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Conformable Fractional Fourier Transformation of Tempered Distributions

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Abstract. In this paper we introduce a new definition of fractional Fourier transformation on the space S of Schwartz test functions and study some of its properties. It turns out that this fractional Fourier transform has many properties with the conformable fractional derivative that the conventional Fourier transform has with the conventional (standard) derivative. We establish some operational formulas for the new transform, and give a left inverse for it. We use duality to define fractional Fourier transform of tempered distributions. Finally, we give two applications to ordinary and partial differential equations.

1. Introduction

The fractional Fourier transformation has been invented and studied several decades ago. Because of the new applications of the transformation, it has been put under considerable study in the last two decades. These applications include signal and image processing, physics, optimal controle, engineering applications, and many other up-to-date applications. Depending on the application, there are in the literature several definitions for the fractional Fourier transform. The recent studies of fractional Fourier transform started with the work of Namias [8], and McBride and Kerr [7]. Zayed [9] has defined fractional Fourier transform on some spaces of functions and extended the definition to generalized functions. He accomplished that using two approaches, one is called the algebraic approach and the other one is called the analytic (embedding) approach. The algebraic approach involves the theory of Boehmians. In the analytic approach, the fractional Fourier transform is defined first for distributions with compact support, and then extended it to larger spaces by continuity of the fractional Fourier transform. The basic idea of the proceedure is the Hilbert space eigenfunction expansion. Khan et al. [4] adopted Zayed's approach and produced a theory of fractional Fourier transform using unbounded differential operator on some subspaces of L^2. Luchko et al. [6] gave a different definition of fractional Fourier transform on the Lizorkin space of test functions, and established several operational relations for the transform. They developed operational relations taking into consideration the Riemann-Liouville fractional derivatives. They also gave application using the fractional Fourier transform to solve some fractional partial differential equations.

Kilbas at al. [5] continued the work done in [6]. They gave a left inverse of their fractional Fourier transform, and studied composition of the fractional Fourier transform with fractional differentiation and

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fractional integration. They solved some fractional partial differential equations as application to their results. In the litrature there are several definitions for the fractional derivative. A recent definition was given by Khalil et al. [3], which they called conformable fractional derivative. The conformable fractional derivative satisfies almost all the differentiation formulas satisfied by the traditional integer order derivative (see [8], [1]). The conformable derivative has been used in many applications including ordinary and partial differential equations and optimal controle problems (see [2])

In this paper we give a new definition for the fractional Fourier transform on the space $\mathcal{S}$ of Schwartz test functions. We called it conformable fractional Fourier transform because we considered the operational relations with the conformable fractional derivative of Khalil et al. [3]. We established some operational relations, and proved that the fractional Fourier transform is continuous from $\mathcal{S}$ to itself. We also give a left inverse for the transformation. As an application, we used the fractional Fourier transform to solve an ordinary differential equation and a one dimentional fractional heat equation. Finally, we use duality to define the fractional Fourier transform of tempered distributions. We also prove continuity of the transform from S' to S' and some other properties.

The rest of the paper goes as follows. In Section 2 we give preliminary definitions and results that will be used in the sequel. In Section 3 we give our definition of conformable fractional Fourier transform, establish some of its properties, give its left inverse, and derive some operational formulas. Section 4 consists of two applications.

2. Preliminaries

The space $\mathcal{S}$ of Scwartz test functions is the space of all functions $\varphi \in C^\infty(\mathcal{R})$ such that

$$\upsilon_k(\varphi) = \sup_{\substack{m\leq k \\ x\in\mathcal{R}}}(1+ \mid x \mid^2)^k \mid D^m\varphi(x) \mid< \infty;\ k = 1, 2, 3, ... \tag{2.1}$$

The space $\mathcal{S}$ with semi-norms υ_k, $k = 1, 2, 3, ...$is a Frechet space and the space $\mathcal{D}$ of test functions of compact support is dense in $\mathcal{S}$. Moreover, the space $\mathcal{S}$ is Montel. The strong dual of $\mathcal{S}$ is the space $\mathcal{S}'$ of tempered distributions, which is provided with the strong dual topology. A sequence (T_j) converges to 0 in $\mathcal{S}'$ if it converges uniformly on every bounded subset of $\mathcal{S}$.

There are several definitions for the fractional derivative of functions. We adopt the definition given by Khalil et al. [3], which is called conformable fractional derivative. This definition has been the base of many publications because, for one reason, the proofs of several results go similar to those of the conventional integer order derivative (see [3]). We consider the fractional derivatives of order $\alpha \in [0, 1)$. If $\beta > 1$, the fractional derivative of order β is defined to be the fractional derivative of order $\alpha \in [0, 1)$, where $\alpha = \beta - n$, where n is the greatest integer equal or less than β.

