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An Abstract and Generalized Formulation of a Theorem by Pelc and Prikry on Invariant Extension of Borel Measure

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Abstract. There are certain countably generated σ-algebras of sets in the real line which do not admit any non-zero, σ-finite, diffused (or, continuous) measure. Such countably generated σ-algebras can be obtained by the use of some special types of infinite matrix known as the Banach-Kuratowski matrix and the same may be used in deriving a generalized version of Pelc and Prikry's theorem as shown by Kharazishvili. Here, in this paper, we develop an abstract and generalized formulation of Pelc and Prikry's theorem in spaces with transformation groups, where instead of using measure type functionals as done by Kharazishvili, we utilize a newly introduced concept which is that of an admissible, diffused k-additive algebra where k is an arbitrary infinite cardinal.

1. Introduction

Banach [1] (see also [4]) asked if there exist two countably generated σ-algebras on the interval $[0, 1)$ such that they carry probability diffused measure, whereas the σ-algebra generated by their union does not. An answer to this was provided in [2] by Grzegorek using Martin axiom and in [3] (see, also [4]) without using any such additional set theoretic assumptions. In [9], Pelc and Prikry obtained an analogue of the result of [2] in translation invariant settings, and, Kharazishvili [7] obtained a generalization of Pelc and Prikry's result by constructing (under continuum hypothesis) certain Banach-Kuratowski matrix consisting of sets that are almost invariant with respect to the group of all isometric transformations.

Let ω and ω_1 denote the first infinite and the first uncountable ordinals and $F = \omega^\omega$ be the family of all functions from ω into ω. For any two functions f and g from F, we write $f \leq g$ to mean that there exists a natural number $n(f, g)$ such that $f(m) \leq g(m)$ for all m such that $n(f, g) \leq m$. The relation $\leq$ so defined is a pre ordering on F and under the assumption of continuum hypothesis, it is not hard to define a subset $E = \{f_\xi : \xi < \omega_1\}$ of F satisfying the following two conditions:
(i) if f is any arbitrary function from F, then there exists an ordinal $\xi < \omega_1$ such that $f < f_\xi$. In otherwords, E is cofinal in F.
(ii) For no two ordinals ξ and ρ satisfying $\xi < \rho < \omega_1$ does the relation $f_\rho < f_\xi$ holds true.

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Now conditions (i) and (ii) imply that card$E = \omega_1$. Further, if for any two natural numbers m and n, we set $E_{m,n} = \{f_\xi \in E : f_\xi(m) \leq n\}$, then the countable double family of sets $(E_{m,n})_{m<\omega,n<\omega}$ satisfies
(a) $E_{m,0} \subseteq E_{m,1} \subseteq \ldots\ldots \subseteq E_{m,n} \subseteq \ldots\ldots$ for any natural number $m < \omega$.
(b) $E = \cup\{E_{m,n} : n < \omega\}$ and
(c) $E_{0,f(0)} \cap E_{1,f(1)} \cap \ldots\ldots E_{m,f(m)} \cap \ldots$ is atmost countable for every $f \in F$.

A matrix $(E_{m,n})_{m<\omega,n<\omega}$ on E having the above three properties is called a Banach-Kuratowski matrix [5], and it can be proved that there does not exist any non-zero, σ-finite, diffused (or, continuous) measure defined simultaneously for all sets $E_{m,n}$. Not only that, the existence of a Banach-Kuratowski matrix on E proves even more [5] and it is this that there does does not exist any non-zero, diffused, admissible functional defined simultaneously for all the above sets $E_{m,n}$, where by a diffused admissible functional [5] (see also [7]) we mean a set valued mapping ν defined on a family of subsets of E which is closed under finite intersection and for which the following set of conditions are fulfilled :
(1) ν is defined on every countable set $X \subseteq E$ with $\nu(X) = 0$.
(2) If $\{Z_n : n < \omega\}$ is an increasing family of sets (with respect to inclusion) from the domain of ν, then their union $\bigcup\limits_{n<\omega} Z_n$ is also a member of the domain and

