Filomat 36:8 (2022), 2761–2774
https://doi.org/10.2298/FIL2208761Z

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Complete Moment Convergence for Weighted Sums of Widely Orthant Dependent Random Variables and its Application in Nonparametric Regression Model

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Abstract. In this paper, the complete moment convergence of weighted sums for widely orthant dependent (WOD, in short) random variables are established. The results obtained in the paper generalize and improve some known ones. As an application of the main results, we present a result on complete consistency for the weighted estimator in a nonparametric regression model based on WOD errors.

1. Introduction

The concept of the complete convergence was introduced by Hsu and Robbins [1]. A sequence of random variables $\{X_n,\ n \geq 1\}$ is said to converge completely to a constant θ if $\sum_{n=1}^{\infty} P(|X_n - \theta| > \varepsilon) < \infty$. In view of Borel-Cantelli lemma, the above result implies that $X_n \to \theta$ almost surely as $n \to \infty$. Therefore the complete convergence is a very important tool in establishing almost sure convergence of summation of random variables as well as weighted sums of random variables.

Hsu and Robbins[1] proved that the sequence of arithmetic means of independent identically distributed random variables converges completely to the expected value of the summands, provided the variances is finite. The converse was proved by Erdös [2]. This Hus-Robbins-Erdös's result was generalized in different ways. Katz [3], Baum and katz [4], and Chow [5] obtained a generalization of complete convergence for a sequence of independent and identically distributed random variables with normalization of Marcinkiewicz-Zygmund type.

Chow [7] presented the following more general concept of the complete moment convergence. Let $\{X_n,\ n \geq 1\}$ be a random variables and $a_n > 0$, $b_n > 0$, $q > 0$. If $\sum_{n=1}^{\infty} a_n E\{b_n^{-1}|X_n| - \varepsilon\}_+^q < \infty$ for all $\varepsilon > 0$, then X_n is said to be complete moment convergence. It is well known that complete moment convergence implies complete convergence. Thus, complete moment convergence is stronger than complete convergence. From then on, many authors have devoted their study to complete convergence, for more details we can refer to

2020 *Mathematics Subject Classification*. Primary 60F15
Keywords. widely orthant dependent random variables; complete moment convergence; nonparametric regression model
Received: 16 March 2018; Revised: 14 November 2018; Accepted: 29 April 2019
Communicated by Biljana Popović
Corresponding author:vYan Shen
This work was supported by Natural Science Foundation of Anhui Province(2008085MA14, 2108085MA06), The Project of excellent courses for graduate students of Anhui University-Advanced Probability Theory, QEP of Anhui Province (2020kfkc046, 2020kfkc057).
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Sung [9], Wang et al. [11], and so on.

The following concept of widely orthant dependence (WOD, in short) random variables was introduced by Wang et al. [14] for risk model as follows.

Definition 1.1. For the random variables $\{X_n, n \geq 1\}$, if there exists a finite positive sequence $\{g_U(n), n \geq 1\}$ satisfying for each $n \geq 1$ and for all $x_i \in (-\infty, +\infty)$, $1 \leq i \leq n$,

$$P(X_1 > x_1, X_2 > x_2, \cdots, X_n > x_n) \leq g_U(n) \prod_{i=1}^{n} P(X_i > x_i),$$

then we say that the random variables $\{X_n, n \geq 1\}$ are widely upper orthant dependent (WUOD, in short); if there exists a finite positive sequence $\{g_L(n), n \geq 1\}$ satisfying for each $n \geq 1$ and for all $x_i \in (-\infty, +\infty)$, $1 \leq i \leq n$,

$$P(X_1 \leq x_1, X_2 \leq x_2, \cdots, X_n \leq x_n) \leq g_L(n) \prod_{i=1}^{n} P(X_i \leq x_i),$$

then we say that the $\{X_n, n \geq 1\}$ are widely lower orthant dependent (WLOD, in short); if they are both WUOD and WLOD, then we say that the $\{X_n, n \geq 1\}$ are WOD random variables, and $g_U(n)$, $g_L(n)$, $n \geq 1$, are called dominating coefficients.

It is easily seen that $g_U(n) \geq 1$, $g_L(n) \geq 1$. If both (1.1) and (1.2) hold for $g_L(n) = g_U(n) = M \geq 1$ for any $n \geq 1$, then $\{X_n, n \geq 1\}$ are extended negatively dependent (END, in short) random variables. If both (1.1) and (1.2) hold for $g_L(n) = g_U(n) = 1$ for any $n \geq 1$, then $\{X_n, n \geq 1\}$ are called negatively orthant dependent (NOD, in short) random variables. It is well known that negatively associated (NA, in short) random variables are NOD random variables. For more details about NOD sequence, we can refer to Shen et al. [15], Wu and Jiang [16], Sung [17], and so on. Hu [12] pointed out that negatively superadditive dependent (NSD, in short) random variables are NOD. For the details about the concept and the probability limit theory of NSD sequence, one can refer to Shen et al. [20], Chen et al. [21], and so forth. Hence, the class of WOD random variables include independent sequence, NA sequence, NSD sequence, NOD sequence and END sequence as special cases. So, it is interesting and necessary to study the convergence properties of WOD random variables.

Many literatures have discussed the probability limiting behavior of WOD random variables and obtained many applications. For example, Wang et al. [14] investigated the complete convergence for WOD random variables and given its applications in nonparametric regression models. Chen et al. [27] considered uniform asymptotics for the finite-time ruin probabilities of two kinds of nonstandard bidimensional renewal risk models with constant interest forces and diffusion generated by Brownian motions. Shen [25] established the Bernstein-type inequality for WOD random variables and gave some applications. Wang and Hu [26] investigated the consistency of the nearest neighbor estimator of the density function based on WOD samples, and so on.

In this paper, we will investigate complete moment convergence for WOD random variables. The results obtained in this paper generalize and improve some known ones. As an application, we present a result on complete consistency for the weighted estimator in a nonparametric regression model based on WOD errors.

The following definitions of stochastic domination and slowly varying function are important tool for the results proof.