Definition 1. *Let $f : [0, \infty) \to R$. Then the conformable fractional derivative of f of order α is defined by*

$$T^\alpha(f)(t) = \lim_{\epsilon\to 0}\frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}$$

for all $t > 0, \alpha \in [0, 1)$. We also denote $T^\alpha(f)$ by $f^{(\alpha)}$.

In the litresure, seversl results for the coformable fractional derivative, like linearity of the new derivative, the derivative of the constant is zero, the product and quotient rules, Rolle's theorem and the mean value theorem. Several authors proved results which are true for the conventional integer derivative. In this direction we have

Theorem 1. *Cauchy Mean Value Theorem for the Conformable Fractional Derivative.*

Let $a > 0$. Let f and g be continuous on $[a, b]$ and $\alpha-$ differentiable on (a, b) for some $\alpha \in (0, 1)$, and assume that $g^{(\alpha)}(x) \neq 0$ for all x in (a, b). Then there exists a point c in (a, b) such that

$$\frac{f^{(\alpha)}(c)}{g^{(\alpha)}(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Proof. *Define the function*

$$h(x) = \frac{f(b) - f(a)}{g(b) - g(a)}(g(x) - g(a)) - (f(x) - f(a)); x \in [a, b].$$

Since f and g are continuous on $[a, b]$ and $\alpha-$ differentiable on (a, b), it follows that the same is true for h. It follows from the fractional form of Rolle's theorem [8] that there exists a point c in (a, b) such that $h^{(\alpha)}(c) = 0$. This gives

$$h^{(\alpha)}(c) = \frac{f(b) - f(a)}{g(b) - g(a)} g^{(\alpha)}(c) - f^{(\alpha)}(c) = 0,$$

which gives the result. □

Theorem 2. *L'Hospital's Rule. Let $0 < a \leq b < \infty$ and let the functions f, g be $\alpha-$differentiable on (a, b), such that $g^{(\alpha)}(x) \neq 0$ for all $x \in (a, b)$. Suppose that*

$$\lim_{x \to a^+} f(x) = 0 = \lim_{x \to a^+} g(x).$$

(a) If $\lim_{x \to a^+} \frac{f^{(\alpha)}(x)}{g^{(\alpha)}(x)} = L \in R$,then $\lim_{x \to a^+} \frac{f(x)}{g(x)} = L$.

(b) If $\lim_{x \to a^+} \frac{f^{(\alpha)}(x)}{g^{(\alpha)}(x)} = L \in [-\infty, \infty]$,then $\lim_{x \to a^+} \frac{f(x)}{g(x)} = L$.

Proof. As in the usual case, the proof uses Rolles' theorem and the Cauchy mean value theorem for $\alpha-$ fractional derivatives, and will be omitted. □

3. Conformable Fractional Fourier Transform

Definition 2. *Let $\varphi \in \mathcal{S}$ and $0 < \alpha < 1$.We define the conformable fractional Fourier transform of φ, denoted by $\mathcal{F}_\alpha(\varphi)$, as follows*

$$\mathcal{F}_\alpha(\varphi)(x) = \int_{-\infty}^{\infty} e^{-i\frac{x^\alpha}{\alpha}t} \varphi(t) dt, x \in \mathcal{R}. \tag{3.1}$$

We note that the integral is absolutely convergent because $\varphi \in \mathcal{S}$, and when the function $\frac{x^\alpha}{\alpha}$ is multi-valued complex function we take its principal branch.

It follows right away from the definition that $\mathcal{F}_\alpha(\varphi)(x) = \mathcal{F}(\varphi)(\frac{x^\alpha}{\alpha})$. This enables one to find the conformable fractional Fourier transform of functions whose conventional Fourier transform is known when the values of α and x are appropriate.

Example 1. *Consider the pulse $f(t) = c$ if $| t | \leq A$, $f(t) = 0$ if $| t | > A$, where c and A are positive constants. We have $\mathcal{F}_\alpha(f(t))(x) = -2ic\alpha \frac{1}{x^\alpha} \sinh(i\frac{x^\alpha}{\alpha} A)$.*

Example 2. *Consider the function* $f(t) = e^{-|t|}$*. By carrying out the calculations one finds that* $\mathcal{F}_\alpha(f(t))(x)$ *is defined for x and* α *which make* $-1 < i\frac{x^\alpha}{\alpha} < 1$*. The conventional Fourier transform of this function is defined for x where* $-1 < ix < 1$.

