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The Generalized Flanders' Theorem in Unit-regular Rings

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Abstract. Let R be a unit-regular ring, and let $a, b, c \in R$ satisfy $aba = aca$. If ac or ba is Drazin invertible, we prove that their Drazin inverses are similar. Furthermore, if ac and ba are group invertible, then ac is similar to ba. For any $n \times n$ complex matrices A, B, C with $ABA = ACA$, we prove that AC and BA are similar if and only if their k-powers have the same rank. These generalize the known Flanders' theorem proved by Hartwig.

1. Introduction

An element $a \in R$ has Drazin inverse if there exists an element $x \in R$ such that

$$ax = xa, \;\; xax = x, \;\; a^{k+1}x = a^k \; for\ some\ k \in \mathbb{N},$$

or equivalently,

$$ax = xa, \;\; xax = x, \;\; a^2x - a \in N(R),$$

where $N(R)$ denotes the set of all nilpotents in R. If a is Drazin invertible, the Drazin inverse of a is unique, denote x by a^D. The least nonnegative k which satisfies formulas above is called the index of a, denoted by $ind(a)$. If $ind(a)$ =1, a is said to be group invertible. In this case, the element x is called the group inverse of a and denoted by $a^\#$, that is ,

$$aa^\# = a^\# a, \;\; a^\# aa^\# = a^\#, \;\; aa^\# a = a.$$

We use $R^\#$ to stand for the set of all group invertible elements of R. Two elements $a, b \in R$ are similar, i.e., $a \sim b$, if there exists an invertible element s such that $a = s^{-1}bs$.

The known Flanders' theorem states that $(AB)^D$ is similar to $(BA)^D$ for any $n \times n$ matrices A and B over a field. In [9], Hartwig extended Flanders' Theorem. Let R be a strongly π-regular unit-regular ring and $a, b \in R$. He proved that $(ab)^D$ and $(ba)^D$ are similar. Cao and Li considered Flanders' theorem in a Bézout

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domain. Let R be a Bézout domain and $A, B \in R^{n\times n}$. If $(AB)^{\#}$ and $(BA)^{\#}$ exist, then AB is similar to BA (see [2, Theorem 3.6]). Afterwards, in [12, Theorem 2.2], Mihallović and Djordjević extended the proceeding results to the general ring setting. Deng [7, Theorem 2.6] also considered the case for operators on Hilbert spaces. Unfortunately, there are some gaps in their proofs. To be specific, for any $a, b \in R$, generally, the condition that $(ab)^{\#}$ and $(ba)^{\#}$ exist do not imply $ab \sim ba$. We will give a counter-example in the next section.

Recall that a ring R is unit-regular provided that for each $a \in R$, there is a unit $u \in R$ such that $aua = a$. For example, the ring of all $n \times n$ complex matrices is unit-regular. The main purpose of this paper is to give a generalized Flander's theorem in unit-regular rings. Let R be unit-regular, and let $a, b, c \in R$ satisfy $aba = aca$. If ac or ba is Drazin invertible, we prove that $(ac)^D \sim (ba)^D$. If ac and ba is group invertible, we further show that $ac \sim ba$. For any $n \times n$ complex matrices A, B, C with $ABA = ACA$, we prove that AC and BA are similar if and only if their k-powers have the same rank. Flanders' theorem is thereby extended to the case of triples (a, b, c) with $aba = aca$.

Throughout this paper, all rings are associative with an identity, the set of all invertible elements of R will be denoted by $U(R)$. $\mathbb{N}$ stands for the set of all natural numbers.

2. Main Results

We begin with a counter-example which infers that [12, Theorem 2.2] and [7, Theorem 2.6] are not true.

Example 2.1. Let V be an infinite dimensional vector space of a field $\mathbb{F}$, and let $R = End_{\mathbb{F}}(V)$. Let $\{x_1, x_2, \cdots, x_n, \cdots\}$ be a basis of V.

Definition

$$\sigma(x_i) = x_{i+1} \text{ for all } i \in \mathbb{N},$$
$$\tau(x_1) = 0, \ \ \tau(x_i) = x_{i-1} \text{ for all } i \geq 2.$$

Then $\sigma, \tau \in R$, and for any $i \in \mathbb{N}$,

$$\tau\sigma(x_i) = \tau(x_{i+1}) = x_i,$$

i.e., $\tau\sigma = 1_V$. Therefore $\tau\sigma$ is invertible in R, hence $\tau\sigma \in R^{\#}$.