Definition 1.2. A sequence of random variables $\{X_n, n \geq 1\}$ is said to be stochastically dominated by random variable X if there exists a positive constant C such that

$$P(|X_n| \geq x) \leq CP(|X| \geq x)$$

for all $x \geq 0$ and all $n \geq 1$.

Definition 1.3. A real-valued function $l(x)$, positive and measurable on $(0, +\infty)$, is said to be slowing varying if

$$\lim_{x\to\infty} \frac{l(x\lambda)}{l(x)} = 1$$

for each $\lambda > 0$.

The layout of this paper is as follows. Some preliminary lemmas are provided in Section 2. Main results and their proofs are stated in Section 3. An application in nonparametric regression model of main result is presented in Section 4. Throughout this paper, let $\{X_n, n \geq 1\}$ be a sequence of WOD random variables. $I(A)$ is the indicator of the set A. Let C be a positive constant which may be different in various places. Denote $X_+ = \max\{0, X\}$, $X_- = \max\{0, -X\}$, and $\log x = \ln \max\{x, e\}$. $a_n = O(b_n)$ stands for $a_n \leq Cb_n$, $\lfloor x \rfloor$ stands for the integer part of x, and $g(n)$= $\max\{g_L(n), g_U(n)\}$.

2. Preliminaries

This section will give some lemmas, which are useful and necessary to the proofs of main results. The first was presented by Wang et al. [14], and the second lemma was presented by Wang et al. [23]. They are basic properties for WOD random variables.

Lemma 2.1. Let $\{X_n, n \geq 1\}$ be WLOD(WUOD) with dominating coefficients $g_L(n), n \geq 1(g_U(n), n \geq 1)$. If $\{f_n(\cdot), n \geq 1\}$ are nondecreasing, then $\{f_n(X_n), n \geq 1\}$ are still WLOD(WUOD) with dominating coefficients $g_L(n), n \geq 1(g_U(n), n \geq 1)$; if $\{f_n(\cdot), n \geq 1\}$ are nonincreasing, then $\{f_n(X_n), n \geq 1\}$ are WUOD(WLOD) with dominating coefficients $g_L(n), n \geq 1(g_U(n), n \geq 1)$.

Lemma 2.2. Let $\{X_n, n \geq 1\}$ be a sequence of WOD random variables with dominating coefficients g_n=$\max\{g_L(n), g_U(n)\}$. If $\{f_n, n \geq 1\}$ is a sequence of real nondecreasing (or nonincreasing) functions, then $\{f_n(X_n), n \geq 1\}$ is still a sequence of WOD random variables with the same dominating coefficients $g(n)$.

According to Shout [6] and Wang et al. [23], we can obtain the following Marcinkiewicz-Zygmund-type maximun inequality and Rosenthal-type maximun inequality for WOD random variables.

Lemma 2.3. Let $q \geq 1$ and $\{X_n, n \geq 1\}$ be a sequence of WOD random variables with $EX_n = 0$, $E|X_n|^q < \infty$ for each $n \geq 1$ and dominating coefficients g_n=$\max\{g_L(n), g_U(n)\}$. Let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of constants. Then there exist positive constants $C_1(q)$ and $C_2(q)$ depending only on q such that

$$E\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} a_{ni}X_i\right|^q\right) \leq [C_1(q) + C_2(q)g(n)] \log^q n \sum_{i=1}^{n} E|a_{ni}X_i|^q,\ 1 < q \leq 2,$$

$$E\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} a_{ni}X_i\right|^q\right) \leq C_1(q) \log^q \sum_{i=1}^{n} E|a_{ni}X_i|^q + C_2(q)g(n) \log^q \left(\sum_{i=1}^{n} E|a_{ni}X_i|^2\right)^{\frac{q}{2}},\ q > 2.$$

Lemma 2.4. Suppose that $\{X_n, n \geq 1\}$ is a sequence of random variables stochastically dominated by a random variable X. Then, for all $q > 0$ and $x > 0$,

$$E|X_n|^q I(|X_n| \leq x) \leq C(E|X|^q I(|X| \leq x) + x^q P(|X| > x)),$$

$$E|X_n|^q I(|X_n| > x) \leq C(E|X|^q I(|X| > x)).$$

The last one is essential in proving our results, which can be found in Sung [33].

Lemma 2.5. Let $\{Y_i, 1 \le i \le n\}$ and $\{Z_i, 1 \le i \le n\}$ be a sequence of random variables. Then for any $q > 1$, $\varepsilon > 0, a > 0$,

$$E\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k}(Y_i+Z_i)\right|-\varepsilon a\right)_+ \le \left(\varepsilon^{-q}+\frac{1}{q-1}\right)a^{1-q}E\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k}Y_i\right|^q\right)+E\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k}Z_i\right|\right).$$

The following one is an important property of stochastic domination. The first statement is due to Adler et al. [8] and the second statement is well known.

3. Main Results and proofs

Theorem 3.1. Let $\alpha > \frac{1}{2}$, $\alpha p > 1$, $p \ge 2$. Let $\{X_n, n \ge 1\}$ be a sequence of WOD random variables which is mean zero and stochastically dominated by a random variable X. Denote the dominating coefficients $g(n)$=max$\{g_L(n), g_U(n)\}$. Let $E|X|^p \log^q(1+|X|) < \infty$ for some $q > \max\{p, \frac{\alpha p-1}{\alpha-\frac{1}{2}}\}$. Assume that $\{a_{ni},\ 1 \le i \le n,\ n \ge 1\}$ is an array of constants satisfying

$$\sum_{i=1}^{n}|a_{ni}|^q = O(n^\beta), \tag{3.1}$$

where $\beta \ge 1$. Then, for every $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{\alpha p-\alpha-\beta-1}E\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k}a_{ni}X_i\right|-\varepsilon(1+g(n))^{\frac{1}{q-1}}n^\alpha\right)_+ < \infty, \tag{3.2}$$

and thus

$$\sum_{n=1}^{\infty} n^{\alpha p-\beta-1}P\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k}a_{ni}X_i\right| > \varepsilon(1+g(n))^{\frac{1}{q-1}}n^\alpha\right) < \infty. \tag{3.3}$$