Next, we examine some properties of the conformable fractional Fourier transform $\mathcal{F}_\alpha$. Being an integral transform, it follows that $\mathcal{F}_\alpha$ is linear. It is also injective, because if $\mathcal{F}_\alpha(\varphi) = \mathcal{F}_\alpha(\psi); \varphi, \psi \in \mathcal{S}$, then $\varphi = \psi$ *a.e.*Hence $\varphi = \psi$ because they are continuous. Next, we consider conformable fractional derivative of translation and dilation. We recall that the translation operator is defined by $(\tau_h\varphi)(x) = \varphi(x-h)$ and the dilation of φ by $\lambda; \lambda > 0$, is defined by $(\Pi_\lambda\varphi)(x) = \varphi(\lambda x)$.

Lemma 1. *Let* $\varphi \in \mathcal{S}$ *and* $0 < \alpha < 1$.*For* $h \in \mathcal{R}$ *and* $\lambda > 0$, *we have*

$$\begin{aligned}\mathcal{F}_\alpha(\tau_h\varphi)(x) &= e^{-i\frac{x^\alpha}{\alpha}h}\mathcal{F}_\alpha(\varphi)(x);\\ \mathcal{F}_\alpha(\Pi_\lambda\varphi)(x) &= \frac{1}{\lambda}\mathcal{F}_\alpha(\varphi)(\frac{x}{\lambda^{\frac{1}{\alpha}}})\end{aligned}$$

Proof. Both formulas follow right away from the definition by simple change of variables. □

Lemma 2. *Let* $\varphi \in \mathcal{S}$ *and* $0 < \alpha < 1$. *Then*

$$D(\mathcal{F}_\alpha(\varphi))(x) = -ix^{\alpha-1}\mathcal{F}_\alpha(t\varphi)(x); x \neq 0. \tag{3.2}$$

Proof. The result follows by interchanging the derivative with the integral, integrating by parts and recalling that $\lim_{|t|\to\infty}\varphi(t) = 0$. □

We remark that further calculations show that

$$D^2(\mathcal{F}_\alpha(\varphi))(x) = -i(\alpha-1)x^{\alpha-2}\mathcal{F}_\alpha(t\varphi)(x) - x^{2\alpha-2}\mathcal{F}_\alpha(t^2\varphi)(x); x \neq 0.$$

One can continue to calculate higher derivatives of $\mathcal{F}_\alpha(\varphi)$. The calculations give more and more terms without a general formula.

Lemma 3. *Let* $\varphi \in \mathcal{S}$, $0 < \alpha < 1$, *and* $m \in N$. *Then*

$$\mathcal{F}_\alpha(D^m\varphi)(x) = (i\frac{x^\alpha}{\alpha})^m\mathcal{F}_\alpha(\varphi)(x). \tag{3.3}$$

Proof. For $m = 1$ the result follows by integrating by parts and recalling that $\lim_{|t|\to\infty}\varphi(t) = 0$. The general result follows by induction. □

Related to the above lemmas one has

$$\mathcal{F}_\alpha(tD\varphi(t))(x) = \mathcal{F}_\alpha(\varphi(t))(x) + i\frac{x^\alpha}{\alpha}\mathcal{F}_\alpha(t\varphi(t))(x)$$

and

$$\mathcal{F}_\alpha(t^2D^2\varphi(t))(x) = -\mathcal{F}_\alpha(\varphi(t))(x) - (i+2)\frac{x^\alpha}{\alpha}\mathcal{F}_\alpha(t\varphi(t))(x) - (\frac{x^\alpha}{\alpha})^2\mathcal{F}_\alpha(t^2\varphi)(x).$$

Theorem 3. *Let* $0 < \alpha < 1$. *Then* $\mathcal{F}_\alpha$ *maps* $\mathcal{S}$ *continuously into itself.*

Proof. Let $\varphi \in \mathcal{S}$ and k, p be any nonnegative integers. We show that there exists $l \in N$ such that for any $\varphi \in \mathcal{S}$,

$$\rho_{k,p}(\varphi) = \sup_{\xi\in\mathcal{R}, m\leq p} (1+ \mid \xi \mid^2)^{\frac{k}{2}} \mid D^m(\mathcal{F}_\alpha(\varphi)(\xi) \mid \leq \rho_l(\varphi).$$