Since $\sigma\tau(x_1) = \sigma(0) = 0$, we have $\sigma\tau \neq 1$. But $(\sigma\tau)^2 = \sigma(\tau\sigma)\tau = \sigma\tau$, $\sigma\tau$ is an idempotent, and so $\sigma\tau \in R^{\#}$.

We claim that $\tau\sigma \nsim \sigma\tau$, otherwise, there exists $s \in U(R)$ such that $\tau\sigma \cdot s = s \cdot \sigma\tau$ which implies $\sigma\tau = 1$, a contradiction. □

Lemma 2.2. *Let $a, b, c \in R$ satisfy $aba = aca$.*

(1) *If $(ac)^D$ or $(ba)^D$ exists, then*

$$(ba)^D = b[(ac)^D]^2 a, \ \ (ac)^D = a[(ba)^D]^2 c,$$

$$a(ba)^D = (ac)^D a, \ \ ab(ac)^D = ac(ac)^D.$$

(2) *If $(ac)^{\#}$ and $(ba)^{\#}$ exist, then*

$$(ba)^{\#} = b[(ac)^{\#}]^2 a, \ \ (ac)^{\#} = a[(ba)^{\#}]^2 c,$$

$$ab(ac)^{\#} = ac(ac)^{\#}, \ \ a(ba)^{\#} = (ac)^{\#} a.$$

Proof. (1) In view of [13, Theorem 2.7], we have $(ba)^D = b[(ac)^D]^2 a$ and $(ac)^D = a[(ba)^D]^2 c$. Moreover, we assume that $(ac)^D$ exists, we get

$$a(ba)^D = ab[(ac)^D]^2 a = ac[(ac)^D]^2 a = (ac)^D a,$$

$$ab(ac)^D = abac[(ac)^D]^2 = acac[(ac)^D]^2 = ac(ac)^D.$$

(2) Suppose that $(ac)^{\#}$ and $(ba)^{\#}$ exist. Then $(ac)^{\#} = (ac)^D$ and $(ba)^{\#} = (ba)^D$, we obtain the result by (1). □

We come now to extend Flanders' theorem to unit-regular rings.

Theorem 2.3. *Let R be a unit-regular ring and let $a, b, c \in R$ with $aba = aca$. If $(ac)^D$ or $(ba)^D$ exists, then $(ac)^D \sim (ba)^D$. In this case, $(ac)^2(ac)^D \sim (ba)^2(ba)^D$.*

Proof. Without loss of generality, assume that $(ac)^D$ exists, by virtue of Lemma 2.2, we have $(ba)^D = b[(ac)^D]^2a = b(ac)^D(ac)^Da$. Let

$$x = b(ac)^D, \quad y = ac(ac)^Da.$$

Then we check that

$$\begin{aligned} x(ac)^Dy &= b(ac)^D(ac)^Dac(ac)^Da = (ba)^D; \\ y(ba)^Dx &= ac(ac)^Da(ba)^Db(ac)^D = ac(ac)^D(ac)^Dac(ac)^D = (ac)^D; \\ xyx &= b(ac)^Dac(ac)^Dab(ac)^D = b(ac)^Dac(ac)^D = x; \\ yxy &= ac(ac)^Dab(ac)^Dac(ac)^Da = ac(ac)^Dac(ac)^Da = y; \end{aligned}$$

Since R is unit-regular, we have $x = xvx$ for some $v \in U(R)$. Set

$$u = (1 - xy - xv)v^{-1}(1 - yx - vx).$$

Since $(1 - yx - vx)^2 = 1$ and $(1 - xy - xv)^2 = 1$, we verify that

$$(1 - xy - xv)v^{-1}(1 - yx - vx)^2v(1 - xy - xv) = 1,$$

$$(1 - yx - vx)v(1 - xy - xv)^2v^{-1}(1 - yx - vx) = 1,$$

i.e., u is invertible in R. Furthermore, we have

$$u^{-1} = (1 - yx - vx)v(1 - xy - xv) = v - vxv + y.$$

We check that

$$\begin{aligned} (ac)^Du^{-1} &= y(ba)^Dxv(1 - xv) + (ac)^Dac(ac)^Da = (ac)^Da, \\ u^{-1}(ba)^D &= (1 - vx)vx(ac)^Dy + ac(ac)^Da(ba)^D = (ac)^Da. \end{aligned}$$

Therefore

$$(ac)^D = u^{-1}(ba)^Du.$$

i.e., $(ac)^D \sim (ba)^D$.