$$\nu(\bigcup_{n<\omega} Z_n) \leq sup\{\nu(Z_n) : n < \omega\}$$

(3) If $\{Z_n : n < \omega\}$ is a decreasing family of sets (with respect to inclusion) from the domain of ν, then their intersection $\bigcap\limits_{n<\omega} Z_n$ is also a member of the domain and

$$\nu(\bigcap_{n<\omega} Z_n) \geq inf\{\nu(Z_n) : n < \omega\}$$

Evidently, every finite measure defined on a σ-algebra of sets satisfies conditions (1) − (3) whereas an admissible functional need not have the property of σ-additivity as required in the definition of any ordinary measure function.

Pelc and Prikry [9] under the assumption of continuum hypothesis and the use of Hamel basis, proved that

Theorem PP : *There exist countably generated σ-algebras $\Re_1$, $\Re_2$ of subsets of the interval* $[0, 1)$ *and probability measures μ_1 and μ_2 on $\Re_1$, $\Re_2$ respectively such that*
(i) $\Re_1$, $\Re_2$ both contain all Borel sets and are translation invariant.
(ii) μ_1 and μ_2 both extend the Lebesgue measure and are translation invariant.
(iii) there is no non-atomic probability measure on any σ-algebra containing $\Re_1 \cup \Re_2$.
(Here non-atomic measure means that it is diffused or continuous)

Kharazishvili [7] by constructing certain Banach-Kuratowski matrix consisting of almost invariant sets (with respect to the group Γ of all isometric transformations of $\mathbb{R}$) obtained the following generalization of Pelc and Prikry's theorem.

Theorem K : *Suppose the continuum hypothesis hold. Then there are two σ-algebras $\mathcal{S}_1$ and $\mathcal{S}_2$ of sets in* $\mathbb{R}$ *having the following properties :*
(i) $\mathcal{S}_1$ and $\mathcal{S}_2$ are countably generated and invariant under the group Γ.
(ii) the σ-algebra of Borel sets $\mathcal{B}(\mathbb{R}) \subseteq \mathcal{S}_1 \cap \mathcal{S}_2$
(iii) there exists a Γ-invariant measure μ_1 on $\mathcal{S}_1$ extending the standard Borel measure on $\mathbb{R}$.
(iv) there exists a Γ-invariant measure μ_2 on $\mathcal{S}_2$ extending the standard Borel measure on $\mathbb{R}$.
(v) there is no non-zero, diffused admissible functional defined on the σ-algebra generated by $\mathcal{S}_1 \cup \mathcal{S}_2$.

Our aim in this paper is to give an abstract generalization of Pelc and Prikry's theorem in space (X, G) with transformation group G, where instead of using diffused, admissible functional as done by

Kharazishvili, we utilize a newly introduced concept which is that of an admissible, k-additive algebra $\mathcal{S}$ on (X, G) defined in connection with some transfinite k-sequence $\{\mathcal{N}_\alpha\}_{0\leq\alpha<k}$ which in the present situation is a modified and generalized version of the system of small sets or small systems originally introduced by Riećan and Neubrunn [10], [11], [12]. It is worth noting here that Riećan and Neubrunn had earlier introduced this notion of small systems to give abstract formulations (with out explicit use of measure) of several well-known classical theorems on Lebesgue measure and integration.

2. Preliminaries and Results

A space X with a transformation group G is a pair (X, G) [4] where X is a nonempty basic set and G is a subgroup of the symmetric group Symm(X) of all bijections from X onto X satisfying the following two conditions :
TG 1) for each $g \in G$, $x \to gx$ is a bijection (or, permutation) of X
TG 2) for all $x \in X$, and $g_1, g_2 \in G$, $g_1(g_2x) = g_1g_2x$.