It follows from Lemmas 2 and 3 and the comments after them that, for any $m \leq p$ there exist three polynomials P, q and Q such that

$$(1+ \mid \xi \mid^2)^{\frac{k}{2}} \mid D^m(\mathcal{F}_\alpha(\varphi)(\xi) \mid \leq \mid P(\xi) \mid\mid \mathcal{F}_\alpha(q(x)\varphi(x))(\xi) \mid \quad (3.4)$$

$$= \mid \mathcal{F}_\alpha(Q(x)\varphi(x))(\xi) \mid = \mid \int_{-\infty}^{\infty} e^{-i\frac{\xi}{\alpha}.x} Q(x)\varphi(x)dx \mid; \quad (3.5)$$

$$\leq \int_{-\infty}^{\infty} \mid Q(x)\varphi(x) \mid dx; \quad (3.6)$$

$$\leq \sup_{x\in R}(1+ \mid x \mid^2)^l \mid \varphi(x) \mid \leq \upsilon_l(\varphi); \quad (3.7)$$

because $Q(x)\varphi(x)$ is in $\mathcal{S}$, where l is a large enough positive integer.This completes the proof of the theorem. □

The next result assures that the conformable fractional Fourier transform behaves like the usual Fourier transform with convolution product. More precisely, the fractional Fourier transform of the convolution of two functions is the product of their fractional Fourier transforms.

Theorem 4. *Let $\varphi, \psi \in \mathcal{S}$ and $0 < \alpha < 1$. Then $\mathcal{F}_\alpha(\varphi * \psi) = \mathcal{F}_\alpha(\varphi)\mathcal{F}_\alpha(\psi)$.*

Proof. By definition we have

$$\mathcal{F}_\alpha(\varphi * \psi) = \int_{-\infty}^{\infty} e^{-i\frac{x^\alpha}{\alpha}t}(\varphi * \psi)(t)dt = \int_{-\infty}^{\infty} e^{-i\frac{x^\alpha}{\alpha}t} \int_{-\infty}^{\infty} \varphi(t-y)\psi(y)dydt.$$

By interchanging the order of integration and making the change of variable $t = y + \tau$, the above equality gives

$$\begin{aligned}\mathcal{F}_\alpha(\varphi * \psi)(x) &= \int_{-\infty}^{\infty} e^{-i\frac{x^\alpha}{\alpha}\tau}\varphi(\tau)d\tau \int_{-\infty}^{\infty} e^{-i\frac{x^\alpha}{\alpha}y}\psi(y)dy \\ &= \mathcal{F}_\alpha(\varphi)(x)\mathcal{F}_\alpha(\psi)(x).\end{aligned}$$

□

Now we examine some operational formulas of the conformable fractional Fourier transform with the conformable fractional derivative.

The first formula gives the conformable fractional derivative of the fractional Fourier transform.

Proposition 1. *Let $\varphi \in \mathcal{S}$ and $0 < \alpha < 1$.Then*

$$T^\alpha(\mathcal{F}_\alpha(\varphi)(x)) = -i\mathcal{F}_\alpha(t\varphi(t))(x).$$

Proof. By definition, and interchanging the conformable derivative with integration, one gets

$$\begin{aligned}
T^{\alpha}(\mathcal{F}_{\alpha}(\varphi)(x)) &= T^{\alpha}\int_{-\infty}^{\infty} e^{-i\frac{x^{\alpha}}{\alpha}t}\varphi(t)dt; \\
&= \int_{-\infty}^{\infty} T^{\alpha}(e^{-i\frac{x^{\alpha}}{\alpha}t}\varphi(t))dt; \\
&= \int_{-\infty}^{\infty} x^{1-\alpha}\frac{d}{dx}e^{-i\frac{x^{\alpha}}{\alpha}t}\varphi(t)dt; \\
&= \int^{\infty} -ite^{-i\frac{x^{\alpha}}{\alpha}t}\varphi(t)dt; \\
&= (-i)(\mathcal{F}_{\alpha}(t\varphi(t)(x).
\end{aligned}$$

□

The second formula gives the conformable fractional Fourier transform of the conformable fractional derivative of functions in $\mathcal{S}$.