Proof. Without loss of generality, we assume that $a_{ni} \ge 0$, $1 \le i \le n$, $n \ge 1$ (otherwise, we can note that $a_{ni} = a_{ni}^+ - a_{ni}^-$). For $n \ge 1$ and $1 \le i \le n$, denote

$$X_{ni} = -n^\alpha I(X_i < -n^\alpha) + X_i I(|X_i| \le n^\alpha) + n^\alpha I(X_i > n^\alpha),$$

$$X_{ni}^* = X_i - X_{ni} = n^\alpha I(X_i < -n^\alpha) - n^\alpha I(X_i > n^\alpha) + X_i I(|X_i| > n^\alpha),$$

and $\widetilde{X}_{ni} = X_{ni} - EX_{ni}$. Thus, $\{X_{ni}^*,\ 1 \le i \le n,\ n \ge 1\}$ are still WOD random variables from Lemma 2.2. It is easily checked that $a_{ni}X_i$=$a_{ni}X_{ni}^* + a_{ni}X_{ni} = a_{ni}X_{ni}^* + a_{ni}EX_{ni} + a_{ni}\widetilde{X}_{ni}$, $1 \le i \le n$.

By Lemma 2.5, we take $a = (1+g(n))^{\frac{1}{q-1}}n^\alpha$, so we can obtain that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-\alpha-\beta-1}E\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k}a_{ni}X_i\right|-\varepsilon(1+g(n))^{\frac{1}{q-1}}n^\alpha\right)_+\\
&\le C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-\beta-1}\frac{1}{1+g(n)}E\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k}a_{ni}\widetilde{X}_{ni}\right|^q\right)+\\
&+C\sum_{n=1}^{\infty} n^{\alpha p-\alpha-\beta-1}E\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k}a_{ni}X_{ni}^*\right|\right)
\end{aligned}$$

$$+ C\sum_{n=1}^{\infty} n^{\alpha p-\alpha-\beta-1}\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} E(a_{ni}X_{ni})\right|\right)$$
$$=: C(H_1 + H_2 + H_3). \tag{3.4}$$

For H_1, note that $g(n) \ge 1$, $p \ge 2$, $q > \frac{\alpha p-1}{\alpha-\frac{1}{2}}$, it is obviously that $q > 2$. We get by Lemma 2.3 that

$$\begin{aligned} H_1 &\le C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-\beta-1}\frac{1}{1+g(n)}\log^q n\left\{\sum_{i=1}^{n} E|a_{ni}\widetilde{X}_{ni}|^q + g(n)\left(\sum_{i=1}^{n} E|a_{ni}\widetilde{X}_{ni}|^2\right)^{\frac{q}{2}}\right\} \\ &\le C\sum_{n=1}^{n} n^{\alpha p-\alpha q-\beta-1}\log^q n\sum_{i=1}^{n} E|a_{ni}\widetilde{X}_{ni}|^q + \sum_{n=1}^{\infty} n^{\alpha p-\alpha q-\beta-1}\log^q n\left(\sum_{i=1}^{n} E|a_{ni}\widetilde{X}_{ni}|^2\right)^{\frac{q}{2}} \\ &=: C(H_{11} + H_{12}). \end{aligned} \tag{3.5}$$

Following from Jensen's inequality, Lemma 2.4 and $E|X|^p\log^q(1+|X|) < \infty$, we can get that

$$\begin{aligned} H_{11} &\le C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-\beta-1}\log^q n\sum_{i=1}^{n} a_{ni}^q(E|X_i|^q I(|X_i| \le n^\alpha) + n^{\alpha q}EI(|X_i| > n^\alpha)) \\ &\le C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-1}\log^q nE|X|^q I(|X| \le n^\alpha) + C\sum_{n=1}^{\infty} n^{\alpha p-1}\log^q nP(|X| > n^\alpha) \\ &\le C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-1}\log^q n\sum_{m=1}^{n} E|X|^q I((m-1)^\alpha < |X| \le m^\alpha) \\ &\quad + C\sum_{n=1}^{\infty} n^{\alpha p-2\alpha-1}\log^q n\sum_{m=n}^{\infty} E|X|^2 I(m^\alpha < |X| \le (m+1)^\alpha) \\ &= C\sum_{m=1}^{\infty} E|X|^q I((m-1)^\alpha < |X| \le m^\alpha)\sum_{n=m}^{\infty} n^{\alpha p-\alpha q-1}\log^q n \\ &\quad + C\sum_{m=1}^{\infty} E|X|^2 I(m^\alpha < |X| \le (m+1)^\alpha)\sum_{n=1}^{m} n^{\alpha p-2\alpha-1}\log^q n \\ &\le C\sum_{m=1}^{n} E|X|^q I((m-1)^\alpha < |X| \le m^\alpha)m^{\alpha p-\alpha q}\log^q m + CE|X|^p\log^q(1+|X|) \\ &\le CE|X|^p\log^q(1+|X|) < \infty. \end{aligned} \tag{3.6}$$

For H_{12}, by Hölder inequality and (3.1), we can know that

$$\left(\sum_{i=1}^{n} a_{ni}^2\right)^{\frac{q}{2}} \le n^{\frac{q}{2}-1}\sum_{i=1}^{n} a_{ni}^q \le Cn^{\frac{q}{2}+\beta-1}. \tag{3.7}$$

Meanwhile, we note the fact $EX_i = 0$, it can be checked that $E\widetilde{X}_{ni}^2 \le CEX^2 < \infty,\ 1 \le i \le n$. Hence, we obtain that for any $q > \max\{p,\ \frac{\alpha p-1}{\alpha-\frac{1}{2}}\}$ that

$$H_{12} = \sum_{n=1}^{\infty} n^{\alpha p-\alpha q-\beta-1}\log^q n\left(\sum_{i=1}^{n} E|a_{ni}\widetilde{X}_{ni}|^2\right)^{\frac{q}{2}} \le C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q+\frac{q}{2}-2}\log^q n(EX^2)^{\frac{q}{2}} < \infty. \tag{3.8}$$