We say that G acts freely on X [4] if $\{x \in X : gx = x\} = \emptyset$ for all $g \in G \setminus \{e\}$ where 'e' is the identity element of G (in fact, $e : X \to X$ is the identity transformation on X). For any $g \in G$ and $E \subseteq X$, we write gE [4] to denote the set $\{gx : x \in E\}$ and call a nonempty family (or, class) $\mathcal{A}$ of sets G-invariant [4] if $gE \in \mathcal{A}$ for every $g \in G$ and $E \in \mathcal{A}$. If $\mathcal{A}$ is a σ-algebra, then a measure μ on $\mathcal{A}$ is called G-invariant [4] if $\mathcal{A}$ is a G-invariant class and $\mu(gE) = \mu(E)$ for every $g \in G$ and $E \in \mathcal{A}$. It is called G-quasiinvariant [4] if $\mathcal{A}$ and the σ-ideal generated by μ-null sets are both G-invariant classes. Obviously, any G-invariant measure is also G-quasiinvariant but not conversely. Any set of the form $Gx = \{gx : g \in G\}$ for some $x \in X$ is called a G-orbit [4] of x. The collection of all G-orbits give rise to a partition of X into mutually disjoint sets. A subset E of X is called a complete G-selector [4] (or, simply, a G-selector) in X if $E \cap Gx$ consists of exactly one point for each $x \in X$. If $Gx = E$ for each $x \in X$, then G is said to act transitively on X [4]. In this situation, for any $x, y \in X$, there exists $g \in G$ such that $y = gx$.

Throughout the paper, we identify every infinite cardinal with the least ordinal representing it and every ordinal with the set of all ordinals preceeding it. We write card$A$ and card$\mathcal{A}$ to represent the cardinality of any set A or any family (or, class) of sets $\mathcal{A}$ and use symbols such as α, β, γ, δ, ξ, η, k etc for infinite cardinals. Moreover, we denote by k^+ and $c(k)$ the successor and cofinality of k. Now given any space (X, G) with transformation group, we define

Definition 2.1 : A nonempty class $\mathcal{S}$ of subsets of X as a k-additive algebra on (X, G) if
(i) $\mathcal{S}$ is an algebra
(ii) $\mathcal{S}$ is k-additive which means that $\mathcal{S}$ is closed with respect to the union of atmost k number of sets from it.
(iii) $\mathcal{S}$ is a G-invariant class.

Hence, a k-additive algebra on (X, G) is a k-additive algebra on X which is also G-invariant. A k-additive algebra $\mathcal{S}$ on X is called diffused if every singleton set $\{x\} \in \mathcal{S}$.

Definition 2.2 : A k-additive algebra $\mathcal{S}$ on (X, G) is called admissible if there is associated with $\mathcal{S}$ a transfinite k-sequence $\{\mathcal{N}_\alpha\}_{0\leq\alpha<k}$ members of which are certain classes of subsets of X satisfying the following set of conditions :
(i) $\emptyset \in \mathcal{N}_\alpha$ and $\mathcal{S} \cap \mathcal{N}'_\alpha \neq \emptyset \neq \mathcal{S} \cap \mathcal{N}_\alpha$ for $0 \leq \alpha < k$ where $\mathcal{N}'_\alpha = \{E \subseteq X : E \notin \mathcal{N}_\alpha\}$.
(ii) For every $\alpha, \beta < k$, there exists $\gamma > \alpha, \beta$ such that $\mathcal{N}_\gamma \subseteq \mathcal{N}_\alpha$ and $\mathcal{N}_\gamma \subseteq \mathcal{N}_\beta$. In other words, with respect to the relation of set inclusion, the system $\{\mathcal{N}_\alpha\}_{0\leq\alpha<k}$ is directed.
(iii) For any $\alpha < k$, there exists $\alpha^* > \alpha$ such that for any one-to-one correspondence $\beta \to \mathcal{N}_\beta$ with $\beta > \alpha^*$, $\cup E_\beta \in \mathcal{N}_\alpha$ whenever $E_\beta \in \mathcal{N}_\beta$.
(iv) Each $\mathcal{N}_\alpha$ is a G-invariant class.
(v) If $E \in \mathcal{N}_\alpha$ and $F \subseteq E$, then $F \in \mathcal{N}_\alpha$. Thus each $\mathcal{N}_\alpha$ is a hereditary class.
(vi) If $\{E_\xi : \xi < k\}$ is a nested sequence of sets from $\mathcal{S}$ such that $\cap_\xi E_\xi \in \mathcal{N}_\alpha$, then there exists some $E_\xi \in \mathcal{N}_\alpha$.