Proposition 2. *Let* $\varphi \in \mathcal{S}$ *and* $0 < \alpha < 1$. *Then*

$$\mathcal{F}_{\alpha}(T^{\alpha}\varphi(t))(x) = ix^{\alpha-1}\mathcal{F}_{\alpha}(t^{2-\alpha}\varphi(t))(x) + \mathcal{F}_{\alpha}(t^{-\alpha}\varphi(t))(x).$$

Proof. By definition, we have

$$\begin{aligned}
\mathcal{F}_{\alpha}(T^{\alpha}\varphi(t))(x) &= \int_{-\infty}^{\infty} e^{-i\frac{x^{\alpha}}{\alpha}t}(T^{\alpha}\varphi)(t)dt; \\
&= \int_{-\infty}^{\infty} e^{-i\frac{x^{\alpha}}{\alpha}t}t^{1-\alpha}\frac{d}{dt}\varphi(t)dt \ \text{ because } \varphi \text{ is differentiable.}
\end{aligned}$$

Integrating by parts and recalling that $\lim_{|t|\to\infty}\varphi(t) = 0$, the above equality becomes

$$\mathcal{F}_{\alpha}(T^{\alpha}\varphi(t))(x) = ix^{\alpha-1}\int_{-\infty}^{\infty} e^{-i\frac{x^{\alpha}}{\alpha}t}t^{2-\alpha}\varphi(t)dt + \int_{-\infty}^{\infty} e^{-i\frac{x^{\alpha}}{\alpha}t}t^{-\alpha}\varphi(t)dt \tag{3.8}$$

$$= ix^{\alpha-1}\mathcal{F}_{\alpha}(t^{2-\alpha}\varphi(t))(x) + \mathcal{F}_{\alpha}(t^{-\alpha}\varphi(t))(x) \tag{3.9}$$

□

We remark that putting $\alpha = 1$ in equality (3.9) does not give the corresponding formula for the usual Fourier transform.

In the next definition we introduce an operator which turns out to be a left inverse of the conformable fractional Fourier transform.

Definition 3. *Let* $0 < \alpha < 1$.*We define the operator* $\mathcal{G}_\alpha$ *on the space* $\mathcal{F}_\alpha(\mathcal{S})$ *as follows*
$\mathcal{G}_\alpha(\psi)(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\frac{x^\alpha}{\alpha}t}x^{\alpha-1}\psi(x)dx.$

We remark that whenever the function x^β is multi-valued then we take its principal branch. The following theorem establishes the fact that $\mathcal{G}_\alpha$ is a left inverse of $\mathcal{F}_\alpha$.

Theorem 5. *Let* $0 < \alpha < 1$. *Then for any* $\varphi \in \mathcal{S}$ *we have*

$$\mathcal{G}_\alpha(\mathcal{F}_\alpha(\varphi)(t) = \varphi(t), \ \ t \in \mathcal{R}.$$

Proof. By definition of $\mathcal{G}_\alpha$ we have,

$$\mathcal{G}_\alpha(\mathcal{F}_\alpha(\varphi)(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\frac{x^\alpha}{\alpha}t}x^{\alpha-1}\mathcal{F}_\alpha(\varphi)(x)dx;$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\frac{x^\alpha}{\alpha}t}x^{\alpha-1}\int_{-\infty}^{\infty} e^{-i\frac{x^\alpha}{\alpha}y}\varphi(y)dydx. \tag{3.10}$$

Making the change of variable $u = \frac{x^\alpha}{\alpha}$ in equality (3.10), we get

$$\begin{aligned}\mathcal{G}_\alpha(\mathcal{F}_\alpha(\varphi)(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iut}\int_{-\infty}^{\infty} e^{-iuy}\varphi(y)dydu;\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iut}\widehat{\varphi}(u)du = \varphi(t);\end{aligned}$$

where $\wedge$ denotes the usual (traditional) Fourier transform. □

The follwing corollary follows from Theorems 4 and 5 .

Corollary 1. *Let* $\varphi, \psi \in \mathcal{S}$ *and* $0 < \alpha < 1$. *Then* $\mathcal{G}_\alpha(\mathcal{F}_\alpha(\varphi)\mathcal{F}_\alpha(\psi)) = \varphi * \psi$.

We remark that $\mathcal{G}_\alpha$ is not a right inverse of $\mathcal{F}_\alpha$. Actually $\mathcal{F}_\alpha\mathcal{G}_\alpha(\psi)(x) = \psi(\frac{x^\alpha}{\alpha})$. One might also think of the operator $\mathcal{H}_\alpha(\varphi)(x) = \int_{-\infty}^{\infty} e^{i\frac{x^\alpha}{\alpha}x}\varphi(x)dx; \varphi \in \mathcal{S}$. We have $\mathcal{F}_\alpha\mathcal{H}_\alpha(\varphi)(x) = \varphi(\frac{x^\alpha}{\alpha})$, but it does not behave any good from the left. Since one of the very important applications of fractional Fourier transform is solving differential equations that can not be solved using other methods, what is useful is the left inverse of $\mathcal{F}_\alpha$ not the right inverse.