For H_2 and H_3, similar to the proof of H_1, following from $EX_i = 0$, C_r-inequality, Jensen's inequality, Lemma 2.3, Lemma 2.4, and $q > \max\{p, \frac{\alpha p-1}{\alpha-\frac{1}{2}}\}$, we have that

$$\begin{aligned}
H_2 &\leq C\sum_{n=1}^{\infty} n^{\alpha p-\alpha-\beta-1}\sum_{i=1}^{n} a_{ni}E|X_i|I(|X_i| > n^{\alpha}) \\
&\leq C\sum_{n=1}^{\infty} n^{\alpha p-\alpha-1}E|X|I(|X| > n^{\alpha}) \\
&\leq C\sum_{m=1}^{\infty} E|X|I(m^{\alpha} < |X| \leq (m+1)^{\alpha})m^{\alpha p-\alpha} \\
&\leq CE|X|^p < \infty. \qquad (3.9)
\end{aligned}$$

$$\begin{aligned}
H_3 &= \sum_{n=1}^{\infty} n^{\alpha p-\alpha-\beta-1}\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{n} E(a_{ni}X_{ni})\right|\right) \\
&\leq C\sum_{n=1}^{\infty} n^{\alpha p-\alpha-\beta-1}\sum_{i=1}^{n} a_{ni}E|X_i|I(|X_i| > n^{\alpha}) \\
&\leq CE|X|^p < \infty. \qquad (3.10)
\end{aligned}$$

Combining (3.4)-(3.10), we get (3.2) immediately.

Now we will show that (3.2) implies (3.3). Denote $S_k = \sum_{i=1}^{k} a_{ni}X_i$. In fact, it can be checked that

$$\begin{aligned}
\infty &> \sum_{n=1}^{\infty} n^{\alpha p-\alpha-\beta-1}E(\max_{1\leq k\leq n}|S_k| - \varepsilon(1+g(n))^{\frac{1}{q-1}}n^{\alpha})_+ \\
&\geq \sum_{n=1}^{\infty} n^{\alpha p-\alpha-\beta-1}\int_0^{\varepsilon(1+g(n))^{\frac{1}{q-1}}n^{\alpha}} P(\max_{1\leq k\leq n}|S_k| - \varepsilon n^{\alpha} > t)dt \\
&\geq C\sum_{n=1}^{\infty} n^{\alpha p-\beta-1}P(\max_{1\leq k\leq n}|S_k| > \varepsilon(1+g(n))^{\frac{1}{q-r}}n^{\alpha}). \qquad (3.11)
\end{aligned}$$

This completes the proof of the theorem. □

According to Bai and Su [28] and Theorem 3.1, we can get Corollary 3.2 as follows.

Corollary 3.2. Suppose that the conditions of Theorem 3.1 hold and $l(x) > 0$ be a slowly varying function. If $E[|X|^p l(|X|^{\frac{1}{\alpha}})] < \infty$, then for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{\alpha p-\alpha-\beta-1}l(n)E\left(\left|\sum_{i=1}^{n} a_{ni}X_i\right| - \varepsilon(1+g(n))^{\frac{1}{q-1}}n^{\alpha}\right)_+ < \infty,$$

and thus

$$\sum_{n=1}^{\infty} n^{\alpha p-\beta-1}l(n)P\left(\left|\sum_{i=1}^{n} a_{ni}X_i\right| > \varepsilon(1+g(n))^{\frac{1}{q-1}}n^{\alpha}\right) < \infty.$$

Theorem 3.3. Let $\alpha > \frac{1}{2}$, $1 < p < 2$. Let $\{X_n, n \geq 1\}$ be a sequence of WOD random variables which is mean zero and stochastically dominated by a random variable X, and $E|X|^p \log^2(1 + |X|) < \infty$. Denote the dominating coefficients $g(n)$=max$\{g_L(n), g_U(n)\}$. Assume further that $\{a_{ni},\ 1 \leq i \leq n,\ n \geq 1\}$ is an array of constants satisfying

$$\sum_{i=1}^{n} |a_{ni}|^2 = O(n^{\beta}), \tag{3.12}$$

where $\beta \geq 1$. Then for every $\varepsilon > 0$

$$\sum_{n=1}^{\infty} n^{\alpha p-\alpha-\beta-1} E\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} a_{ni}X_i\right| - \varepsilon(1 + g(n))n^{\alpha}\right)_+ < \infty, \tag{3.13}$$

and thus

$$\sum_{n=1}^{\infty} n^{\alpha p-\beta-1} P\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} a_{ni}X_i\right| > \varepsilon(1 + g(n))n^{\alpha}\right) < \infty. \tag{3.14}$$

Proof. Similarly, in order to prove Theorem 3.3, we use the same notation in the proof of Theorem 3.1 and assume that $a_{ni} \geq 0$. We take $r = 1$, $a = (1 + g(n))n^{\alpha}$, and $q = 2$ in Lemma 2.5, so we can obtain that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-\alpha-\beta-1} E\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} a_{ni}X_i\right| - \varepsilon(1 + g(n))n^{\alpha}\right)_+ \\
&\leq C\sum_{n=1}^{\infty} n^{\alpha p-2\alpha-\beta-1} \frac{1}{1+g(n)} E\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} a_{ni}\widetilde{X}_{ni}\right|^2\right) \\
&\quad + C\sum_{n=1}^{\infty} n^{\alpha p-\alpha-\beta-1} E\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} a_{ni}X^*_{ni}\right|\right) \\
&\quad + C\sum_{n=1}^{\infty} n^{\alpha p-\alpha-\beta-1} E\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} a_{ni}X_{ni}\right|\right) \\
&=: C(Q_1 + Q_2 + Q_3).
\end{aligned}$$