An admissible k-additive algebra $\mathcal{S}$ is called diffused if every singleton set $\{x\} \in \mathcal{S} \cap \mathcal{N}_\alpha$ for $0 \leq \alpha < k$. We further define

Definition 2.3 : An admissible k-additive algebra on (X, G) (admissible with respect to some $\{\mathcal{N}_\alpha\}_{0\leq\alpha<k}$) as satisfying k-chain condition if for any $\alpha < k$, the cardinality of the maximal family of disjoint sets from $\mathcal{S} \setminus \mathcal{N}_\alpha$ is atmost k.
and

Definition 2.4 : An admissible k-additive algebra $\mathcal{S}$ on (X, G) (admissible with respect to some $\{\mathcal{N}_\alpha\}_{0\leq\alpha<k}$) as regular if given any k-sequence $\{E_\alpha\}_{0\leq\alpha<k}$ satisfying $\cap_\alpha E_\alpha \in \mathcal{N}_\alpha$, there exists $F_\alpha \supseteq E_\alpha$ such that $F_\alpha \in \mathcal{S}$ and $\cap_\alpha F_\alpha \in \mathcal{N}_\alpha$.

In the family F of all functions from k into k, let us set up a partial preordering as follows:
$f, g \in k^k$, $f \leq g$ iff card$(\{\alpha < k : g(\alpha) < f(\alpha)\}) < k$. If the generalized continuum hypothesis is assumed, then it is not hard to define a subfamily $E = \{f_\xi : \xi < k^+\}$ of F which is cofinal in F in the sense that for every $f \in F$, there is some $f_\xi \in E$ such that $f \leq f_\xi$ and which also satisfies the property that the relation $f_\eta < f_\xi$ is not true for any ξ and η such that $\xi < \eta$.
Now upon setting $E_{\alpha,\beta} = \{f_\xi \in E : f_\xi(\alpha) \leq \beta\}$, we find that the double family $(E_{\alpha,\beta})_{0\leq\alpha<k,0\leq\beta<k}$ of sets satisfies the following three conditions :
(i) $E_{\alpha,\beta} \subseteq E_{\alpha,\gamma}$ for any $0 \leq \alpha < k$ and $0 \leq \beta < \gamma < k$.
(ii) $E = \bigcup\limits_{0\leq\beta<k} E_{\alpha,\beta}$ and
(iii) for any $f \in k^k$, card $(\bigcap\limits_{0\leq\alpha<k} E_{\alpha,f(\alpha)}) \leq k$.

One may note that the family $(E_{\alpha,\beta})_{0\leq\alpha<k,0\leq\beta<k}$ is a direct generalization of the Banach-Kuratowski matrix. Now using this $k \times k$ transfinite matrix and the concept of an admissible k-additive algebra on (X, G), we prove the following theorem. For both theorem 2.5 and theorem 2.6 we assume generalized continuum hypothesis.

Theorem 2.5 : *Let (X, G) be a space with a transformation group G where $k^+ =$ card $G \leq$ card X and G acts freely on X. Let L be a G-selector in X. Then there exists a family $(F_{\alpha,\beta})_{0\leq\alpha<k,0\leq\beta<k}$ of sets in X which is not contained in any admissible k-additive algebra $\mathcal{S}$ on (X, G) such that $L \in \mathcal{S}$ and which satisfies the k-chain condition.*