Problem 1. *Let* $\varphi, \psi \in \mathcal{S}$ *and* $0 < \alpha < 1$. *Is it true that* $\mathcal{F}_\alpha(\varphi\psi) = \mathcal{F}_\alpha(\varphi) * \mathcal{F}_\alpha(\psi)$. *This problem is interesting because if it is true then, together with Theorem 4, we will have a convolution theorem for the conformable* $\alpha-$*fractional Fourier transform.*

The next result establishes continuity of the conformable fractional Fourier transform as function of the index.

Theorem 6. *Let $\alpha, \beta \in (0,1)$. Then $\lim_{\beta\to\alpha} \mathcal{F}_\beta(\varphi) = \mathcal{F}_\alpha(\varphi)$ in $\mathcal{S}$ for all $\varphi \in \mathcal{S}$.*

Proof. It suffices to show that $\lim_{\beta\to\alpha} | x^m D^n(\mathcal{F}_\beta(\varphi)(x) - x^m D^n(\mathcal{F}_\alpha(\varphi)(x) |= 0$, where m, n are any nonnegative integers. We recall that $x^m D^n(\mathcal{F}_\alpha(\varphi)(x) = \int_{-\infty}^{\infty} e^{-i\frac{x^\alpha}{\alpha}t} D^m(t^{k_n}\varphi(t))dt$;where k_n is a positive integer which depends on n. Indeed

$$\begin{aligned}
\lim_{\beta\to\alpha} \quad | \quad & x^m D^n(\mathcal{F}_\beta(\varphi)(x) - x^m D^n(\mathcal{F}_\alpha(\varphi)(x) |; \\
= \quad & \lim_{\beta\to\alpha} | \int_{-\infty}^{\infty} (e^{-i\frac{x^\beta}{\beta}t} - e^{-i\frac{x^\alpha}{\alpha}t}) D^m(t^{k_n}\varphi(t))dt | \\
= \quad & \lim_{\beta\to\alpha} | \int_{-\infty}^{\infty} (\sum_{l=0}^{\infty} \frac{(-i)^l}{l!}((\frac{x^\beta}{\beta})^l - (\frac{x^\alpha}{\alpha})^l) | t |^l | D^m(t^{k_n}\varphi(t)) | dt; \\
\leq \quad & \sum_{l=0}^{\infty} \lim_{\beta\to\alpha} | \int_{-\infty}^{\infty} \frac{1}{l!} | (\frac{x^\beta}{\beta})^l - (\frac{x^\alpha}{\alpha})^l) || D^m(t^{k_n}\varphi(t)) | dt.
\end{aligned}$$

Since $((\frac{x^\beta}{\beta})^l - (\frac{x^\alpha}{\alpha})^l)$ converges to 0 as $\beta \to \alpha$ and $D^m(t^{k_n}\varphi(t))$ is in $\mathcal{S}$ it follows that the right hand side of the last inequality converges to 0 as $\beta \to \alpha$. □

Now, we are in a position to define the conformabel fractional Fourier transform of tempered distributions. As in the case of usual Fourier transform it is done by duality.

Definition 4. *Let $0 < \alpha < 1$ and let $T \in \mathcal{S}'$.Then the conformable fractional Fourier transform of T is defined by the equality*

$$\langle \mathcal{F}_\alpha T, \varphi \rangle = \langle T, \mathcal{F}_\alpha \varphi \rangle, \varphi \in \mathcal{S}.$$

Theorem 7. *The conformable fractional Fourier transform is continuous on $\mathcal{S}'$.*

Proof. Since $\mathcal{S}'$ is Montel space it suffices to show that $\mathcal{F}_\alpha$ is sequentially continuous on $\mathcal{S}'$. Let (T_j) be a sequence which converges to 0 in $\mathcal{S}'$. We show that the sequence $(\mathcal{F}_\alpha T_j)$ converges to 0 in $\mathcal{S}'$. Let B be a bounded subset of $\mathcal{S}$. Since $\mathcal{F}_\alpha$ is continuous on $\mathcal{S}$ (Theorem 1) it follows that $\mathcal{F}_\alpha(B)$ is bounded in $\mathcal{S}$.Thus

$$\left\langle \mathcal{F}_\alpha T_j, \varphi \right\rangle = \left\langle T_j, \mathcal{F}_\alpha \varphi \right\rangle \to 0 \ \text{uniformly in } \varphi \in B.$$

□

We can also define $\mathcal{G}_\alpha$ on $\mathcal{S}'$by duality, and we have

$$\langle \mathcal{F}_\alpha(\mathcal{G}_\alpha(T), \varphi \rangle = \langle T, \mathcal{G}_\alpha(\mathcal{F}_\alpha(\varphi)) \rangle = \langle T, \varphi \rangle \quad \text{for all } \varphi \text{ in } \mathcal{S}.$$

That is $\mathcal{G}_\alpha$ is a right inverse of $\mathcal{F}_\alpha$ on $\mathcal{S}'$.

