not an interior point of $\sigma(T)$. Also we prove that T is generalized Drazin-g-meromorphic invertible if and only if there exists a projection $P \in L(X)$ such that P commutes with T, TP is g-meromorphic and $T + P$ is invertible. We characterize bounded linear operators which can be expressed as a direct sum of a g-meromorphic operator and a bounded below (resp. surjective, upper (lower) semi-Fredholm, Fredholm, upper (lower) semi-Weyl, Weyl) operator. In particular, we characterize the single-valued extension property at a point $\lambda_0 \in \mathbb{C}$ in the case that $\lambda_0 - T$ admits a generalized Kato-g-meromorphic decomposition, and in that way we extend [2, Theorem 2., Theorem 2.5], [8, Theoem 3.5, Theorem 3.9], [15, Corollary 2.1], [16, Corollary 1, Corollary 2]. In the forth section we investigate corresponding spectra. In particular we give some results regarding boundaries, connected hulls and isolated points of corresponding spectra, and improve [2, Theorem 2.10 and Corollary 2.11], [8, Theorem 3.12 and Corollary 3.13], [15, Theorems 3.14 and 3.15] and [16, Theorems 13, 14].

2. Preliminary results

The following preliminary assertions will be needed in the sequel.

Lemma 2.1. *([15, Lemma 2.1]) Let $T \in L(X)$ and $(M, N) \in Red(T)$. The following statements hold:*
(i) *$T \in \mathbf{R}_i$ if and only if $T_M \in \mathbf{R}_i$ and $T_N \in \mathbf{R}_i$, $1 \leq i \leq 6$, and in that case* $\operatorname{ind}(T) = \operatorname{ind}(T_M) + \operatorname{ind}(T_N)$;
(ii) *If $T_M \in \mathbf{R}_i$ and $T_N \in \mathbf{R}_i$, then $T \in \mathbf{R}_i$, $7 \leq i \leq 9$.*
(iii) *If $T \in \mathbf{R}_i$ and T_N is Weyl, then $T_M \in \mathbf{R}_i$, $7 \leq i \leq 9$.*

Lemma 2.2. *Let $E, F \subset \mathbb{C}$. Then:*
(i) *If $\partial F \subset E \subset F$, then* $\operatorname{iso} F \subset \operatorname{iso} E$.
(ii) *If $\partial F \subset E$ and F is closed, then $\partial F \cap \operatorname{iso} E \subset \operatorname{iso} F$.*

Proof. See [5, Lemma 2.2]. □

Lemma 2.3. *([16, Lemma 4]) Let $X = X_1 \oplus X_2 \cdots \oplus X_n$ where $X_1, X_2, \ldots, X_n$ are closed subspaces of X and let M_i be a closed subset of X_i, $i = 1, \ldots, n$. Then the set $M_1 \oplus M_2 \oplus \cdots \oplus M_n$ is closed.*

Lemma 2.4. *Let $T, U \in L(X)$ and let U be invertible such that $TU = UT$. Then T is generalized Drazin invertible if and only if TU is generalized Drazin invertible.*

Proof. Since generalized Drazin invertibles form a regularity [10, Theorem 1.2], applying [13, Proposition 6.2 (iii)] we obtain the desired conclusion. □

Lemma 2.5. *Let $T \in L(X)$ and let $(M, N) \in Red(T)$. Then T is generalized Drazin invertible if and only if T_M and T_N are generalized Drazin invertible.*

Proof. For any $K_1, K_2 \subset \mathbb{C}$ it holds

$$\operatorname{acc}(K_1 \cup K_2) = \operatorname{acc} K_1 \cup \operatorname{acc} K_2$$

Really, from $K_i \subset K_1 \cup K_2$ it follows that $\operatorname{acc} K_i \subset \operatorname{acc}(K_1 \cup K_2)$, $i = 1, 2$. Hence $\operatorname{acc} K_1 \cup \operatorname{acc} K_2 \subset \operatorname{acc}(K_1 \cup K_2)$. Let $\lambda \notin \operatorname{acc} K_1 \cup \operatorname{acc} K_2$. Then there is an $\epsilon > 0$ such that $(D(\lambda, \epsilon) \setminus \{\lambda\}) \cap K_1 = (D(\lambda, \epsilon) \setminus \{\lambda\}) \cap K_2 = \emptyset$. Consequently, $(D(\lambda, \epsilon) \setminus \{\lambda\}) \cap (K_1 \cup K_2) = \emptyset$, and so $\lambda \notin \operatorname{acc}(K_1 \cup K_2)$. Applying [9, Theorem 4.2] we get

$$\sigma_{gD}(T) = \operatorname{acc} \sigma(T) = \operatorname{acc}(\sigma(T_M) \cup \sigma(T_N)) = \operatorname{acc} \sigma(T_M) \cup \operatorname{acc} \sigma(T_N) = \sigma_{gD}(T_M) \cup \sigma_{gD}(T_N).$$

It implies that T is generalized Drazin invertible if and only if $0 \notin \sigma_{gD}(T)$ if and only if $0 \notin \sigma_{gD}(T_M)$ and $0 \notin \sigma_{gD}(T_N)$, i.e. T_M and T_N are generalized Drazin invertible. □

3. $GD(g\mathcal{M})R_i$ operators and g-meromorphic operators

We start with some properties of g-meromorphic operators.
From Definition 1.1 it is clear that

$$T \text{ is } g-\text{meromorphic} \Longleftrightarrow \operatorname{acc} \sigma(T) \subset \{0\}. \tag{3.1}$$

Therefore, $T \in L(X)$ is g-meromorphic if and only if $\sigma(T)$ is finite or countable with $\sigma(T) = \{\lambda_n : n \in \mathbb{N}\} \cup \{0\}$, where (λ_n) is a sequence of isolated points of $\sigma(T)$ which converges to 0.

Theorem 3.1. *Let $T \in L(X)$. Then the following conditions are equivalent:*
(i) *T is g-meromorphic;*
(ii) *$\sigma_{gD}(T) \subset \{0\}$;*
(iii) *$\sigma_{gD\mathbf{R_i}}(T) \subset \{0\}$ for some $i \in \{1, \ldots, 9\}$;*
(iv) *$\sigma_{gD\mathbf{R_i}}(T) \subset \{0\}$ for every $i \in \{1, \ldots, 9\}$;*
(v) *$\sigma_{gK}(T) \subset \{0\}$.*

Proof. (i)$\Longleftrightarrow$(ii): Since $\sigma_{gD}(T) = \operatorname{acc}\sigma(T)$ [9, Theorem 4.2], from (3.1) it follows that T is g-meromorphic if and only if $\sigma_{gD}(T) \subset \{0\}$.

(ii)$\Longleftrightarrow$(iii)$\Longleftrightarrow$(iv): From [5, Proposition 5.6] it follows that $\sigma_{gD}(T)$ is finite if and only if $\sigma_{gD\mathbf{R_i}}(T)$ is finite, where $i \in \{2, \dots, 9\}$, and this is also equivalent to the fact that $\sigma_{gK}(T)$ is finite, whereby $\sigma_{gK}(T) = \sigma_{gD\mathbf{R_i}}(T)$ for every $i \in \{1, \dots, 9\}$. Hence, $\sigma_{gD\mathbf{R_i}}(T) \subset \{0\}$ for some $i \in \{1, \dots, 9\}$ if and only if $\sigma_{gK}(T) = \sigma_{gD\mathbf{R_i}}(T) \subset \{0\}$ for every $i \in \{1, \dots, 9\}$. □

Proposition 3.2. *Let $T \in L(X)$ and let $(M, N) \in Red(T)$. Then $T \in (g\mathcal{M})$ if and only if $T_M \in (g\mathcal{M})$ and $T_N \in (g\mathcal{M})$.*

Proof. From the equality

$$\sigma_{gD}(T) = \sigma_{gD}(T_M) \cup \sigma_{gD}(T_N)$$

it follows that $\sigma_{gD}(T) \subset \{0\}$ if and only if $\sigma_{gD}(T_M) \subset \{0\}$ and $\sigma_{gD}(T_N) \subset \{0\}$. Consequently, T is g-meromorphic if and only if T_M and T_N are g-meromorphic. □

Lemma 3.3. *Let $T \in L(X)$. Then T is g-meromorphic if and only if T' is g-meromorphic.*

Proof. It follows from the equality $\sigma(T) = \sigma(T')$. □

Proposition 3.4. *Let $T \in L(X)$ be g-meromorphic and let $f \in \mathrm{Holo}_1(\sigma(a))$ and $f(0) = 0$. Then $f(T)$ is g-meromorphic.*

Proof. By using [10, Theorem 1.4] and Theorem 3.1 we conclude that

$$\sigma_{gD}(f(T)) = f(\sigma_{gD}(T)) \subset f(0) = 0,$$

and so $f(T)$ is g-meromorphic. □

Proposition 3.5. *Let $T, S \in L(X)$. Then TS is g-meromorphic if and only if ST is g-meromorphic.*

Proof. From [14, Theorem 2.3] it follows that $\lambda - TS$ is generalized Drazin invertible if and only if $\lambda - ST$ is generalized Drazin invertible, for every $\lambda \neq 0$. Hence $\sigma_{gD}(TS) \cup \{0\} = \sigma_{gD}(ST) \cup \{0\}$, which implies that $\sigma_{gD}(TS) \subset \{0\}$ if and only if $\sigma_{gD}(ST) \subset \{0\}$. Thus TS is g-meromorphic if and only if ST is g-meromorphic. □

Remark 3.6. It is clear that every meromorphic operator is g-meromorphic, and so every Riesz operator is g-meromorphic. In contrast to Riesz operators, and as in the case of meromorphic operators, the sum of a pair of commuting g-meromorphic operators may not be a g-meromorphic operator. For example, if A is a Riesz operator with infinite spectrum, then A is g-meromorphic, the indentity operator I is g-meromorphic and commutes with A. As $\sigma_{gD}(A) = \{0\}$, we have that $\sigma_{gD}(I + A) = \{1\}$, and so $I + A$ is not g-meromorphic. Also, the product of two commuting operators, one of which is g-meromorphic, may not be meromorphic. For example, I and $I + A$ commute, I is g-meromorphic, but their product $I + A$ is not g-meromorphic.