Accordingly, by [9, Theorem 1], $(ac)^2(ac)^D \sim (ba)^2(ba)^D$. □

Corollary 2.4. *Let R be a unit-regular ring and let $a, b, c \in R$ with $aba = aca$. If $(ac)^\#$ and $(ba)^\#$ exist, then $(ac)^\# \sim (ba)^\#$.*

Proof. Since $(ac)^\#$ and $(ba)^\#$ exist, then $(ac)^D = (ac)^\#$ and $(ba)^D = (ba)^\#$. So this is a direct consequence of Theorem 2.3. □

Corollary 2.5. *Let $A, B, C \in \mathbb{C}^{n\times n}$ with $ABA = ACA$. If $(AC)^\#$ and $(BA)^\#$ exist, then $(AC)^\# \sim (BA)^\#$.*

Proof. By [4, Corollary 4.5], $\mathbb{C}^{n\times n}$ is unit-regular. Therefore we complete the proof by Corollary 2.4. □

Contract to Corollary 2.4, we now derive

Theorem 2.6. *Let R be a unit-regular ring and let $a, b, c \in R$ satisfy $aba = aca$. If $(ac)^\#$ and $(ba)^\#$ exist, then $ac \sim ba$.*

Proof. Assume that $(ac)^\#$ and $(ba)^\#$ exist. Let $x = b(ac)^\#$ and $y = ac(ac)^\#a$. Since R is unit-regular, there exists some $v \in U(R)$ such that $x = xvx$. Set

$$u = (1 - xy - xv)v^{-1}(1 - yx - vx).$$

As in the proof of Theorem 2.3, we prove that $u \in U(R)$. Moreover,

$$(ac)^\#u^{-1} = (ac)^\#a \text{ and } u^{-1}(ba)^\# = (ac)^\#a.$$

Multiplying the first equality by $(ac)^2$ from the left, we obtain

$$(ac)u^{-1} = aca.$$

Multiplying the second equality by $(ba)^2$ from the right, we have

$$u^{-1}(ba) = (ac)^{\#}a(ba)^2 = (ac)^{\#}acaca = aca.$$

This implies that $ac \sim ba$, as asserted. □

Corollary 2.7. *Let $A, B, C \in \mathbb{C}^{n\times n}$ with $ABA = ACA$. If $(AC)^{\#}$ and $(BA)^{\#}$ exist, then $AC \sim BA$.*

Proof. Since $\mathbb{C}^{n\times n}$ is unit-regular, the result follows by Theorem 2.6. □

Bu and Cao proved that $AB \sim BA$ if AB and BA have group inverses(see [1, Corollary 4]). Corollary 2.7 is a nontrivial generalization of this result as the following shows.

Example 2.8. Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$ in $\mathbb{C}^{2\times 2}$. Then $ABA = \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix} = ACA$. We compute that

$$AC = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},\ BA = \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix};$$

$$(AC)^{\#} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix}, (BA)^{\#} = \begin{bmatrix} 0 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix}.$$

Then $AC = S^{-1}BAS$, $(AC)^{\#} = S^{-1}(BA)^{\#}S$, where $S = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$. Therefore $AC \sim BA$, $(AC)^{\#} \sim (BA)^{\#}$. But $B \neq C$. □

Theorem 2.9. *Let R be a unit-regular ring and let $a, b, c \in R$ satisfy $aba = aca$. If $(ac)^D$ or $(ba)^D$ exists, then $(ac)^s \sim (ba)^s$ for all $s \geq \max\{ind(ac), ind(ba)\}$.*