In order to prove (3.13), it suffices to prove $Q_1 < \infty$, $Q_2 < \infty$ and $Q_3 < \infty$. In view of proof of Theorem 3.1, it is easily checked that $Q_2 < \infty$, $Q_3 < \infty$, so we omitted the details. Note that $g(n) \geq 1$, we get by Lemma 2.3, Lemma 2.4, C_r-inequality, Jensen's inequality that

$$\begin{aligned}
Q_1 &\leq C\sum_{n=1}^{\infty} n^{\alpha p-2\alpha-\beta-1} \frac{1}{1+g(n)} \log^2 n\left(\sum_{i=1}^{n} E|a_{ni}\widetilde{X}_{ni}|^2 + g(n)\sum_{i=1}^{n} E|a_{ni}\widetilde{X}_{ni}|^2\right) \\
&\leq C\sum_{n=1}^{n} n^{\alpha p-2\alpha-\beta-1} \log^2 n \sum_{i=1}^{n} E|a_{ni}\widetilde{X}_{ni}|^2 \\
&\leq C\sum_{n=1}^{\infty} n^{\alpha p-2\alpha-1} \log^2 nE|X|^2 I(|X| \leq n^{\alpha}) + C\sum_{n=1}^{\infty} n^{\alpha p-1} \log^2 nP(|X| > n^{\alpha}) \\
&\leq C\sum_{m=1}^{\infty} E|X|^2 I((m-1)^{\alpha} < |X| \leq m^{\alpha}) \sum_{n=m}^{\infty} n^{\alpha p-\alpha-1} \log^2 n + CE|X|^p \log^2(1 + |X|)
\end{aligned}$$

$$\leq CE|X|^p \log^2(1+|X|) < \infty. \tag{3.15}$$

Therefore the desired result (3.13) obtained.

Similar to the proof of (3.11). Denote $S_k = \sum_{i=1}^k a_{ni}X_i$. We can get that

$$\begin{aligned}\infty &> \sum_{n=1}^{\infty} n^{\alpha p-\alpha-\beta-1} E(\max_{1\leq k\leq n}|S_k| - \varepsilon(1+g(n))n^{\alpha})_+ \\ &\geq \sum_{n=1}^{\infty} n^{\alpha p-\alpha-\beta-1} \int_0^{\varepsilon(1+g(n))n^{\alpha}} P(\max_{1\leq k\leq n}|S_k| - \varepsilon(1+g(n))n^{\alpha} > t)dt \\ &\geq C\sum_{n=1}^{\infty} n^{\alpha p-2} P(\max_{1\leq k\leq n}|S_k| > \varepsilon(1+g(n))n^{\alpha}).\end{aligned} \tag{3.16}$$

This completes the proof of the theorem. □

Corollary 3.4. Suppose that the conditions of Theorem 3.3 hold and $l(x) > 0$ be a slowly varying function. If $E[|X|^p l(|X|^{\frac{1}{\alpha}})] < \infty$, then for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{\alpha p-\alpha-\beta-1} l(n) E\left(\left|\sum_{i=1}^{n} a_{ni}X_i\right| - \varepsilon(1+g(n))n^{\alpha}\right)_+ < \infty,$$

and thus

$$\sum_{n=1}^{\infty} n^{\alpha p-\beta-1} l(n) P\left(\left|\sum_{i=1}^{n} a_{ni}X_i\right| > \varepsilon(1+g(n))n^{\alpha}\right) < \infty.$$

Theorem 3.5. Let $\frac{1}{2} < \alpha < \frac{1+\beta}{2}$, and $\{X_n, n \geq 1\}$ be a sequence of WOD random variables which is mean zero and stochastically dominated by a random variable X. Assume that $\{a_{ni},\ 1 \leq i \leq n,\ n \geq 1\}$ is an array of constants satisfying (3.1). Denote the dominating coefficients $g(n)$=max$\{g_L(n), g_U(n)\}$. Let $E|X|^{\frac{1+\beta}{\alpha}} \log^q(1+|X|) < \infty$, then for some $q > \frac{\beta}{\alpha-\frac{1}{2}}$ and every $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{-\alpha} E\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} a_{ni}X_i\right| - \varepsilon(1+g(n))^{\frac{1}{q-1}} n^{\alpha}\right)_+ < \infty, \tag{3.17}$$

and thus

$$\sum_{n=1}^{\infty} P\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} a_{ni}X_i\right| > \varepsilon(1+g(n))^{\frac{1}{q-1}} n^{\alpha}\right) < \infty. \tag{3.18}$$

Proof. By the fact that $\beta \geq 1$, $\alpha < \frac{1+\beta}{2}$, we take $\alpha p = 1+\beta$, according to Theorem 3.1, one gets (3.17) and (3.18) immediately. □

For random variables $\{X,\ X_n,\ n \geq 1\}$ and constants $\{a_n,\ n \geq 1\}$, by Theorem 3.5 and Borel-Cantelli lemma, we have the following weighted version of Marcinkiewicz-Zygmund-type strong law of large numbers. □

Corollary 3.6. Suppose that the conditions in Theorem 3.5 hold. Assume that $\{a_n,\ n \geq 1\}$ is a sequences of constants satisfying $\sum_{i=1}^n |a_i|^q = O(n^{\beta})$ for some $\beta \geq 1$. Assume further that the dominating coefficients $g(n)$=$O(\log n)$. Then

$$\frac{1}{n^{\alpha}(\log n)^{1/(q-1)}} \sum_{i=1}^{n} a_iX_i \to 0 \ \ a.s., \ \ n \to \infty.$$

Theorem 3.7. Let $\alpha > 0$, and $\{X_n, n \geq 1\}$ be a sequence of WOD random variables which is mean zero and stochastically dominated by a random variable X. Assume that $\{a_{ni},\ 1 \leq i \leq n,\ n \geq 1\}$ is an array of constants satisfying (3.12). Denote the dominating coefficients $g(n)$=max$\{g_L(n), g_U(n)\}$. Let $E|X|\log^3(1+|X|) < \infty$, then for every $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{-\beta-1} E\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} a_{ni}X_i\right| - \varepsilon(1+g(n))n^{\alpha}\right)_+ < \infty, \tag{3.19}$$

and thus

$$\sum_{n=1}^{\infty} n^{\alpha-\beta-1} P\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} a_{ni}X_i\right| > \varepsilon(1+g(n))n^{\alpha}\right) < \infty. \tag{3.20}$$