Theorem 3.7. *The following conditions are equivalent for $T \in L(X)$ and $1 \leq i \leq 9$:*
(i) *There exists $(M, N) \in Red(T)$ such that $T_M \in \mathbf{R}_i$ and $T_N \in (g\mathcal{M})$, that is $T \in GD(g\mathcal{M})\mathbf{R_i}$;*
(ii) *T admits a $GK(g\mathcal{M})D$ and $0 \notin \operatorname{acc}\sigma_{gD\mathbf{R}_i}(T)$;*
(iii) *T admits a $GK(g\mathcal{M})D$ and $0 \notin \operatorname{int}\sigma_{gD\mathbf{R}_i}(T)$;*
(iv) *T admits a $GK(g\mathcal{M})D$ and $0 \notin \operatorname{int}\sigma_{\mathbf{R}_i}(T)$.*

Proof. (i)$\Longrightarrow$(ii): Let there exists $(M, N) \in Red(T)$ such that $T_M \in \mathbf{R}_i$ and $T_N \in (g\mathcal{M})$. For $1 \leq i \leq 3$, T_M is Kato, and so T admits a $GK(g\mathcal{M})D$. For $4 \leq i \leq 9$, from [13, Theorem 16.20] there exists $(M_1, M_2) \in Red(T_M)$ such that $\dim M_2 < \infty$, T_{M_1} is Kato and T_{M_2} is nilpotent. Then for $N_1 = M_2 \oplus N$ we have that N_1 is a closed subspace and $T_{N_1} = T_{M_2} \oplus T_N \in (g\mathcal{M})$ by Proposition 3.2. So T admits a $GK(g\mathcal{M})D$.

From $T_M \in \mathbf{R}_i$ it follows that there exists $\epsilon > 0$ such that for every $\lambda \in \mathbb{C}$ satisfying $|\lambda| < \epsilon$ we have $T_M - \lambda I_M \in \mathbf{R_i}$. Since $T_N \in (g\mathcal{M})$, according to Theorem 3.1 we have that $T_N - \lambda I_N$ is generalized Drazin invertible for every $\lambda \in \mathbb{C}$ such that $\lambda \neq 0$, and hence it is a direct sum of a quasinilpotent operator and an invertible operator. By using Lemma 2.1 (i), (ii) we conclude that $T-\lambda I \in GD\mathbf{R_i}$ for every $\lambda \in \mathbb{C}$ such that $0<|\lambda|<\epsilon$, and so $0 \notin \operatorname{acc} \sigma_{gD\mathbf{R}_i}(T)$.

(ii)$\Longrightarrow$(iii): It is obvious.

(iii)$\Longleftrightarrow$(iv): From [16, Corollary 4] it follows that $\operatorname{int}\sigma_{gD\mathbf{R_i}}(T) = \operatorname{int}\sigma_{\mathbf{R_i}}(T)$.

(iv)$\Longrightarrow$(i): Suppose that T admits a $GK(g\mathcal{M})D$ and $0 \notin \operatorname{int}\sigma_{\mathbf{R}_i}(T)$. Then there exists a decomposition $(M,N) \in Red(T)$ such that T_M is Kato and $T_N \in (g\mathcal{M})$. Fix $\epsilon>0$. From $0 \notin \operatorname{int}\sigma_{\mathbf{R}_i}(T)$ it follows that there exists $\lambda \in \mathbb{C}$ such that $0<|\lambda|<\epsilon$ and $T-\lambda I \in \mathbf{R}_i$. We prove that $T_M - \lambda I_M \in \mathbf{R_i}$. For $1 \leq i \leq 6$ it follows from Lemma 2.1 (i). Suppose that $7 \leq i \leq 9$. Since $T_N \in (g\mathcal{M})$ we have that $T_N - \lambda I_N$ is generalized Drazin invertible and therefore it is a direct sum of a quasinilpotent operator and an invertible operator, that is there exists $(N_1,N_2) \in Red(T_N)$ such that $T_{N_1}-\lambda I_{N_1}$ is invertible and $T_{N_2}-\lambda I_{N_2}$ is quasinilpotent. According to Lemma 2.1 (i) from $T-\lambda I \in \mathbf{R}_i$ it follows that $T_{N_2}-\lambda I_{N_2}$ is semi-Fredholm. Consequently, $\sigma_{\Phi_\pm}(T_{N_2}-\lambda I_{N_2})=\emptyset$ and hence $\sigma_{\Phi}(T_{N_2}-\lambda I_{N_2})=\emptyset$ according to [13, Theorem 21.11 (iii)]. It implies that $\dim N_2<\infty$ and hence $T_{N_2}-\lambda I_{N_2}$ is Weyl. Now Lemma 2.1 (ii) ensures that $T_N-\lambda I_N = (T_{N_1}-\lambda I_{N_1}) \oplus (T_{N_2}-\lambda I_{N_2})$ is Weyl. From Lemma 2.1 (iii) it follows that $T_M-\lambda I_M \in \mathbf{R_i}$. Consequently, $0 \notin \operatorname{int}\sigma_{\mathbf{R}_i}(T_M)$. As T_M is Kato, from [15, Proposition 2.1] it follows that $T_M \in \mathbf{R_i}$. □

Proposition 3.8. *Let $(M,N) \in Red(T)$. Then*

T admits a $GK(g\mathcal{M})D(M,N)$ if and only if T' admits a $GK(g\mathcal{M})D(N^\perp,M^\perp)$.

Proof. Let T admit a $GK(g\mathcal{M})D(M,N)$. Then T_M is Kato, $T_N \in (g\mathcal{M})$ and $(N^\perp,M^\perp) \in Red(T')$. Let P_N be the projection of X onto N along M. Then $(M,N) \in Red(TP_N)$, $TP_N = P_NT$, $TP_N = 0 \oplus T_N$, and Proposition 3.2 ensures that $TP_N \in (g\mathcal{M})$. According to Lemma 3.3 we have that $T'P'_N = P'_NT' \in (g\mathcal{M})$. As $(N^\perp,M^\perp) \in Red(T'P'_N)$ and since $R(P'_N) = N(P_N)^\perp = M^\perp$, according to Proposition 3.2 we conclude that $(T'P'_N)_{M^\perp} = T'_{M^\perp} \in (g\mathcal{M})$. From the proof of Theorem 1.43 in [3] it follows that $T'_{N^\perp}$ is Kato. Therefore, $(N^\perp,M^\perp)$ is a $GK(g\mathcal{M})D$ for T'.

Suppose that T' admits a $GK(\mathcal{M})D(N^\perp,M^\perp)$. Then $T'_{N^\perp}$ is Kato and $T'_{M^\perp} \in (g\mathcal{M})$. Since $(N^\perp,M^\perp) \in Red(T'P'_N)$, then $T'P'_N = (T'P'_N)_{N^\perp} \oplus (T'P'_N)_{M^\perp} = 0 \oplus T'_{M^\perp}$, and from Proposition 3.2 it follows that $T'P'_N \in (g\mathcal{M})$. According to Lemma 3.3 we have that $TP_N \in (g\mathcal{M})$. Since $TP_N = 0 \oplus T_N$, Proposition 3.2 ensures that $T_N \in (g\mathcal{M})$. From the proof of [16, Theorem 4] it follows that T_M is Kato. Consequently, T admits a $GK(g\mathcal{M})D(M,N)$. □

Definition 3.9. An operator $T \in B(X)$ is *g-meromorphic quasi-polar* if there exists a bounded projection Q satisfying

$$TQ = QT,\ T(I-Q) \in (g\mathcal{M}),\ Q \in (L(X)T) \cap (TL(X)). \tag{3.2}$$

Theorem 3.10. *The following conditions are mutually equivalent for operators $T \in L(X)$:*

(i) *There exists $(M,N) \in Red(T)$ such that T_M is invertible and $T_N \in (g\mathcal{M})$.*

(ii) *T admits a $GK(g\mathcal{M})D$ and $0 \notin \operatorname{int}\sigma(T)$.*

(iii) *T admits a $GK(g\mathcal{M})D$ and, T and T' have SVEP at 0.*

(iv) *T is generalized Drazin-g-meromorphic invertible.*

(v) *T is g-meromorphic quasi-polar.*

(vi) *There exists a projection $P \in L(X)$ such that P commutes with T, $TP \in (g\mathcal{M})$ and $T+P$ is generalized Drazin invertible.*

(vii) *There exists a projection $P \in L(X)$ which commutes with T and such that $TP \in (g\mathcal{M})$ and $T(I-P)+P$ is generalized Drazin invertible.*

(viii) *There exists $(M,N) \in Red(T)$ such that T_M is generalized Drazin invertible and $T_N \in (g\mathcal{M})$.*

(ix) *$0 \notin \operatorname{acc} \sigma_{gD}(T)$.*

(x) *There exists a projection $P \in L(X)$ such that P commutes with T, $TP \in (g\mathcal{M})$ and $T+P$ is invertible.*

(xi) *There exists a projection $P \in L(X)$ which commutes with T and such that $TP \in (g\mathcal{M})$ and $T(I-P)+P$ is invertible.*

Proof. The equivalence (i)$\Longleftrightarrow$(ii) is already proved in Theorem 3.7.