Proof. By Lemma 2.2, if $(ac)^D$ or $(ba)^D$ exists, then both $(ac)^D$ and $(ba)^D$ exist. Construct u as in Theorem 2.3, by the proof of Theorem 2.3,

$$(ac)^D u^{-1} = (ac)^D a,\ \ u^{-1}(ba)^D = (ac)^D a.$$

Multiplying the first equality by $(ac)^{s+1}$ from the left, we obtain

$$(ac)^s u^{-1} = (ac)^s a.$$

Multiplying the second equality by $(ba)^2$ from the right, we have

$$u^{-1}(ba)^s = (ac)^s a.$$

Therefore $(ac)^s \sim (ba)^s$. □

Corollary 2.10. *Let $A, B, C \in \mathbb{C}^{n\times n}$ with $ABA = ACA$. If $s \geq \max\{ind(AC), ind(BA)\}$, then $(AC)^s \sim (BA)^s$.*

Proof. Since $\mathbb{C}^{n\times n}$ is unit-regular, $A, B, C \in \mathbb{C}^{n\times n}$ with $ABA = ACA$, AC and BA are Drazin invertible, by using Theorem 2.9, $(AC)^s \sim (BA)^s$ for all $s \geq \max\{ind(AC), ind(BA)\}$. □

Corollary 2.11. *Let $A, B \in \mathbb{C}^{n\times n}$. If $s \geq \max\{ind(AB), ind(BA)\}$, then $(AB)^s \sim (BA)^s$.*

Proof. This is obvious by choosing "$B = C$" in Corollary 2.10. □

The following example illustrates Corollary 2.10 is a nontrivival generalization of [9, Corollary 2].

Example 2.12. Let $A = B = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in \mathbb{C}^{2\times 2}$. Then $ABA = ACA$, while $B \neq C$. In this case, $ind(AC) = ind(BA) = 1$. In view of Corollary 2.10, $(AC)^s \sim (BA)^s$ for all $s \geq 1$. Evidently, $AC = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ and $BA = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ are idempotent. Let $U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Then $(AC)^s = U^{-1}(BA)^s U$ for all $s \in \mathbb{N}$. □

As an application of Theorem 2.3, we now ready to prove

Theorem 2.13. *Let $A, B, C \in \mathbb{C}^{n\times n}$ with $ABA = ACA$. Then $AC \sim BA$ if and only if $rank(AC)^k = rank(BA)^k$ for $k = 1, 2, \cdots$.*

Proof. The necessity is obvious. For the sufficiency, there exist two invertible matrices P and Q such that

$$PACP^{-1} = \begin{pmatrix} U_1 & \\ & N_1 \end{pmatrix}, \quad QBAQ^{-1} = \begin{pmatrix} U_2 & \\ & N_2 \end{pmatrix}$$

where U_1, U_2 are invertible, and N_1, N_2 are nilpotent. Choose $s = ind(N_1) + ind(N_2)$, then

$$P(AC)^s P^{-1} = \begin{pmatrix} U_1^s & \\ & O \end{pmatrix}, \quad Q(BA)^s Q^{-1} = \begin{pmatrix} U_2^s & \\ & O \end{pmatrix}.$$

Since $rank(AC)^s = rank(BA)^s$, we see that U_1 and U_2 have the same rank. It is easy to check that

$$(AC)^D = P^{-1}\begin{pmatrix} U_1^{-1} & \\ & O \end{pmatrix}P, \quad (BA)^D = Q^{-1}\begin{pmatrix} U_2^{-1} & \\ & O \end{pmatrix}Q.$$

In view of Theorem 2.3, we have

$$(AC)^D \sim (BA)^D,$$

i.e.,

$$\begin{pmatrix} U_1^{-1} & \\ & O \end{pmatrix} \sim \begin{pmatrix} U_2^{-1} & \\ & O \end{pmatrix},$$

which follows $U_1^{-1} \sim U_2^{-1}$, and so $U_1 \sim U_2$. Moreover, as $rank(AC)^k = rank(BA)^k$, $rank(N_1)^k = rank(N_2)^k$ for all positive integers k. Since N_1, N_2 are nilpotent, by the Jordan forms of N_1, N_2, we have that N_1 and N_2 have the same Jordan forms. Hence $N_1 \sim N_2$. Therefore $AC \sim BA$, as asserted . □

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