Proof. Similar to the prove of Theorem 3.3, we use the same notation in the proof of Theorem 3.1. We take $r = 1$, $a = (1 + g(n))n^{\alpha}$, and $q = 2$ in Lemma 2.5, so we can get that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{-\beta-1} E\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} a_{ni}X_i\right| - \varepsilon(1+g(n))n^{\alpha}\right)_+ \\
&\leq C\sum_{n=1}^{\infty} n^{-\alpha-\beta-1}\frac{1}{1+g(n)} E\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} a_{ni}\widetilde{X}_{ni}\right|^2\right) \\
&\quad + C\sum_{n=1}^{\infty} n^{-\beta-1} E\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} a_{ni}X^*_{ni}\right|\right) \\
&\quad + C\sum_{n=1}^{\infty} n^{-\beta-1} E\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} E(a_{ni}X_{ni})\right|\right) \\
&=: C(J_1 + J_2 + J_3).
\end{aligned} \tag{3.21}$$

In order to prove (3.21), it suffices to prove $J_1 < \infty$, $J_2 < \infty$ and $J_3 < \infty$. Note that $g(n) \geq 1$ and $E|X|\log^3(1 + |X|) < \infty$, we get by Lemma 2.3 that

$$\begin{aligned}
J_1 &\leq C\sum_{n=1}^{\infty} n^{-\alpha-\beta-1}\frac{1}{1+g(n)}\log^2 n\left(\sum_{i=1}^{n} E|a_{ni}\widetilde{X}_{ni}|^2 + g(n)\sum_{i=1}^{n} E|a_{ni}\widetilde{X}_{ni}|^2\right) \\
&\leq C\sum_{n=1}^{n} n^{-\alpha-\beta-1}\log^2 n\sum_{i=1}^{n} E|a_{ni}\widetilde{X}_{ni}|^2 \\
&\leq C\sum_{n=1}^{\infty} n^{-\alpha-1}\log^2 nE|X|^2 I(|X| \leq n^{\alpha}) + C\sum_{n=1}^{\infty} n^{\alpha-1}\log^2 nP(|X| > n^{\alpha}) \\
&\leq CE|X|\log^2(1+|X|) + C\sum_{m=1}^{\infty} E|X|I(m^{\alpha} < |X|(m+1)^{\alpha})\sum_{n=1}^{m} n^{-1}\log^2 n \\
&\leq CE|X|\log^3(1+|X|) < \infty.
\end{aligned} \tag{3.22}$$

For J_2 , following from $EX_i = 0$, C_r-inequality, Jensen's inequality, Lemma 2.3, Lemma 2.4, we have that

$$J_2 \leq C\sum_{n=1}^{\infty} n^{-\beta-1}\sum_{i=1}^{n} |a_{ni}|E|X_i|I(|X_i| > n^{\alpha})$$

$$\leq C\sum_{m=1}^{\infty} E|X|I(m^{\alpha} < |X| \leq (m+1)^{\alpha})\sum_{n=1}^{m} n^{-1}$$
$$\leq CE|X|\log(1+|X|) < \infty. \tag{3.23}$$

Similarly, by the proof of (3.10), for J_3 one has

$$J_3 \leq C\sum_{i=1}^{\infty} n^{-1}E|X|I(|X| > n^{\alpha}) \leq CE|X|\log(1+|X|). \tag{3.24}$$

Combining (3.21)-(3.24), we obtain (3.19) immediately.

By the proof of (3.16), it is easily checked that (3.19) can implies (3.20), so we omitted the details.

Hence, this completes the proof of the theorem. □

4. Complete consistency for the estimator in a nonparametric regression model based on WOD errors

4.1. *Complete consistency*

We consider the following nonparametric regression model based on WOD errors:

$$Y_{nk} = f(x_{nk}) + \varepsilon_{nk}, \ \ k = 1,\ 2, ..., n,\ n \geq 1, \tag{4.1}$$

where x_{nk} are known fixed design points from A where $A \subset \mathbb{R}^m$ is a given compact set for some $m \geq 1$, $f(\cdot)$ is an unknown regression function defined on A, and the ε_{nk} are random errors such that $(\varepsilon_{n1}, \varepsilon_{n2}, ..., \varepsilon_{nn})$ has the same distribution as $(\varepsilon_1, \varepsilon_2, ..., \varepsilon_n)$. As an estimator of $f(\cdot)$, consider the following weighted regression estimator:

$$f_n(x) = \sum_{k=1}^{n} \omega_{nk} Y_{nk}, \ \ x \in A \subset \mathbb{R}^m, \tag{4.2}$$

where $\omega_{nk} = \omega_{nk}(x; x_{n1}, x_{n2}, ..., x_{nn})$, $k = 1, 2, ..., n$ are the weighted function.

The above estimator was first proposed by Stone [19] and subsequently have been studied by many authors. For more details about the property of the above estimator, one can refer to Roussas [10], Yang et al. [22], Shen [25], and so forth. The purpose of this section is to further investigate the complete consistency for the estimator in the nonparametric regression model based on WOD errors.

For the function $f(x)$, we use $f(g)$ to denote all continuity points of the function f on A. The norm $\| x \|$ is the Eucledean norm. For any fixed design point $x \in A$, the following assumptions on weight function $\omega_{nk}(x)$ will be used:

(A_1) $\sum_{k=1}^{n} \omega_{nk}(x) \to 1$ as $n \to \infty$;

(A_2) $\sum_{k=1}^{n} | \omega_{nk}(x) | \leq C < \infty$ for all n;

(A_3) $\sum_{k=1}^{n} | \omega_{nk}(x) | \cdot | f(x_{nk}) - f(x) | I(\| x_{nk} - x \| > a) \to 0$ as $n \to \infty$ for all $a > 0$.

Wang et al. [29] pointed out that the design assumptions $(A_1) - (A_3)$ are general and satisfied for nearest neighbour weights. Based on the assumptions above, we present the following result on complete consistency of nonparametric regression estimator $g_n(x)$.