(ii)$\Longrightarrow$(iii): Let T admits a $GK(g\mathcal{M})D$ and $0 \notin \operatorname{int}\sigma(T)$. Then $0 \notin \sigma(T)$ or $0 \in \partial\sigma(T)$. In both cases T and T' have SVEP at 0.

(iii)$\Longrightarrow$(iv): Suppose that $(M, N) \in Red(T)$, $T_N \in (g\mathcal{M})$ and T_M is Kato. From Proposition 3.8 it follows that $T'_{N^\perp}$ is Kato. Since T and T' have SVEP at 0, it follows that T_M and $T'_{N^\perp}$ also have SVEP at 0. According to [3, Theorem 2.9] we conclude that T_M and $T'_{N^\perp}$ are injective. As in the proof of [3, Lemma 3.13] it can be proved that T_M is surjective, and so T_M is invertible. Let $S = T_M^{-1} \oplus 0$. Then we have

$$ST = TS,\ STS = S,\ TST - T = 0 \oplus (-T_N)$$

and according to Proposition 3.2 we conclude that $TST - T \in (g\mathcal{M})$.

(iv)$\Longrightarrow$(v): Suppose that T is generalized Drazin-g-meromorphic invertible and let S be its generalized Drazin-g-meromorphic inverse. Let $Q = TS = ST$. Then Q is a projector and

$$QT = TQ,\ Q \in TL(X) \cap L(X)T \text{ and } T(I - Q) \in (g\mathcal{M}), \tag{3.3}$$

and so T is g-meromorphic quasi-polar.

(v)$\Longrightarrow$(vi): Suppose that there exists a projector $Q \in L(X)$ such that (3.3) holds. Set $P = I - Q$. Then $TP \in (g\mathcal{M})$ and for $N = P(X)$ and $M = (I - P)(X)$ we have

$$PT = TP,\ T_N \in (g\mathcal{M}) \text{ and } I - P = UT = TV$$

for some $U, V \in L(X)$. Let $U, V \in L(M \oplus N)$ have the (2×2 matrix) representations $U = [U_{ij}]_{i,j=1}^2$ and $V = [V_{ij}]_{i,j=1}^2$. Then

$$\begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}\begin{bmatrix} T_M & 0 \\ 0 & T_N \end{bmatrix} = \begin{bmatrix} T_M & 0 \\ 0 & T_N \end{bmatrix}\begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} = \begin{bmatrix} I_M & 0 \\ 0 & 0 \end{bmatrix} : (M \oplus N) \to (M \oplus N)$$

and it implies that T_M is invertible, $U_{21} = 0 = V_{12}$, $U_{12}T_N = U_{22}T_N = 0 = T_N V_{21} = T_N V_{22}$, and hence $UTV + P = T_M^{-1} \oplus I_N$ is invertible with $(UTV + P)^{-1} = T_M \oplus I_N = T(I - P) + P$. As $TP \in (g\mathcal{M})$, we have that $I + TP$ is generalized Drazin invertible, and hence according to Lemma 2.4 it follows that

$$T + P = (I + TP)(UTV + P)^{-1} = (UTV + P)^{-1}(I + TP)$$

is generalized Drazin invertible.

(vi)$\Longrightarrow$(vii): Suppose that there exists a projection $P \in B(X)$ such that P commutes with T, $TP \in (g\mathcal{M})$ and $T + P$ is generalized Drazin invertible. Then for $M = (I - P)X$ and $N = PX$ we have that $(M, N) \in Red(T)$, $T + P = T_M \oplus (T_N + I_N)$. According to Lemma 2.5 it follows that T_M is generalized Drazin invertible. Since $T(I - P) + P = T_M \oplus I_N$, again from Lemma 2.5 it follows that $T(I - P) + P$ is generalized Drazin invertible.

(vii)$\Longrightarrow$(viii): Suppose that (vii) holds. Set $P(X) = N$ and $(I - P)X = M$. Then $(M, N) \in Red(T)$ and $T_N \in (g\mathcal{M})$. Since $T(I - P) + P = T_M \oplus I_N$ is generalized Drazin invertible, from Lemma 2.5 it follows that T_M is generalized Drazin invertible.

(viii)$\Longrightarrow$(ix): Suppose that there exists $(M, N) \in Red(T)$ such that T_M is generalized Drazin invertible and $T_N \in (g\mathcal{M})$. Then there exists a decomposition $M = M_1 \oplus M_2$ of M such that T_{M_1} is invertible and T_{M_2} is quasi-nilpotent [9]. Set $M_2 \oplus N = N_1$ and define T_{N_1} by $T_{N_1} = T_{M_2} \oplus T_N$. Then N_1 is closed by Lemma 2.3, $(M_1, N_1) \in Red(T)$ and $T_{N_1} \in (g\mathcal{M})$ according to Proposition 3.2. Now from Theorem 3.7 it follows that $0 \notin \operatorname{acc}\, \sigma_{gD}(T)$.

(ix)$\Longrightarrow$(iv) Suppose that $0 \notin \operatorname{acc}\, \sigma_{gD}(T)$. There are two cases:

1. If $0 \notin \operatorname{acc}\, \sigma(T)$, then from [9, Theorem 4.2] it follows that there exists $S \in L(X)$ such that $TS = ST$, $STS = S$ and $TST - T$ is quasinilpotent and hence $TST - T$ is g-meromorphic. Consequently, T is generalized Drazin-g-meromorphic invertible.

2. If $0 \in \operatorname{acc}\sigma(T)$, then $0 \in \operatorname{acc}\sigma(T) \setminus \operatorname{acc}\sigma_{gD}(T) = \sigma_{gD}(T) \setminus \operatorname{acc}\sigma_{gD}(T) = \operatorname{iso}\, \sigma_{gD}(T)$. Hence there exists an $\epsilon > 0$ such that $(D(0, \epsilon) \setminus \{0\}) \cap \sigma_{gD}(T) = \emptyset$ and so $(D(0, \epsilon) \setminus \{0\}) \cap \sigma(T) \subset \operatorname{iso}\, \sigma(T)$. As $0 \in \operatorname{acc}\sigma(T)$, it

follows that the set $(D(0,\epsilon) \setminus \{0\}) \cap \sigma(T)$ is countable. Thus there exists a sequence (λ_n) of isolated points of $\sigma(T)$ which converges to 0, where $\{\lambda_n : n \in \mathbb{N}\} = (D(0,\epsilon) \setminus \{0\}) \cap \sigma(T)$ and $|\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq \dots$. There is $n_0 \in \mathbb{N}$ such that for $n \in \mathbb{N}$, $n \geq n_0$ implies that $0 < |\lambda_n| < 1$. Then $\sigma_{n_0} = \{0, \lambda_{n_0}, \lambda_{n_0+1}, \dots\}$ is a spectral set of T. Let $P_{\sigma_{n_0}}$ be the spectral projection of T associated with σ_{n_0}. From [7, Theorem 49.1] it follows that $(R(P_{\sigma_{n_0}}), N(P_{\sigma_{n_0}})) \in Red(T)$, $\sigma(T_{R(P_{\sigma_{n_0}})}) = \sigma_{n_0}$ and $\sigma(T_{N(P_{\sigma_{n_0}})}) = \sigma(T) \setminus \sigma_{n_0}$. Since the spectral radius $r(T_{R(P_{\sigma_{n_0}})}) = \sup\{|\lambda_{n_0}|, |\lambda_{n_0+1}|, \dots\} = |\lambda_{n_0}| < 1$, it follows that $T_{R(P_{\sigma_{n_0}})} - I_{R(P_{\sigma_{n_0}})}$ is invertible, and since $0 \notin \sigma(T_{N(P_{\sigma_{n_0}})})$, we have that $T_{N(P_{\sigma_{n_0}})}$ is invertible. Now from

$$T - P_{\sigma_{n_0}} = (T_{R(P_{\sigma_{n_0}})} - I_{R(P_{\sigma_{n_0}})}) \oplus T_{N(P_{\sigma_{n_0}})}$$

we conclude that $T - P_{\sigma_{n_0}}$ is invertible. Then

$$S_{\sigma_{n_0}} = (T - P_{\sigma_{n_0}})^{-1}(I - P_{\sigma_{n_0}})$$

is a generalized Drazin-g-meromorphic inverse for T.