Proof : We write $G = \bigcup\limits_{\rho<k^+} G_\rho$ where $\{G_\rho : \rho < k^+\}$ is an increasing family of subgroups of G satisfying $G_\rho \neq \bigcup\limits_{\eta<\rho} G_\eta$ and card$(G_\rho) \leq k$ for every $\rho < k^+$ (for this representation, see [6], Exercise 19, Ch 3). Since by hypothesis, G acts freely on X so the above increasing family yields a disjoint covering $\{\Omega_\gamma : \gamma < k^+\}$ of X where $\Omega_\gamma = (G_\gamma \setminus \bigcup\limits_{\eta<\gamma} G_\eta)L$. Let $\mathcal{S}$ be any admissible k-additive algebra on (X, G) (admissible with respect to $\{\mathcal{N}_\alpha\}_{0\leq\alpha<k}$) satisfying k-chain condition such that $L \in \mathcal{S}$. Using the $k \times k$ transfinite matrix $(E_{\alpha,\beta})_{0\leq\alpha<k,0\leq\beta<k}$, we define $F_{\alpha,\beta} = \bigcup\limits_{\gamma\in E_{\alpha,\beta}} \Omega_\gamma$. Then $F_{\alpha,\beta} \subseteq F_{\alpha,\gamma}$ whenever $0 \leq \beta < \gamma < k$ and $X = \bigcup\limits_{0\leq\beta<k} F_{\alpha,\beta}$.

We observe that there exists δ such that no subset M of X can belong to $\mathcal{N}_\delta$ if its complement in X is in $\mathcal{N}_\delta$. This follows since $\mathcal{S} \cap \mathcal{N}'_\alpha \neq \emptyset$ (by **(i)**) and so $X \notin \mathcal{N}_\alpha$ as $X \in \mathcal{S}$ and also because that there exists $\beta, \gamma > \alpha$ (by **(iii)**) such that $\mathcal{N}_\beta \cup \mathcal{N}_\gamma \subseteq \mathcal{N}_\alpha$ and $\delta > \beta, \gamma$ (by **(ii)**) such that $\mathcal{N}_\delta \subseteq \mathcal{N}_\beta$, $\mathcal{N}_\delta \subseteq \mathcal{N}_\gamma$. Now, if possible, let $\{F_{\alpha,\beta} : 0 \leq \alpha < k, 0 \leq \beta < k\} \subseteq \mathcal{S}$.

By virtue of condition **(iii)** and **(vi)**, there exists $\delta^* > \alpha$ and an one-to-one correspondence $f : \alpha \to \beta_\alpha$ $(\beta_\alpha > \delta^*)$ such that $G_{\alpha,\beta_\alpha} \in \mathcal{N}_{\beta_\alpha} \cap \mathcal{S}$ and $\bigcup\limits_{0\leq\alpha<k} G_{\alpha,\beta_\alpha} \in \mathcal{N}_\delta \cap \mathcal{S}$ where $G_{\alpha,\beta_\alpha} = X \setminus F_{\alpha,\beta_\alpha}$. Therefore, $\bigcap\limits_{0\leq\alpha<k} F_{\alpha,\beta_\alpha} \in$

$\mathcal{S} \cap \mathcal{N}'_\delta$ from the above observation. But $\bigcap_{0\le\alpha<k} F_{\alpha,f(\alpha)}$ by construction is also the union of atmost k translates of L. Again, L cannot belong to any $\mathcal{N}'_\alpha$ because card(G)= k^+, G acts freely on X and $\mathcal{S}$ satisfies the k-chain condition. So, $L \in \mathcal{S} \cap (\bigcap_{0\le\alpha<k} \mathcal{N}_\alpha)$ and consequently $\bigcap_{0\le\alpha<k} F_{\alpha,f(\alpha)} \in \mathcal{S} \cap (\bigcap_{0\le\alpha<k} \mathcal{N}_\alpha)$ since by virtue of conditions **(iii)** and **(iv)**, $\bigcap_{0\le\alpha<k} \mathcal{N}_\alpha$ is k-additive and also G-invariant. Thus $\bigcap_{0\le\alpha<k} F_{\alpha,f(\alpha)} \in \mathcal{S} \cap \mathcal{N}'_\delta$ and also $\bigcap_{0\le\alpha<k} F_{\alpha,f(\alpha)} \in \mathcal{S} \cap (\bigcap_{0\le\alpha<k} \mathcal{N}_\alpha)$ - a contradiction.