Theorem 4.1. Let $\frac{1}{2} < \alpha < \frac{1+\beta}{2}$, and $\{\varepsilon_n, n \geq 1\}$ be a sequence of rowwise WOD random variables with mean zero which is stochastically dominated by a random variable X. Suppose that the conditions $(A_1) - (A_3)$ hold, the dominating coefficients $f(n)$=$O(\log n)$, and

$$\sum_{k=1}^{n} | \omega_{nk}(x) |^q = O(n^\beta)$$

for some $\beta \geq 1$ and $q > \frac{\beta}{\alpha - \frac{1}{2}}$. If $E|X|^{\frac{1+\beta}{\alpha}} < \infty$, then for all $x \in c(g)$,

$$f_n(x) \to f(x), \text{ completely.} \tag{4.3}$$

Proof. For $a > 0$ and $x \in c(f)$, we obtain from (4.1) and (4.2) that

$$\begin{aligned} | Ef_n(x) - f(x) | \leq & \sum_{k=1}^{n} | \omega_{nk}(x) | \cdot | f(x_{nk}) - f(x) | I(\| x_{nk} - x \| \leq a) \\ & + \sum_{k=1}^{n} | \omega_{nk}(x) | \cdot | f(x_{nk}) - f(x) | I(\| x_{nk} - x \| > a) \\ & + | f(x) | \cdot \left| \sum_{k=1}^{n} \omega_{nk}(x) - 1 \right|. \end{aligned} \tag{4.4}$$

It follows from $x \in c(f)$ that for all $\varepsilon > 0$, there exists a constant $\delta > 0$ such that for all x' which satisfy $\| x' - x \| < \delta$, we have $| f(x') - f(x) | < \varepsilon$. Hence we take $0 < a < \delta$ in (4.4) and obtain that

$$\begin{aligned} | Ef_n(x) - f(x) | \leq & \sum_{k=1}^{n} \varepsilon | \omega_{nk}(x) | + \sum_{k=1}^{n} | \omega_{nk}(x) | \cdot | f(x_{nk}) - f(x) | I(\| x_{nk} - x \| > a) \\ & + | f(x) | \cdot \left| \sum_{k=1}^{n} \omega_{nk}(x) - 1 \right|. \end{aligned}$$

We have by assumption $(A_1) - (A_3)$ and the arbitrariness of $\varepsilon > 0$ that for all $x \in c(f)$,

$$\lim_{n \to \infty} Ef_n(x) = f(x). \tag{4.5}$$

In view of (4.5), to prove (4.3), it suffices to prove

$$f_n(x) - Ef_n(x) = \textstyle\sum_{k=1}^{n} \omega_{nk}(x)\varepsilon_k \to 0, \text{ completely.}$$

In other words, we need to verify that

$$\sum_{n=1}^{\infty} P\left(\left| \sum_{k=1}^{n} \omega_{nk}(x)\varepsilon_k \right| > \varepsilon \right) < \infty. \tag{4.6}$$

Applying Theorem 3.5 with $X_k = \varepsilon_k$, $a_{nk} = \omega_{nk} n^\alpha (\log n)^{1/(q-1)}$, we immediately obtain the desired result. The proof is completed. □

4.2. *Numerical simulation*

In this subsection, we will present a simulation to study the numerical performance of the consistency for the nearest neighbor weight function estimators $f_n(x)$ in nonparametric regression model and the data are generated from model (4.1). First let us recall the concept of the nearest neighbor weight function as follows.

Let $A = [0,1]$ and $x_{nk} = k/n,\ k = 1,2,...,n$. For any $x \in A$, we rewrite $|x_{n1} - x|, |x_{n2} - x|, ..., |x_{nn} - x|$ as follows:

$$|x_{n,R_1(x)} - x| \le |x_{n,R_2(x)} - x| \le ... \le |x_{n,R_n(x)} - x|,$$

if $|x_{ni} - x| = |x_{nj} - x|$, then $|x_{ni} - x|$ is located before $|x_{nj} - x|$ when $x_{ni} < x_{nj}$.

Let $1 \le k_n \le n$, the nearest neighbor weight function is defined as follows:

$$W_{nk}(x) = \begin{cases} 1/k_n, & \text{if } |x_{nk} - x| \le |x_{n,R_{k_n}(x)} - x|, \\ 0, & \text{otherwise.} \end{cases}$$

For any fixed $n \ge 3$, let normal random vector $(\varepsilon_{n1}, \varepsilon_{n2}, ..., \varepsilon_{nn}) \sim N_n(0, \Sigma)$, where 0 represents zero vector and

$$\Sigma = \begin{bmatrix} 1+\delta^2 & -\delta & 0 & \cdots & 0 & 0 & 0 \\ -\delta & 1+\delta^2 & -\delta & \cdots & 0 & 0 & 0 \\ 0 & -\delta & 1+\delta^2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1+\delta^2 & -\delta & 0 \\ 0 & 0 & 0 & \cdots & -\delta & 1+\delta^2 & -\delta \\ 0 & 0 & 0 & \cdots & 0 & -\delta & 1+\delta^2 \end{bmatrix}_{n\times n},$$

where $0 < \delta < 1$. It is easily checked that $(\varepsilon_{n1}, \varepsilon_{n2}, \cdots, \varepsilon_{nn})$ is a NA vector for each $n \ge 3$ with finite moment of any order. Thus, it is a WOD vector with $f(n) = O(\log n)$ when $\delta = 0.3$. Let $\alpha = 3/5$, $\beta = 1$, $q = 11$, $k_n = \lfloor n^{3/5}(\log n)^{1/10} \rfloor$. Thus, all conditions of Theorem 4.1 are satisfied. As is stated in Wang et al. [23], conditions $(A_1) - (A_3)$ hold true. Take the points $x = 0.3, 0.5, 0.7$ and the sample sizes n as $n = 100, 200, 300, 400$ respectively. We use R software to compute $f_n(x) - f(x)$ with $f(x) = \sin x - x$ and $f(x) = x^2 + x$, respectively, for 1000 times and obtain the boxplots of $f_n(x) - f(x)$ in Figures 1-6 and the Mean Square Error (MSE, in short) of $f_n(x)$ in Table 1.