Indeed, T commutes with $S_{\sigma_{n_0}}$,

$$TS_{\sigma_{n_0}} = T(T - P_{\sigma_{n_0}})^{-1}(I - P_{\sigma_{n_0}}) = (T - P_{\sigma_{n_0}})(T - P_{\sigma_{n_0}})^{-1}(I - P_{\sigma_{n_0}}) = I - P_{\sigma_{n_0}},$$

and hence,

$$S_{\sigma_{n_0}} TS_{\sigma_{n_0}} = S_{\sigma_{n_0}}(I - P_{\sigma_{n_0}}) = S_{\sigma_{n_0}},$$

and

$$T - TS_{\sigma_{n_0}}T = T - (I - P_{\sigma_{n_0}})T = P_{\sigma_{n_0}}T = TP_{\sigma_{n_0}}.$$

We have that T, as well as $TP_{\sigma_{n_0}}$, is completely reduced by the pair $(R(P_{\sigma_{n_0}}), N(P_{\sigma_{n_0}}))$, $T = T_{R(P_{\sigma_{n_0}})} \oplus T_{N(P_{\sigma_{n_0}})}$ and

$$TP_{\sigma_{n_0}} = T_{R(P_{\sigma_{n_0}})} \oplus 0. \tag{3.4}$$

Since $T - \lambda_k$ is generalized Drazin invertible, Lemma 2.5 ensures that $T_{R(P_{\sigma_{n_0}})} - \lambda_k$ is generalized Drazin invertible for every $k \in \mathbb{N}$. From (3.4) it follows that

$$TP_{\sigma_{n_0}} - \lambda_k = (T_{R(P_{\sigma_{n_0}})} - \lambda_k) \oplus (-\lambda_k I_{N(P_{\sigma_{n_0}})}),$$

and so again using Lemma 2.5 we obtain that $TP_{\sigma_{n_0}} - \lambda_k$ is generalized Drazin invertible, for every $k \in \mathbb{N}$. As $\sigma(TP_{\sigma_{n_0}}) = \sigma_{n_0}$, it follows that $\sigma_{gD}(TP_{\sigma_{n_0}}) = \{0\}$, and therefore $TP_{\sigma_{n_0}}$ is g-meromorphic.

(ix)⟹(x): It follows from the proof of the implication (ix)⟹(iv).

(x)⟹(xi): Analogously to the proof of the implication (vi)⟹(vii).

(xi)⟹(vii): It is clear. □

Remark 3.11. Let $0 \in \operatorname{acc} \sigma(T) \setminus \operatorname{acc} \sigma_{gD}(T)$ and let σ_{n_0} a spectral set as in the proof of the implication (ix)⟹(iv) in Theorem 3.10. If $f = 1$ in a neighborhood U_0 of σ_{n_0} and $f = 0$ in a neighborhood U_1 of $\sigma(T) \setminus \sigma_{n_0}$, then for the function

$$g(\lambda) = (\lambda - f(\lambda))^{-1}(1 - f(\lambda)) = \begin{cases} 0, & \lambda \in U_0, \\ \frac{1}{\lambda}, & \lambda \in U_1 \end{cases}$$

we have that $g(T) = (T - P_{\sigma_{n_0}})^{-1}(1 - P_{\sigma_{n_0}}) = S_{\sigma_{n_0}}$ and according to the spectral mapping theorem it follows that

$$\sigma(S_{\sigma_{n_0}}) = g(\sigma(T)) = \{0\} \cup \{\frac{1}{\lambda} : \lambda \in \sigma(T) \setminus \sigma_{n_0}\}. \tag{3.5}$$

If $\sigma_{n_0+1} = \{0\} \cup \{\lambda_{n_0+1}, \lambda_{n_0+2}, \dots\}$, then we have that $S_{\sigma_{n_0+1}} = (T - P_{\sigma_{n_0+1}})^{-1}(1 - P_{\sigma_{n_0+1}})$ is also a generalized Drazin-g-meromorphic inverse of T and

$$\sigma(S_{\sigma_{n_0+1}}) = \{0\} \cup \{\frac{1}{\lambda} : \lambda \in \sigma(T) \setminus \sigma_{n_0+1}\}. \tag{3.6}$$

As $1/\lambda_{n_0} \in \sigma(S_{\sigma_{n_0+1}}) \setminus \sigma(S_{\sigma_{n_0}})$, we conclude that $S_{\sigma_{n_0}} \neq S_{\sigma_{n_0+1}}$. Therefore, if T is generalized Drazin-g-meromorphic invertible, then its generalized Drazin-g-meromorphic inverse may not be unique. This also follows from [1, Theorem 2.3] since every generalized Drazin-Riesz invertible operator is generalized Drazin-g-meromorphic invertible, but the proof above is more direct.

Corollary 3.12. *For $T \in L(X)$, $T \in L(X)^{gD(g\mathcal{M})} \setminus L(X)^{gD}$ if and only if there exist a spectral set $\sigma \subset \sigma(T)$ and a sequence (λ_n) of nonzero isolated points of $\sigma(T)$ which converges to 0 such that*

$$\sigma(T) = \{0\} \cup \{\lambda_n : n \in \mathbb{N}\} \cup \sigma.$$

Proof. From [9, Theorem 4.2] and Theorem 3.10 it follows that $T \in L(X)^{gD(g\mathcal{M})} \setminus L(X)^{gD}$ if and only if $0 \in \text{acc}\, \sigma(T) \setminus \text{acc}\, \sigma_{gD}(T)$. The rest follows from the proof of the implication (ix)$\Longrightarrow$(iv) in Theorem 3.10. □

Theorem 3.13. *The following conditions are mutually equivalent for operators $T \in L(X)$:*
(i) *There exists $(M, N) \in Red(T)$ such that T_M is bounded below and $T_N \in (g\mathcal{M})$;*
(ii) *T admits a $GK(g\mathcal{M})D$ and $0 \notin \text{int}\, \sigma_{ap}(T)$;*
(iii) *T admits a $GK(g\mathcal{M})D$ and T has SVEP at 0;*
(iv) *T admits a $GK(g\mathcal{M})D$ and $0 \notin \text{acc}\, \sigma_{gD\mathcal{J}}(T)$;*
(v) *T admits a $GK(g\mathcal{M})D$ and $0 \notin \text{int}\, \sigma_{gD\mathcal{J}}(T)$.*

Proof. The equivalences (i)$\Longleftrightarrow$(ii)$\Longleftrightarrow$(iv)$\Longleftrightarrow$(v) follow from Theorem 3.7.
(i)$\Longleftrightarrow$(iii): Similarly to the proof of the implications (i)$\Longrightarrow$(iii) and (iii)$\Longrightarrow$(iv) in Theorem 3.10. □

Theorem 3.14. *The following conditions are mutually equivalent for operators $T \in L(X)$:*
(i) *There exists $(M, N) \in Red(T)$ such that T_M is surjective and $T_N \in (g\mathcal{M})$;*
(ii) *T admits a $GK(g\mathcal{M})D$ and $0 \notin \text{int}\, \sigma_{su}(T)$;*
(iii) *T admits a $GK(g\mathcal{M})D$ and T' has SVEP at 0;*
(iv) *T admits a $GK(g\mathcal{M})D$ and $0 \notin \text{acc}\, \sigma_{gD\mathcal{S}}(T)$;*
(v) *T admits a $GK(g\mathcal{M})D$ and $0 \notin \text{int}\, \sigma_{gD\mathcal{S}}(T)$.*

Proof. The equivalences (i)$\Longleftrightarrow$(ii)$\Longleftrightarrow$(iv)$\Longleftrightarrow$(v) follow from Theorem 3.7.
(i)$\Longleftrightarrow$(iii): Similarly to the proof of the implications (i)$\Longrightarrow$(iii) and (iii)$\Longrightarrow$(iv) in Theorem 3.10. □

P. Aiena and E. Rosas [2, Theorems 2.2 and 2.5] characterized the SVEP at a point λ_0 in the case that $\lambda_0 - T$ is of Kato type. Q. Jiang and H. Zhong [8, Theorems 3.5 and 3.9] gave further characterizations of the SVEP at λ_0 in the case that $\lambda_0 - T$ admits a generalized Kato decomposition. In [15, Corollary 2.1] ([16, Corollary 1, Corollary 2]) the SVEP at λ_0 is characterized in the case that $\lambda_0 - T$ admits a generalized Kato-Riesz decomposition (a generalized Kato-meromorphic decomposition). Now we give characterizations for the case that $\lambda_0 - T$ admits generalized Kato-g-meromorphic decomposition.