Example 3. *Find $\mathcal{F}_\alpha(\delta)$, where $0 < \alpha < 1$ and δ is the Dirac function.*

Let $\varphi \in \mathcal{S}$. Then

$$\langle \mathcal{F}_\alpha(\delta), \varphi \rangle = \langle \delta, \mathcal{F}_\alpha(\varphi) \rangle = \mathcal{F}_\alpha(\varphi)(0) = \int_{-\infty}^{\infty} \varphi(x)dx = \langle 1, \varphi \rangle .$$

Thus $\mathcal{F}_\alpha(\delta) = 1$ as tempered distributions.

In the next result we examine the conformable fractional Fourier transform of convolution operators on $\mathcal{S}'$. A tempered distribution S is a convolution operator on $\mathcal{S}'$ if $S * T \in \mathcal{S}'$ for all $T \in \mathcal{S}'$ and the linear mapping $T \to S * T$ from $\mathcal{S}'$ into itself is continuous. The space of all convolution operators on $\mathcal{S}'$ is denoted by $O_c'(\mathcal{S}' : \mathcal{S}')$. In the proof, the following characterization of the members of $O_c'(\mathcal{S}' : \mathcal{S}')$ will be used (see [1] Theorem 6.6). A tempered distribution S is in $O_c'(\mathcal{S}' : \mathcal{S}')$, if and only if for every $k > 0$ there is an integer $m = m(k)$ such that

$$S = \sum_{0 \le l \le m} D^l f_l; \tag{3.11}$$

where, for each l, f_l is a continuous functions such that $(1+ | x |^2)^{\frac{k}{2}} f_l \in L^\infty$.

Theorem 8. *Let $S \in O_c'(\mathcal{S}' : \mathcal{S}')$ and $0 < \alpha < 1$. Then $\mathcal{F}_\alpha(S)$ is bounded by a polynomial in $\frac{x^\alpha}{\alpha}$.*

Proof. *From equality (3.11), linearity of $\mathcal{F}_\alpha$ and Lemma 3, we have $\mathcal{F}_\alpha(S) = \sum_{l \le m} (i)^l \mathcal{F}_\alpha(f_l)(x)(\frac{x^\alpha}{\alpha})^l$, where $\mathcal{F}_\alpha(f_l)(x)$ is bounded by a constant C_l.* □

We remark that Theorem 8 is similar to a corresponding result when $\alpha = 1$.

In some applications conformable cosine and sine Fourier transforms might be needed. These are defined similar to the corresponding ones for the traditional Fourier transform. They are

$$\begin{aligned} \mathcal{F}_\alpha^c(\varphi)(x) &= \sqrt{\frac{2}{\pi}} \int_0^\infty \varphi(t) \cos(\frac{x^\alpha}{\alpha} t) dt; \\ \mathcal{F}_\alpha^s(\varphi)(x) &= \sqrt{\frac{2}{\pi}} \int_0^\infty \varphi(t) \sin(\frac{x^\alpha}{\alpha} t) dt. \end{aligned}$$

The following properties of $\mathcal{F}_\alpha^c$ and $\mathcal{F}_\alpha^s$ follow from the above definition, integration by parts and change of variable.