This proves the theorem.

Now mainly based on the above result, we establish an abstract and generalized formulation of Pelc and Prikry's theorem.

Theorem 2.6 : *Let (X, G) be a space with transformation group G where k^+ = card(G)= card(X) and G acts freely and transitively on X. Let $\mathcal{S}_0$ be an admissible (with respect to $\{\mathcal{N}_\alpha^{\ 0}\}_{0\le\alpha<k}$ say), diffused and regular k-additive algebra satisfying k-chain condition on (X, G) and also fulfilling the requirement that $E \in \mathcal{N}_\alpha^{\ 0}$ and $F \in \bigcap_{0\le\alpha<k} \mathcal{N}_\alpha^{\ 0}$ implies $E\Delta F \in \mathcal{N}_\alpha^{\ 0}$. Then there exists k-additive algebras $\mathcal{S}_1$ and $\mathcal{S}_2$ on (X, G) such that*
(i) both $\mathcal{S}_1$ and $\mathcal{S}_2$ properly contain $\mathcal{S}_0$.
(ii) $\mathcal{S}_1$ and $\mathcal{S}_2$ are diffused and admissible.
but (iii) the k-additive algebra on (X, G) generated by $\mathcal{S}_1 \cup \mathcal{S}_2$ is not admissible.

Proof: Since G acts freely and transitively on X, so for any arbitrary but fixed choice of $x \in X$, the increasing family $\{G_\rho : \rho \in k^+\}$ (as constructed above) of subgroups yields a disjoint covering $\{\Lambda_\gamma : \gamma < k^+\}$ of X where $\Lambda_\gamma = (G_\gamma \setminus \bigcup_{\eta<\gamma} G_\eta)x$. Again as $\{x\} \in \mathcal{S}_0$ and $\mathcal{S}_0$ satisfies the k-chain condition with respect to $\{\mathcal{N}_\alpha^{\ 0}\}_{0\le\alpha<k}$, so $\{x\} \in \bigcap_{0\le\alpha<k} \mathcal{N}_\alpha^{\ 0}$. Now following an exactly similar pattern of argument as used in the case of theorem 2.5 with L replaced by $\{x\}$, we may find that $\{\mathcal{H}_{\alpha,\beta} : 0 \le \alpha < k, 0 \le \beta < k\} \nsubseteq \mathcal{S}_0$ where $\mathcal{H}_{\alpha,\beta} = \bigcup_{\gamma\in E_{\alpha,\beta}} \Lambda_\gamma$.

Consequently, there exists $\mathcal{H}_{\alpha_0,\beta_0} \notin \mathcal{S}_0$. We set $\mathcal{S}_1$ = the k-additive algebra on X generated by $\mathcal{S}_0 \cup \{\mathcal{H}_{\alpha_0,\beta_0}\} \cup \{(X \setminus \mathcal{H}_{\alpha_0,\beta_0}) \cap \mathcal{H}_{\alpha,\beta} : (\alpha, \beta) \neq (\alpha_0, \beta_0)\}$

and $\mathcal{S}_2$ = the k-additive algebra on X generated by $\mathcal{S}_0 \cup \{X \setminus \mathcal{H}_{\alpha_0,\beta_0}\} \cup \{\mathcal{H}_{\alpha_0,\beta_0} \cap \mathcal{H}_{\alpha,\beta} : (\alpha, \beta) \neq (\alpha_0, \beta_0)\}$

Since $\mathcal{S}_0$ is diffused and also for any $g \in G$ and any set $\Sigma \subseteq k^+$,
card $(g(\bigcup_{\gamma\in\Sigma}\Lambda_\gamma)\Delta (\bigcup_{\gamma\in\Sigma}\Lambda_\gamma)) \le k$, therefore both $\mathcal{S}_1$ and $\mathcal{S}_2$ are k-additive algebra on (X, G).