Corollary 3.15. *Let $T \in L(X)$ and let $\lambda_0 - T$ admit a $GK(g\mathcal{M})D$. Then the following statements are equivalent:*
(i) *T has the SVEP at λ_0;*
(ii) *λ_0 is not an interior point of $\sigma_{ap}(T)$;*
(iii) *$\sigma_{gD\mathcal{J}}(T)$ does not cluster at λ_0.*

Proof. It follows from the equivalences (ii)$\Longleftrightarrow$(iii)$\Longleftrightarrow$(iv) in Theorem 3.13. □

Corollary 3.16. *Let $T \in L(X)$ and let $\lambda_0 - T$ admit a $GK(g\mathcal{M})D$. Then the following statements are equivalent:*
(i) *T' has the SVEP at λ_0;*
(ii) *λ_0 is not an interior point of $\sigma_{su}(T)$;*
(iii) *$\sigma_{gD\mathcal{S}}(T)$ does not cluster at λ_0.*

Proof. It follows from Theorem 3.14. □

Theorem 3.17. *Let $T \in L(X)$. The following statements are equivalent:*

(i) *$T = T_M \oplus T_N$ where T_M is invertible and T_N is g-meromorphic with infinite spectrum;*

(ii) *T admits a $GK(g\mathcal{M})D$ and there exists a sequence of nonzero isolated points of $\sigma(T)$ which converges to 0.*

Proof. (i) $\Longrightarrow$ (ii): Suppose that $T = T_M \oplus T_N$ where T_M is invertible and T_N is g-meromorphic with infinite spectrum. Then T admits a $GK(g\mathcal{M})D(M,N)$ and $\sigma(T_N) = \{0, \mu_1, \mu_2, \ldots\}$ where μ_n, $n \in \mathbb{N}$, are nonzero points of $\sigma(T_N)$, all of them are isolated points of $\sigma(T_N)$ and

$$\lim_{n\to\infty} \mu_n = 0. \tag{3.7}$$

From Theorem 3.10 we have that $0 \notin \text{acc}\,\sigma_{gD}(T)$, i.e. there exists $\epsilon > 0$ such that $\mu \notin \sigma_{gD}(T)$ for $0 < |\mu| < \epsilon$. From (3.7) it follows that there exists $n_0 \in \mathbb{N}$ such that $0 < |\mu_n| < \epsilon$ for $n \geq n_0$. Hence $\mu_n \in \sigma(T) \setminus \sigma_{gD}(T) = \sigma(T) \setminus \text{acc}\,\sigma(T) = \text{iso}\,\sigma(T)$ for all $n \geq n_0$. Thus $(\mu_n)_{n=n_0}^{\infty}$ is the sequence of nonzero isolated points of $\sigma(T)$ which converges to 0.

(ii) $\Longrightarrow$ (i): Suppose that $T = T_M \oplus T_N$ where T_M is Kato, T_N is g-meromorphic and let (λ_n) be the sequence of isolated points of $\sigma(T)$ which converges to 0. Since $\lambda_n \notin \sigma_{gD}(T)$ for all $n \in \mathbb{N}$, it follows that $0 \notin \text{int}\,\sigma_{gD}(T)$. As in the proof of the implications (iii)$\Longrightarrow$ (iv)$\Longrightarrow$(i) of Theorem 3.7 we conclude that T_M is invertible. Thus there exists an $\epsilon > 0$ such that $D(0,\epsilon) \cap \sigma(T_M) = \emptyset$ and there exists $n_0 \in \mathbb{N}$ such that $\lambda_n \in D(0,\epsilon)$ for all $n \geq n_0$. Consequently, $\lambda_n \notin \sigma(T_M)$ for all $n \geq n_0$ and and hence $\lambda_n \in \sigma(T_N)$ for all $n \geq n_0$, which implies that the spectrum of T_N is infinite. □

Theorem 3.18. *Let $T \in GD(g\mathcal{M})\mathbf{R_i}$, $f \in \text{Holo}_1(\sigma(a))$ and $f^{-1}(0) \cap \sigma_{\mathbf{R_i}}(T) = \{0\}$, $1 \leq i \leq 9$. Then $f(T) \in GD(g\mathcal{M})\mathbf{R_i}$.*

Proof. It is known that $f(\sigma_{\mathbf{R_i}}(T)) = \sigma_{\mathbf{R_i}}(f(T))$ for all f holomorphic on a neighbourhood of $\sigma(T)$ and $1 \leq i \leq 6$. The corresponding inclusion for $7 \leq i \leq 9$ is $\sigma_{\mathbf{R_i}}(f(T)) \subset f(\sigma_{\mathbf{R_i}}(T))$. If $T \in GD(g\mathcal{M})\mathbf{R_i}$, then there exists a decomposition $(M,N) \in Red(T)$ such that $T_M \in \mathbf{R_i}$ and $T_N \in (g\mathcal{M})$. Furthermore $f(T) = f(T_M) \oplus f(T_N)$. Since $f(0) = 0$, from Proposition 3.4 it follows that $f(T_N) \in (g\mathcal{M})$. Observe that $0 \notin \sigma_{\mathbf{R_i}}(T_M)$ and since $f^{-1}(0) \cap \sigma_{\mathbf{R_i}}(T) = \{0\}$ we conclude that $0 \notin f(\sigma_{\mathbf{R_i}}(T_M))$. As $f(\sigma_{\mathbf{R_i}}(T_M)) \supset \sigma_{\mathbf{R_i}}(f(T_M))$ for all $1 \leq i \leq 9$, we conclude that $0 \notin \sigma_{\mathbf{R_i}}(f(T_M))$, and so $f(T_M) \in \mathbf{R_i}$. Therefore, $f(T) \in GD(g\mathcal{M})\mathbf{R_i}$. □

4. Spectra

For $T \in L(X)$, set

$$\sigma_{gK(g\mathcal{M})}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ does not admit generalized Kato-}g\text{-meromorphic decomposition}\}$$

and

$$\sigma_{gD(g\mathcal{M})\mathbf{R}_i}(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin gD(g\mathcal{M})\mathbf{R}_i(X)\},\ \ 1 \leq i \leq 9.$$

In the following we shorten, for convenience, $\sigma_{gD(g\mathcal{M})L(X)^{-1}}(T)$ to

$$\sigma_{gD(g\mathcal{M})}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not generalized Drazin-}g\text{-meromorphic invertible}\}.$$

Corollary 4.1. *Let $T \in L(X)$. Then*

(i) $\sigma_{gD(g\mathcal{M})}(T) = \text{acc}\,\sigma_{gD}(T)$;

(ii) $\sigma_{gD(g\mathcal{M})\mathbf{R}_i}(T) = \sigma_{gK(g\mathcal{M})}(T) \cup \text{acc}\ \sigma_{gD\mathbf{R}_i}(T)$, $2 \leq i \leq 9$;

(iii) $\sigma_{gD(g\mathcal{M})\mathbf{R}_i}(T) = \sigma_{gK(g\mathcal{M})}(T) \cup \text{int}\,\sigma_{\mathbf{R}_i}(T)$, $1 \leq i \leq 9$.

Proof. (i) It follows from the equivalence (iv)$\Longleftrightarrow$(ix) in Theorem 3.10.

(ii), (iii): It follows from the equivalences (i)$\Longleftrightarrow$(ii)$\Longleftrightarrow$(iv) in Theorem 3.7. □

Corollary 4.2. *For $T \in L(X)$ if $\sigma_{\mathbf{R}_i}(T)$ is countable or contained in a line, then*

$$\sigma_{gD(g\mathcal{M})\mathbf{R}_i}(T) = \sigma_{gK(g\mathcal{M})}(T),\ 1 \le i \le 9.$$

Proof. It follows from Corollary 4.1 (iii). □

Theorem 4.3. *Let $T \in L(X)$ and let T admits a $GK(g\mathcal{M})D(M,N)$. Then there exists $\epsilon > 0$ such that $T - \lambda$ admits a GKD for each λ such that $0 < |\lambda| < \epsilon$.*

Proof. If $M = \{0\}$, then T is g-meromorphic and hence $T - \lambda$ is generalized Drazin invertible for all $\lambda \neq 0$. From [9, Theorem 7.1] it follows that $T - \lambda$ can be decomposed into a direct sum of an invertible operator and a quasinilpotent operator for all $\lambda \neq 0$. Hence $T - \lambda$ admits a GKD for all $\lambda \neq 0$.

Suppose that $M \neq \{0\}$. From [3, Theorem 1.31] it follows that for $|\lambda| < \gamma(T_M)$, $T_M - \lambda$ is Kato. As T_N is g-meromorphic, $T_N - \lambda$ is generalized Drazin invertible for all $\lambda \neq 0$. Hence $T_N - \lambda$ can be decomposed into a direct sum of an invertible operator and a quasinilpotent operator for all $\lambda \neq 0$. Let $\epsilon = \gamma(T_M)$. Using Lemma 2.3 and the fact that a direct sum of two Kato operators is Kato [4, Theorem 1.46], we conclude that $T - \lambda$ admits a GKD for each λ such that $0 < |\lambda| < \epsilon$. □

Corollary 4.4. *Let $T \in L(X)$. Then*

(i) $\sigma_{gK(g\mathcal{M})}(T)$ *is compact;*

(ii) *The set $\sigma_{gK}(T) \setminus \sigma_{gK(g\mathcal{M})}(T)$ consists of at most countably many points.*

Proof. (i): According to Theorem 4.3, $\sigma_{gK(g\mathcal{M})}(T)$ is closed, and since $\sigma_{gK(g\mathcal{M})}(T) \subset \sigma(T)$, $\sigma_{gK(g\mathcal{M})}(T)$ is bounded. Hence $\sigma_{gK(g\mathcal{M})}(T)$ is compact.