Table 1. MSE of the estimator $f_n(x)$

$f(x)$	x	n=100	n=200	n=300	n=400
	0.3	0.02318809	0.01775874	0.01555241	0.01072344
$\sin x - x$	0.5	0.02271258	0.01667872	0.01578463	0.01091291
	0.7	0.02410608	0.01738176	0.01411086	0.00985498
	0.3	0.02357551	0.01836288	0.01541625	0.01442039
$x^2 + x$	0.5	0.02360975	0.01737076	0.01622930	0.01001196
	0.7	0.02276254	0.01689222	0.01585582	0.01145534

Figures 1-3 are the boxplots of $f_n(x) - f(x)$ for $f(x) = \sin x - x$ and Figures 4-6 are the boxplots of $f_n(x) - f(x)$ for $f(x) = x^2 + x$ with $x = 0.3, 0.5, 0.7$. We can see that no matter $f(x) = \sin x - x$ or $f(x) = x^2 + x$, for $x = 0.3, 0.5, 0.7$, the differences $f_n(x) - f(x)$ fluctuate to zero and the variation range decreases markedly as the sample n increases. In other words, this show a good fit of our result.

5. Conclusions

In this paper, we obtain some complete moment convergence results for weighted sums of WOD random variables without the assumption of an identical distribution. In view of the proof of Theorem 3.1-3.7, the essential tools are the Rosenthal-type maximun inequality, the Marcinkiewicz-Zygmund-type maximun inequality and truncation method. It is well known that the class of WOD random variables contain many dependent structure random variables, such as NA random variables, NOD random variables, END random variables, and so on. That is to say, our results are also available for these sequences.

Moreover, if we take place (3.1) with an array of independent random variables A_{ni} which satisfying that $\{A_{ni},\ 1 \le i \le n\}$ is independent of $\{X_n,\ n \ge 1\}$ and $\sum_{i=1}^{n} E|A_{ni}|^q = O(n^\beta)$ in Theorem 3.1, our results can implies the results of Zhao et al. [18] for the case of NOD random variables when $g(n) = 1$ and $\beta = 1$; our results can implies the results of Li et al. [34] for the case of END random variables when $g_n = M \ge 1$. If $a_{ni} = 1$ and $l(n) = 1$ in Corollary 3.2, we can obtain the result of Theorem 2.1 in Liu et al. [32] for the case of WOD random variables.

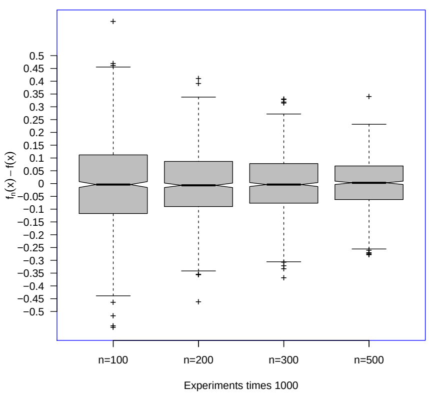

Fig 1: Boxplots of $f_n(x) - f(x)$ with x=0.3 and f(x)=sinx−x

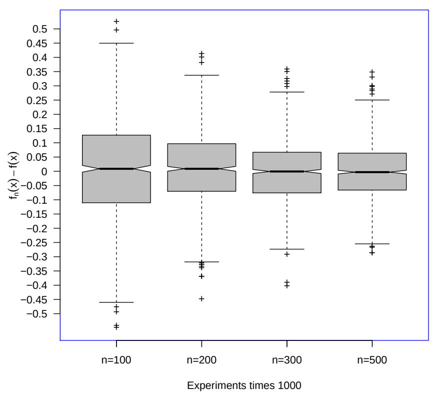

Fig 2: Boxplots of $f_n(x) - f(x)$ with x=0.5 and f(x)=sinx−x

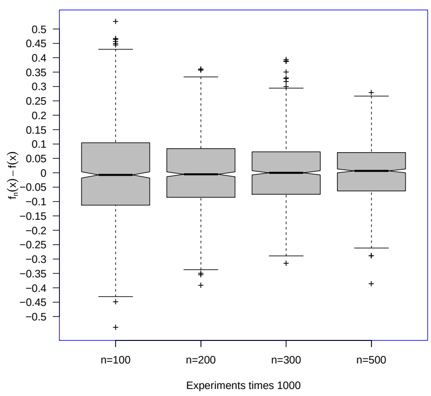

Fig 3: Boxplots of $f_n(x) - f(x)$ with x=0.7 and f(x)=sinx−x

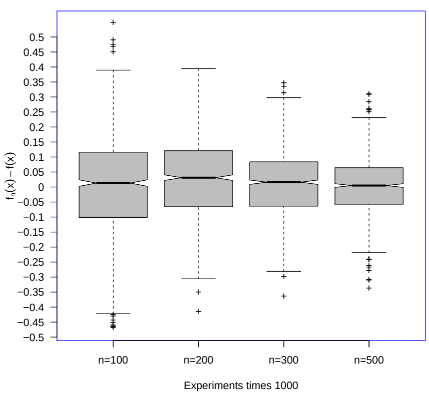

Fig 4: Boxplots of $f_n(x) - f(x)$ with x=0.3 and f(x)=x^2+x

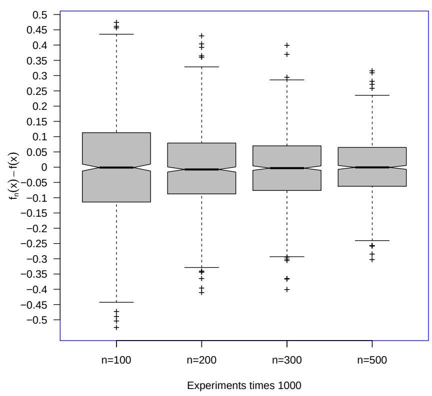

Fig 5: Boxplots of $f_n(x) - f(x)$ with x=0.5 and f(x)=x^2+x

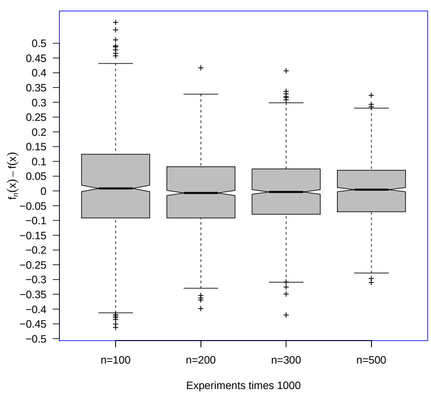

Fig 6: Boxplots of $f_n(x) - f(x)$ with x=0.7 and f(x)=x^2+x

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