Proposition 3. *Let $\varphi \in \mathcal{S}$ and $0 < \alpha < 1$. Then*

$$\begin{aligned} \mathcal{F}_\alpha^c(D\varphi)(x) &= \frac{x^\alpha}{\alpha} \mathcal{F}_\alpha^s(\varphi)(x) - \sqrt{\frac{2}{\pi}} \varphi(0), \\ \mathcal{F}_\alpha^s(D\varphi)(x) &= -\frac{x^\alpha}{\alpha} \mathcal{F}_\alpha^c(\varphi)(x), \\ \mathcal{F}_\alpha^c(D^2\varphi)(x) &= -(\frac{x^\alpha}{\alpha})^2 \mathcal{F}_\alpha^c(\varphi)(x) - \sqrt{\frac{2}{\pi}} \varphi'(0), \\ \mathcal{F}_\alpha^s(D^2\varphi)(x) &= -(\frac{x^\alpha}{\alpha})^2 \mathcal{F}_\alpha^s(\varphi)(x) + \sqrt{\frac{2}{\pi}} \frac{x^\alpha}{\alpha} \varphi'(0), \\ \mathcal{F}_\alpha^c(T^\alpha\varphi)(x) &= -\mathcal{F}_\alpha^c(\frac{\varphi(t)}{t^\alpha})(x) + \mathcal{F}_\alpha^s(\varphi(t)t^{1-\alpha})(x), \\ \mathcal{F}_\alpha^s(T^\alpha\varphi)(x) &= -\mathcal{F}_\alpha^s(\frac{\varphi(t)}{t^\alpha})(x) - \frac{x^\alpha}{\alpha} \mathcal{F}_\alpha^c(\varphi(t)t^{1-\alpha})(x), \\ T^\alpha \mathcal{F}_\alpha^c(\varphi)(x) &= -(\frac{1}{\alpha})^{1-\alpha} x^{2\alpha-\alpha^2-1} \mathcal{F}_\alpha^s(\varphi(t)t^{2-\alpha})(x). \end{aligned}$$

4. Applications

In this section we discuss two examples which show how conformable fractional Fourier transform is used to solve differential equations. The first example is a second order ordinary differential equation and the second example is a conformable heat equation.

Example 4. *Use conformable fractional Fourier transform to solve the differential equation*

$$u^{''}(x) + b^2x^2u(x) = 0,\ b > 0.$$

By taking the conformable fractional Fourier transform of both sides of the equationm, we get

$$(i\frac{x^\alpha}{\alpha})^2U(\xi) + b^2(\alpha^3\xi^{-2\alpha-1} - \alpha^2\xi^{-2\alpha})D_\xi(U(\xi)) = 0,$$

which gives the ordinary differential equation in $U(\xi)$

$$\frac{U^{'}(\xi)}{U(\xi)} = -\frac{\xi^{4\alpha+1}}{\alpha^2b^2(\alpha^3 - \alpha^2\xi)}.$$

By solving the last differential equation for $U(\xi)$ and taking $\alpha = \frac{1}{2}$ we get

$$U(\xi) = C\exp(-\int \frac{16\xi^3}{b^2(\frac{1}{2} - \xi)}d\xi).$$

Applying the fractional operator $G_{\frac{1}{2}}$ from the left to $U(\xi)$,we get

$$u(x) = G_{\frac{1}{2}}(C\exp(-\int \frac{16\xi^3}{b^2(\frac{1}{2} - \xi)}d\xi).$$

Remark 1. *One might try to solve the the above differential equation using traditional Fourier transform. Doing so, we get a second order differential equation with variable coefficients in $\widehat{u}$ the Fourier transform of u. Solving this equation using power sries, and taking the inverse Fourier transform of the power series solution gives a differential operator of infinite order. One then has to determine in which space of distributions the differential operator is convergent.*

Example 5. *Use conformable fractional Fourier transform to solve the conformable fractional heat equation*

$$\begin{aligned} \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} &= k\frac{\partial^2 u(x,t)}{\partial x^2};\ -\infty < x < \infty, t \geq 0, \\ u(0,t) &= 0, t \geq 0,\ u(x.0) = f(x). \end{aligned}$$

By taking the conformable fractional Fourier transform of both sides with respect to x, we get

$$t^{1-\alpha}\frac{d}{dt}\widehat{u}(\omega,t) = -k\frac{\omega^{2\alpha}}{\alpha^2}\widehat{u}(\omega,t),$$

where $\widehat{u}(\omega,t)$ is the conformable fractional Fourier transform of $u(x,t)$ with respect to x. The last differential equation gives

$$\widehat{u}(\omega,t) = C\exp(-k\frac{\omega^{2\alpha}}{\alpha^2}\frac{t^\alpha}{\alpha}).$$

Next, we take the fractional Fourier transform of the initial condition $u(x,0) = f(x)$, we get $\widehat{u}(\omega,0) = \widehat{f}(\omega)$. Substituting this in the last equality, one gets $C = \widehat{f}(\omega)$. Hence

$$\widehat{u}(\omega,t) = \widehat{f}(\omega)\exp(-k\frac{\omega^{2\alpha}}{\alpha^2}\frac{t^\alpha}{\alpha}).$$

Finally we apply the fractional operator G_α from the left and get

$$u(x,t) = G_\alpha(\widehat{f}(\omega)\exp(-k\frac{\omega^{2\alpha}}{\alpha^2}\frac{t^\alpha}{\alpha})).$$

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