(ii): Suppose that $\lambda_0 \in \sigma_{gK}(T) \setminus \sigma_{gK(g\mathcal{M})}(T)$. Then $T - \lambda_0$ admits a $GK(g\mathcal{M})D$ and according to Theorem 4.3 there exists $\epsilon > 0$ such that $T - \lambda$ admits a GKD for each λ such that $0 < |\lambda - \lambda_0| < \epsilon$. This implies that $\lambda_0 \in \text{iso}\, \sigma_{gK}(T)$. Therefore, $\sigma_{gK}(T) \setminus \sigma_{gK(g\mathcal{M})}(T) \subset \text{iso}\, \sigma_{gK}(T)$, which implies that $\sigma_{gK} \setminus \sigma_{gK(g\mathcal{M})}(T)$ is at most countable. □

Corollary 4.5. *Let $T \in L(X)$ and $1 \le i \le 9$. Then*

(i) $\sigma_{gD(g\mathcal{M})\mathbf{R}_i}(T) \subset \sigma_{\mathbf{R}_i}(T)$;

(ii) $\sigma_{gD(g\mathcal{M})\mathbf{R}_i}(T)$ *is compact;*

(iii) $\text{int}\,\sigma_{gD(g\mathcal{M})\mathbf{R}_i}(T) = \text{int}\,\sigma_{\mathbf{R}_i}(T)$;

(iv) $\partial\sigma_{gD(g\mathcal{M})\mathbf{R}_i}(T) \subset \partial\sigma_{\mathbf{R}_i}(T)$;

(v) $\sigma_{gD\mathbf{R}_i}(T) \setminus \sigma_{gD(g\mathcal{M})\mathbf{R}_i}(T) = (\text{iso}\ \sigma_{gD\mathbf{R}_i}(T)) \setminus \sigma_{gK(g\mathcal{M})}(T)$;

(vi) *The set $\sigma_{gD\mathbf{R}_i}(T) \setminus \sigma_{gD(g\mathcal{M})\mathbf{R}_i}(T)$ consist of at most countably many points.*

Proof. (i): Obvious.

(ii): From Corollary 4.1 (ii) and Corollary 4.4 (i) it follows that $\sigma_{gD(g\mathcal{M})\mathbf{R}_i}(T)$ is closed as the union of two closed sets, while from (i) it follows that $\sigma_{gD(g\mathcal{M})\mathbf{R}_i}(T)$ is bounded, and so it is compact.

(iii): From Corollary 4.1 (iii) we have that $\text{int}\,\sigma_{\mathbf{R}_i}(T) \subset \sigma_{gD(g\mathcal{M})\mathbf{R}_i}(T)$, and hence $\text{int}\,\sigma_{\mathbf{R}_i}(T) \subset \text{int}\,\sigma_{gD(g\mathcal{M})\mathbf{R}_i}(T)$, while from the inclusion (i) it follows that $\text{int}\,\sigma_{gD(g\mathcal{M})\mathbf{R}_i}(T) \subset \text{int}\,\sigma_{\mathbf{R}_i}(T)$. Consequently, $\text{int}\,\sigma_{gD(g\mathcal{M})\mathbf{R}_i}(T) = \text{int}\,\sigma_{\mathbf{R}_i}(T)$.

(iv): Let $\lambda \in \partial\sigma_{gD(g\mathcal{M})\mathbf{R}_i}(T)$. Since $\partial\sigma_{gD(g\mathcal{M})\mathbf{R}_i}(T) \subset \sigma_{gD(g\mathcal{M})\mathbf{R}_i}(T) \subset \sigma_{\mathbf{R}_i}(T)$, from $\lambda \notin \text{int}\,\sigma_{gD(g\mathcal{M})\mathbf{R}_i}(T) = \text{int}\,\sigma_{\mathbf{R}_i}(T)$ we conclude $\lambda \in \partial\sigma_{\mathbf{R}_i}(T)$. So, $\partial\sigma_{gD(g\mathcal{M})\mathbf{R}_i}(T) \subset \partial\sigma_{\mathbf{R}_i}(T)$.

(v): It follows from Corollary 4.1 (ii).

(vi) It follows from (v). □

Corollary 4.6. *Let $T \in L(X)$ and $1 \le i \le 9$. Then*

$$\partial\,\sigma_{gD\mathbf{R}_i}(T) \cap \text{acc}\ \sigma_{gD\mathbf{R}_i}(T) \subset \partial\,\sigma_{\mathbf{R}_i}(T) \cap \text{acc}\ \sigma_{gD\mathbf{R}_i}(T) \subset \partial\,\sigma_{gK(g\mathcal{M})}(T). \tag{4.1}$$

Proof. Let $T-\lambda I$ admit a $GK(g\mathcal{M})D$ and let $\lambda \in \partial\,\sigma_{\mathbf{R}_i}(T)$. Then $\lambda \notin \operatorname{int}\sigma_{\mathbf{R}_i}(T)$ and according to the equivalence (ii)$\Longleftrightarrow$(iv) in Theorem 3.7 it follows that $\lambda \notin \operatorname{acc}\,\sigma_{gD\mathbf{R}_i}(T)$. Therefore,

$$\partial\,\sigma_{\mathbf{R}_i}(T) \cap \operatorname{acc}\,\sigma_{gD\mathbf{R}_i}(T) \subset \sigma_{gK(g\mathcal{M})}(T). \tag{4.2}$$

Suppose that $\lambda \in \partial\,\sigma_{\mathbf{R}_i}(T) \cap \operatorname{acc}\,\sigma_{gD\mathbf{R}_i}(T)$. Then there exists a sequence (λ_n) which converges to λ and such that $T-\lambda_n \in \mathbf{R_i}$ for every $n \in \mathbb{N}$. According to [13, Theorem 16.21] it follows that $T-\lambda_n$ admits a $GK(g\mathcal{M})D$, and so $\lambda_n \notin \sigma_{gK(g\mathcal{M})}(T)$ for every $n \in \mathbb{N}$. Since $\lambda \in \sigma_{gK(g\mathcal{M})}(T)$ by (4.2), we conclude that $\lambda \in \partial\,\sigma_{gK(g\mathcal{M})}(T)$. This proves the second inclusion in (4.1).

From [16, Corollary 4 (iii)] it follows that $\operatorname{int}\sigma_{gD\mathbf{R}_i}(T) = \operatorname{int}\sigma_{\mathbf{R}_i}(T)$, and hence $\partial\,\sigma_{gD\mathbf{R}_i}(T) \subset \partial\,\sigma_{\mathbf{R}_i}(T)$. It implies the first inclusion in (4.1). □

Corollary 4.7. *Let $T \in L(X)$.*

(i) *If T has the SVEP, then all accumulation points of $\sigma_{gD\mathcal{J}}(T)$ belong to $\sigma_{gK(g\mathcal{M})}(T)$.*

(ii) *If T' has the SVEP, then all accumulation points of $\sigma_{gD\mathcal{S}}(T)$ belong to $\sigma_{gK(g\mathcal{M})}(T)$.*

(iii) *If T and T' have the SVEP, then all accumulation points of $\sigma_{gD}(T)$ belong to $\sigma_{gK(g\mathcal{M})}(T)$.*

Proof. (i): It follows from the equivalence (iii)$\Longleftrightarrow$(iv) of Theorem 3.13.
(ii): It follows from the equivalence (iii)$\Longleftrightarrow$(iv) of Theorem 3.14.
(iii): It follows from the equivalence (iii)$\Longleftrightarrow$(ix) of Theorem 3.10. □

The next corollary extends [3, Corollary 3.118] and [16, Corollary 7].