We now show that $\mathcal{S}_1$ is admissible with respect to the transfinite k-sequence $\{\mathcal{N}_\alpha^{\ 1}\}_{0\le\alpha<k}$ where $\mathcal{N}_\alpha^{\ 1} = \{E \subseteq X : E \cap \mathcal{H}_{\alpha_0,\beta_0} \in \mathcal{N}_\alpha^{\ 0}\}$

To prove this, we need only verify conditions **(iv)** and **(vi)** for the remaining conditions can be verified using the definition of $\{\mathcal{N}_\alpha^{\ 1}\}_{0\le\alpha<k}$ and a simple observation that $\mathcal{N}_\alpha^{\ 0} \subseteq \mathcal{N}_\alpha^{\ 1}$ for $0 \le \alpha < k$. Let $E \in \mathcal{N}_\alpha^{\ 1}$. Then $E \cap \mathcal{H}_{\alpha_0,\beta_0} \in \mathcal{N}_\alpha^{\ 0}$ by definition and therefore $g(E \cap \mathcal{H}_{\alpha_0,\beta_0}) \in \mathcal{N}_\alpha^{\ 0}$ by condition **(iv)**. Now $gE \cap \mathcal{H}_{\alpha_0,\beta_0} \subseteq (g(\mathcal{H}_{\alpha_0,\beta_0})\Delta\mathcal{H}_{\alpha_0,\beta_0})\Delta(g(E\cap\mathcal{H}_{\alpha_0,\beta_0}))$ where $g(\mathcal{H}_{\alpha_0,\beta_0})\Delta\mathcal{H}_{\alpha_0,\beta_0} \in \bigcap_{0\le\alpha<k} \mathcal{N}_\alpha^{\ 0}$ because card $(g(\mathcal{H}_{\alpha_0,\beta_0})\Delta\mathcal{H}_{\alpha_0,\beta_0}) \le k$ and

$\mathcal{S}_0$ is diffused satisfying k-chain condition with respect to $\{\mathcal{N}_\alpha{}^0\}_{0\leq\alpha<k}$. Moreover, $\mathcal{N}_\alpha{}^0 \Delta(\bigcap_{0\leq\alpha<k} \mathcal{N}_\alpha{}^0) \subseteq \mathcal{N}_\alpha{}^0$ by hypothesis. Hence $gE \cap \mathcal{H}_{\alpha_0,\beta_0} \in \mathcal{N}_\alpha{}^0$ and so $gE \in \mathcal{N}_\alpha{}^1$. This proves that $\{\mathcal{N}_\alpha{}^1\}_{0\leq\alpha<k}$ satisfies condition **(iv)**.

Now let $\{E_\xi : \xi < k\}$ be a nested k-sequence of sets from $\mathcal{S}_1$ such that $\bigcap_\xi E_\xi \in \mathcal{N}_\alpha{}^1$. Then $\{E_\xi \cap \mathcal{H}_{\alpha_0,\beta_0} : \xi < k\}$ is also a nested k-sequence of sets from $\mathcal{S}_1$ such that $\bigcap_\xi (E_\xi \cap \mathcal{H}_{\alpha_0,\beta_0}) \in \mathcal{N}_\alpha{}^0$. As $\mathcal{S}_0$ is regular with respect to $\{\mathcal{N}_\alpha{}^0\}_{0\leq\alpha<k}$, so there exists $F_\alpha \supseteq E_\alpha \cap \mathcal{H}_{\alpha_0,\beta_0}$ that $F_\alpha \in \mathcal{S}_0$ and $\bigcap_\alpha F_\alpha \in \mathcal{N}_\alpha{}^0$. But $\mathcal{S}_0$ is admissible with respect to $\{\mathcal{N}_\alpha{}^0\}_{0\leq\alpha<k}$. So by condition **(vi)**, there exists $F_\alpha \in \mathcal{N}_\alpha{}^0$ and therefore $E_\alpha \cap \mathcal{H}_{\alpha_0,\beta_0} \in \mathcal{N}_\alpha{}^0$ by condition **(v)**. Hence $E_\alpha \in \mathcal{N}_\alpha{}^1$.This proves that $\mathcal{S}_1$ is admissible with respect to $\{\mathcal{N}_\alpha{}^1\}_{0\leq\alpha<k}$. In a similar manner, we can show $\mathcal{S}_2$ is admissible with respect to the transfinite k-sequence $\{\mathcal{N}_\alpha{}^2\}_{0\leq\alpha<k}$ where $\mathcal{N}_\alpha{}^2 = \{E \subseteq X : E \cap (X \setminus \mathcal{H}_{\alpha_0,\beta_0}) \in \mathcal{N}_\alpha{}^0\}$.