Corollary 4.8. *Let T be unilateral weighted right shift operator on $\ell_p(\mathbb{N})$, $1 \le p < \infty$, with weight (ω_n), and let $c(T) = \lim\limits_{n\to\infty} \inf(\omega_1 \cdot \dots \cdot \omega_n)^{1/n} = 0$. Then $\sigma_{gK(g\mathcal{M})}(T) = \sigma_{gD(g\mathcal{M})\mathbf{R}_i}(T) = \sigma(T) = \overline{D(0, r(T))}$, $1 \le i \le 9$.*

Proof. From [3, Corollary 3.118] we have that $\sigma(T) = \overline{D(0, r(T))}$, and T and T' have the SVEP. The equivalence (ii)$\Longleftrightarrow$(iii) in Theorem 3.10 ensures that $D(0, r(T)) = \operatorname{int}\sigma(T) \subset \sigma_{gK(g\mathcal{M})}(T)$. As $\sigma_{gK(g\mathcal{M})}(T)$ is closed, it follows that

$$\overline{D(0, r(T))} \subset \sigma_{gK(g\mathcal{M})}(T) \subset \sigma_{gD(g\mathcal{M})\mathbf{R}_i}(T) \subset \sigma(T) = \overline{D(0, r(T))},$$

and so $\sigma_{gK(g\mathcal{M})}(T) = \sigma_{gD(g\mathcal{M})\mathbf{R}_i}(T) = \sigma(T) = \overline{D(0, r(T))}$. □

The *connected hull* of a compact subset K of the complex plane $\mathbb{C}$, denoted by ηK, is the complement of the unbounded component of $\mathbb{C} \setminus K$ [6, Definition 7.10.1]. A hole of K is a bounded component of $\mathbb{C} \setminus K$, and so a hole of K is a component of $\eta K \setminus K$. We recall that, for compact subsets $H, K \subset \mathbb{C}$, the following implication holds ([6, Theorem 7.10.3]):

$$\partial H \subset K \subset H \Longrightarrow \partial H \subset \partial K \subset K \subset H \subset \eta K = \eta H\,, \tag{4.3}$$

and H can be obtained from K by filling in some holes of K. Evidently, if $K \subseteq \mathbb{C}$ is at most countable, then $\eta K = K$. Therefore, for compact subsets $H, K \subseteq \mathbb{C}$, if $\eta K = \eta H$, then H is at most countable if and only if K is at most countable, and in that case $H = K$.

Theorem 4.9. *Let $T \in L(X)$. Then*

(i)

$$\begin{array}{ccccccccc}
 & & \partial\sigma_{gD(g\mathcal{M})\mathcal{J}}(T) & \subset & \partial\sigma_{gD(g\mathcal{M})\mathcal{W}_+}(T) & \subset & \partial\sigma_{gD(g\mathcal{M})\Phi_+}(T) & & \\
 & \subset & & \subset & & \subset & & \subset & \\
\partial\sigma_{gD(g\mathcal{M})}(T) & \subset & \partial\sigma_{gD(g\mathcal{M})\mathcal{W}}(T) & \subset & \partial\sigma_{gD(g\mathcal{M})\Phi}(T) & \subset & & & \partial\sigma_{gK(g\mathcal{M})}(T), \\
 & \subset & & \subset & & \subset & & \subset & \\
 & & \partial\sigma_{gD(g\mathcal{M})\mathcal{S}}(T) & \subset & \partial\sigma_{gD(g\mathcal{M})\mathcal{W}_-}(T) & \subset & \partial\sigma_{gD(g\mathcal{M})\Phi_-}(T) & &
\end{array}$$

(ii)

$$\begin{array}{ccccccccc} & & \text{iso}\,\sigma_{gD(g\mathcal{M})\mathcal{J}}(T) & \subset & \text{iso}\,\sigma_{gD(g\mathcal{M})\mathcal{W}_+}(T) & \subset & \text{iso}\,\sigma_{gD(g\mathcal{M})\Phi_+}(T) & & \\ & \subset & & \subset & & \subset & & \subset & \\ \text{iso}\,\sigma_{gD(g\mathcal{M})}(T) & \subset & \text{iso}\,\sigma_{gD(g\mathcal{M})\mathcal{W}}(T) & \subset & \text{iso}\,\sigma_{gD(g\mathcal{M})\Phi}(T) & \subset & & & \text{iso}\,\sigma_{gK(g\mathcal{M})}(T), \\ & \subset & & \subset & & \subset & & \subset & \\ & & \text{iso}\,\sigma_{gD(g\mathcal{M})\mathcal{S}}(T) & \subset & \text{iso}\,\sigma_{gD(g\mathcal{M})\mathcal{W}_-}(T) & \subset & \text{iso}\,\sigma_{gD(g\mathcal{M})\Phi_-}(T) & & \end{array}$$

(iii) $\eta\sigma_{gD(g\mathcal{M})}(T) = \eta\sigma_{gD(g\mathcal{M})\mathbf{R_i}}(T) = \eta\sigma_{gK(g\mathcal{M})}(T)$, $2 \le i \le 9$.
(iv) *The set* $\sigma_{gD(g\mathcal{M})}(T)$ *consists of* $\sigma_*(T)$ *and possibly some holes in* $\sigma_*(T)$ *where* $\sigma_* \in \{\sigma_{gK(g\mathcal{M})}, \sigma_{gD(g\mathcal{M})\mathcal{W}}, \sigma_{gD(g\mathcal{M})\Phi}, \sigma_{gD(g\mathcal{M})\mathcal{W}_+}, \sigma_{gD(g\mathcal{M})\Phi_+}, \sigma_{gD(g\mathcal{M})\mathcal{J}}, \sigma_{gD(g\mathcal{M})\mathcal{W}_-}, \sigma_{gD(g\mathcal{M})\Phi_-}, \sigma_{gD(g\mathcal{M})\mathcal{S}}\}$.
(v) *If one of* $\sigma_{gK(g\mathcal{M})}(T), \sigma_{gD(g\mathcal{M})}(T), \sigma_{gD(g\mathcal{M})\mathcal{W}}(T), \sigma_{gD(g\mathcal{M})\Phi}(T), \sigma_{gD(g\mathcal{M})\mathcal{W}_+}(T), \sigma_{gD(g\mathcal{M})\Phi_+}(T), \sigma_{gD(g\mathcal{M})\mathcal{J}}(T), \sigma_{gD(g\mathcal{M})\mathcal{W}_-}(T), \sigma_{gD(g\mathcal{M})\Phi_-}(T), \sigma_{gD(g\mathcal{M})\mathcal{S}}(T)$ *is finite (countable), then all of them are equal and hence finite (countable).*

Proof. Since $\sigma_{gK(g\mathcal{M})}(T)$ and $\sigma_{gD(g\mathcal{M})\mathbf{R_i}}(T)$, $1 \le i \le 9$, are compact, according to (4.3), Lemma 2.2 (i) and the inclusions

$$\begin{array}{ccccccccc} & & \sigma_{gD(g\mathcal{M})\Phi_+}(T) & \subset & \sigma_{gD(g\mathcal{M})\mathcal{W}_+}(T) & \subset & \sigma_{gD(g\mathcal{M})\mathcal{J}}(T) & & \\ & \subset & & \subset & & \subset & & \subset & \\ \sigma_{gK(g\mathcal{M})}(T) & \subset & & \subset & \sigma_{gD(g\mathcal{M})\Phi}(T) & \subset & \sigma_{gD(g\mathcal{M})\mathcal{W}}(T) & \subset & \sigma_{gD(g\mathcal{M})}(T). \\ & \subset & & \subset & & \subset & & \subset & \\ & & \sigma_{gD(g\mathcal{M})\Phi_-}(T) & \subset & \sigma_{gD(g\mathcal{M})\mathcal{W}_-}(T) & \subset & \sigma_{gD(g\mathcal{M})\mathcal{S}}(T) & & \end{array}$$

it is enough to prove that

$$\partial\,\sigma_{gD(g\mathcal{M})\mathbf{R}_i}(T) \subset \sigma_{gK(g\mathcal{M})}(T),\ 1 \le i \le 9. \tag{4.4}$$

Since $\sigma_{gD(g\mathcal{M})\mathbf{R}_i}(T)$ is closed, it follows that

$$\partial\,\sigma_{gD(g\mathcal{M})\mathbf{R}_i}(T) \subset \sigma_{gD(g\mathcal{M})\mathbf{R}_i}(T),\ 1 \le i \le 9. \tag{4.5}$$

According to Corollary 4.1 (iii) and Corollary 4.5 (iii) we have that

$$\sigma_{gD(g\mathcal{M})\mathbf{R}_i}(T) = \sigma_{gK(g\mathcal{M})}(T) \cup \operatorname{int}\sigma_{gD(g\mathcal{M})\mathbf{R}_i}(T). \tag{4.6}$$

Now from (4.5) and (4.6) it follows (4.4). □

Corollary 4.10. *Let* $T \in L(X)$ *and let* $\mathbb{C} \setminus \sigma_{gK(g\mathcal{M})}(T)$ *has only one component. Then*

$$\sigma_{gK(g\mathcal{M})}(T) = \sigma_{gD(g\mathcal{M})}(T).$$

Proof. Since $\mathbb{C} \setminus \sigma_{gK(g\mathcal{M})}(T)$ has only one component, it follows that $\sigma_{gK(g\mathcal{M})}(T)$ has no holes, and so from Theorem 4.9 (iv) it follows that $\sigma_{gD(g\mathcal{M})}(T) = \sigma_{gK(g\mathcal{M})}(T)$. □

Corollary 4.11. *Let* $T \in L(X)$ *and* $1 \le i \le 9$. *Then*

$$\text{iso}\ \sigma_{gK(g\mathcal{M})}(T) \subset \text{iso}\ \sigma_{gD(g\mathcal{M})\mathbf{R_i}}(T) \cup \operatorname{int}\sigma_{\mathbf{R_i}}(T).$$

Proof. Suppose that $\lambda_0 \in \text{iso}\ \sigma_{gK(g\mathcal{M})}(T) \setminus \operatorname{int}\sigma_{\mathbf{R_i}}(T)$. Then $\lambda_0 \in \sigma_{gD(g\mathcal{M})\mathbf{R_i}}(T)$ and there exists a sequence (λ_n) converging to λ_0, such that $T - \lambda_n$ admits a $GK(g\mathcal{M})D$ and $\lambda_n \notin \sigma_{\mathbf{R_i}}(T)$ for all $n \in \mathbb{N}$. Hence $\lambda_n \notin \operatorname{int}\sigma_{\mathbf{R_i}}(T)$, that is $0 \notin \operatorname{int}\sigma_{\mathbf{R_i}}(T - \lambda_n)$ for all $n \in \mathbb{N}$. Theorem 3.7 ensures that $T - \lambda_n \in GD(g\mathcal{M})\mathbf{R_i}$ for all $n \in \mathbb{N}$. Consequently, $\lambda_0 \in \partial\,\sigma_{gD(g\mathcal{M})\mathbf{R_i}}(T) \cap \text{iso}\ \sigma_{gK(g\mathcal{M})}(T)$, which according to (4.4), Corollary 4.5 (ii) and Lemma 2.2 (ii) implies that $\lambda_0 \in \text{iso}\ \sigma_{gD(g\mathcal{M})\mathbf{R_i}}(T)$. □

We shall say that an operator $T \in L(X)$ is *polynomially g-meromorphic* if there exists a nonzero complex polynomial $p(z)$ such that $p(T)$ is g-meromorphic.