Now let T be the k-additive algebra on (X, G) generated by $\mathcal{S}_1 \cup \mathcal{S}_2$. Then T is not admissible, for otherwise, if it is admissible with respect to some k-sequence $\{\mathcal{N}_\alpha\}_{0\leq\alpha<k}$ then as T contains the family $(F_{\alpha,\beta})_{0\leq\alpha<k, 0\leq\beta<k}$ it would contradict theorem 2.5.

Hence the theorem.

Remarks : If $X = \mathbb{R}$ and $G = (\mathbb{R}, +)$ (the usual additive group of Real numbers). Then with respect to this particular transformation group $(\mathbb{R}, \mathbb{R})$, $(\mathbb{R}, +)$ acts freely and transitively on $\mathbb{R}$. Let $\mathcal{B}$ denotes the σ-algebra of Borel sets with μ^* as the outer measure generated by the Borel measure μ. Then $\mathcal{B}$ is an ω-additive algebra on $(\mathbb{R}, \mathbb{R})$ which is admissible with respect to the ω-sequence $\{\mathcal{M}_n\}_{n<\omega}$ where $\mathcal{M}_n = \{E \subseteq \mathbb{R} : \mu^*(E) \leq \frac{1}{n}\}$. It is also diffused, regular and satisfies ω-chain (or, countable chain) condition. This idea is generalized in this paper using the concept of an admissible k-additive algebra (for any arbitrary infinite cardinal k) on a space (X, G) with transformation group G which is completely abstract and devoid of the use of any measure type functionals. It is worth mentioning here that Kharazishvili gave another generalization [8] of Pelc and Prikry's theorem using the fact that under Martin's axiom there exist absolutely nonmeasurable functions.

References

[1] S.Banach , *Sur nes snites d'ensembles exchant l' existence d' une measure (note posthnme avec preface et commentaire de E. Marezewski)* , collaq.Math.1(1948),103 – 108.

[2] E.Grzegorek, *Remarks on σ-fields without continuous measures*, collaq.Math. Vol 39 (1978),73 – 75.

[3] E.Grzegorek, *Solution of a problem of Banach on σ-fields without continuous measures*, Bull. Acad. Polon. Sci., Ser. Sci. Math., Vol. 28 (1980), 7 – 10.

[4] A.B.Kharazishvili, *Transformation groups and invariant measurse*, World Scientific, 1988.

[5] A.B.Kharazishvili, *Nonmeasurable sets and functions*, Elsevier, 2004.

[6] A.B.Kharazishvili, *Topics in measure theory and Real Analysis*, Atlantis Press , World Scientific , 2009.

[7] A.B.Kharazishvili, *On countably generated invariant σ-algebras which do not admit measure type functionals* , Real. Analysis. Exchange, Vol 23 (1), 1998-99, 287 – 294.

[8] A.B.Kharazishvili, *On absolutely nonmeasurable additive functions*, Georgian Mathematical Journal, Vol 11 (2004), No 2, 301 – 306.

[9] A.Pelc and K.Prikry, *On a problem of Banach*, Proc. Amer. Math. Soc. Vol 89, No 4, Dec 1983.

[10] B.Riećan, *Abstract formulation of some theorems of measure theory*, Math. Slovaca. Vol 16 (1966), No 3, 268 – 273.

[11] B.Riećan, *Abtract formulations of some theorems of measure theory II*, Math. Casopis, Vol 19 (1969), No 2, 138 – 144

[12] B.Riećan and T.Neubraunn, *Integral, Measure and Ordering*, Kluwer Academic Publisher, Bratislava, 1977.