Theorem 4.12. *Let* $T \in L(X)$. *The following statements are equivalent:*
(i) $\sigma_{gK(g\mathcal{M})}(T) = \emptyset$;
(ii) $\sigma_{gD(g\mathcal{M})}(T) = \emptyset$;
(iii) $\sigma_{gD}(T)$ *is a finite set;*
(iv) $\sigma(T)$ *has finitely many accumulation points;*
(v) T *is polynomially g-meromorphic.*

Proof. The equivalence (i)$\Longleftrightarrow$(ii) follows from Theorem 4.9.

(ii)$\Longleftrightarrow$(iii): From Corollary 4.1 (i) it follows that

$$\sigma_{gD(g\mathcal{M})}(T) = \emptyset \Longleftrightarrow \text{acc } \sigma_{gD}(T) = \emptyset \Longleftrightarrow \sigma_{gD}(T) \text{ is finite.}$$

(iii)$\Longleftrightarrow$(iv): It is clear since $\sigma_{gD}(T) = \text{acc}\,\sigma(T)$ [9, Theorem 4.2].

(iii)$\Longrightarrow$(v): Suppose that $\sigma_{gD}(T)$ is finite and let $\sigma_{gD}(T) = \{\lambda_1, \dots, \lambda_n\}$. According to the spectral mapping theorem for the generalized Drazin spectrum [10, Theorem 1.4], for $p(z) = (z - \lambda_1) \cdot \dots \cdot (z - \lambda_n)$ we have $\{0\} = p(\sigma_{gD}(T)) = \sigma_{gD}(p(T))$. From Theorem 3.1 it follows that $p(T)$ is g-meromorphic.

(v)$\Longrightarrow$(iii): Let T be polynomially g-meromorphic. Then there exists a nonzero complex polynomial $p(z)$ such that $p(T)$ is g-meromorphic, and so $\sigma_{gD}(p(T)) \subset \{0\}$. As $p(\sigma_{gD}(T)) = \sigma_{gD}(p(T))$ we obtain that $\sigma_{gD}(T)$ is contained in the set of zeros of p, and hence it is finite. □

P. Aiena and E. Rosas [2, Theorem 2.10] proved that if $T \in L(X)$ be an operator for which $\sigma_{ap}(T) = \partial\sigma(T)$ and every $\lambda \in \partial\sigma(T)$ is not isolated in $\sigma(T)$, then $\sigma_{ap}(T) = \sigma_{Kt}(T)$, while Q. Jiang and H. Zhong [8, Theorem 3.12] improved this result by proving that under the same conditions it holds $\sigma_{ap}(T) = \sigma_{gK}(T)$. Later it was proved that $\sigma_{ap}(T) = \sigma_{gKR}(T) = \sigma_{gK(\mathcal{M})}(T)$ [15, Theorem 3.14], [16, Theorem 13]. The next theorem extends these results.

Theorem 4.13. *For $T \in L(X)$ suppose that $\sigma_{ap}(T) = \partial\sigma(T)$ and every $\lambda \in \partial\sigma(T)$ is not isolated in $\sigma(T)$. Then*

$$\sigma_{gK(g\mathcal{M})}(T) = \sigma_{gD(g\mathcal{M})\Phi_+}(T) = \sigma_{gD(g\mathcal{M})\mathcal{W}_+}(T) = \sigma_{gD(g\mathcal{M})\mathcal{J}}(T) = \sigma_{ap}(T). \tag{4.7}$$

Proof. From the proof of [5, Corollary 5.11] we have that

$$\sigma_{ap}(T) = \text{acc } \sigma_{ap}(T) = \partial\, \sigma_{ap}(T). \tag{4.8}$$

Also from [5, Corollary 5.11] it follows that $\sigma_{ap}(T) = \sigma_{gD\mathcal{J}}(T)$, which together with (4.8) implies that

$$\partial\, \sigma_{ap}(T) \cap \text{acc } \sigma_{gD\mathcal{J}}(T) = \sigma_{ap}(T). \tag{4.9}$$

According to the inclusion (4.2) it holds

$$\partial\, \sigma_{ap}(T) \cap \text{acc } \sigma_{gD\mathcal{J}}(T) \subset \sigma_{gK(g\mathcal{M})}(T). \tag{4.10}$$

Now from (4.9) and (4.10) we conclude that $\sigma_{ap}(T) \subset \sigma_{gK(g\mathcal{M})}(T)$, which together with the inclusions $\sigma_{gK(g\mathcal{M})}(T) \subset \sigma_{gD(g\mathcal{M})\Phi_+}(T) \subset \sigma_{gD(g\mathcal{M})\mathcal{W}_+}(T) \subset \sigma_{gD(g\mathcal{M})\mathcal{J}}(T) \subset \sigma_{ap}(T)$ gives the equalities (4.7). □

The following theorem is an improvement of [2, Corollary 2.11], [8, Corollary 3.13], [15, Theorem 3.15] and [16, Theorem 14].

Theorem 4.14. *Let $T \in L(X)$ be an operator for which $\sigma_{su}(T) = \partial\sigma(T)$ and every $\lambda \in \partial\sigma(T)$ is not isolated in $\sigma(T)$. Then*

$$\sigma_{gK(g\mathcal{M})}(T) = \sigma_{gD(g\mathcal{M})\Phi_-}(T) = \sigma_{gD(g\mathcal{M})\mathcal{W}_-}(T) = \sigma_{gD(g\mathcal{M})\mathcal{S}}(T) = \sigma_{su}(T).$$

Proof. Follows from [5, Corollary 5.11] and the inclusion (4.2), analogously to the proof of Theorem 4.13. □

Example 4.15. For the *Cesáro operator* C_p defined on the classical Hardy space $H_p(\mathbf{D})$, $\mathbf{D}$ the open unit disc and $1 < p < \infty$, by

$$(C_p f)(\lambda) = \frac{1}{\lambda}\int_0^\lambda \frac{f(\mu)}{1-\mu}\, d\mu, \ \text{ for all } f \in H_p(\mathbf{D}) \text{ and } \lambda \in \mathbf{D},$$

it is known that its spectrum is the closed disc Γ_p centered at $p/2$ with radius $p/2$, $\sigma_{gK(\mathcal{M})}(C_p) = \sigma_{gKR}(C_p) = \sigma_{gK}(C_p) = \sigma_{Kt}(C_p) = \sigma_{ap}(C_p) = \partial\Gamma_p$ and also $\sigma_\Phi(C_p) = \partial\Gamma_p$ [11], [2], [15], [16]. From Theorem 4.13 it follows that

$$\sigma_{gK(g\mathcal{M})}(C_p) = \sigma_{gD(g\mathcal{M})\Phi_+}(C_p) = \sigma_{gD(g\mathcal{M})\mathcal{W}_+}(C_p) = \sigma_{gD(g\mathcal{M})\mathcal{J}}(C_p) = \sigma_{ap}(C_p) = \partial\Gamma_p,$$

and since $\text{int}\,\sigma_\Phi(C_p) = \text{int}\,\sigma_{\Phi_-}(C_p) = \emptyset$, according to Corollary 4.1 (iii) we have that

$$\sigma_{gD(g\mathcal{M})\Phi}(C_p) = \sigma_{gD(g\mathcal{M})\Phi_-}(C_p) = \sigma_{gK(g\mathcal{M})}(C_p) = \partial\Gamma_p.$$

As $\sigma_{\mathcal{W}_-}(C_p) = \sigma_{\mathcal{W}}(C_p) = \Gamma_p$, from Corollary 4.1 (iii) we conclude that $\sigma_{gD(g\mathcal{M})\mathcal{W}_-}(C_p) = \sigma_{gD(g\mathcal{M})\mathcal{W}}(C_p) = \sigma_{gD(g\mathcal{M})\mathcal{S}}(C_p) = \sigma_{gD(g\mathcal{M})}(C_p) = \Gamma_p$.

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