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Continuity of Spectra on Class of p-$wA(s,t)$ Operators

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Abstract. In this paper, we show that spectrum, Weyl spectrum and Browder spectrum are continuous on the set of p-$wA(s,t)$ operators with $0 < p \leq 1$ and $0 < s, t, s+t \leq 1$.

1. Introduction

Let $\mathcal{B}(\mathcal{H})$ denote the algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$. Finding subsets $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$ for which the spectrum σ is continuous when restricted to $\mathcal{A}$ is one of challenging problems in operator theory.

Every operator $T \in \mathcal{B}(\mathcal{H})$ can be decomposed into $T = U|T|$ with a partial isometry U where $|T|$ is the square root of T^*T. If U is determined uniquely by the kernel condition $\ker U = \ker |T|$, then this decomposition is called the polar decomposition of T. In this paper, $T = U|T|$ denotes the polar decomposition satisfying the kernel condition $\ker U = \ker |T|$. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be hyponormal if $T^*T \geq TT^*$. The Aluthge transformation introduced by Aluthge [1] is defined by $T(\frac{1}{2},\frac{1}{2}) = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ where $T = U|T|$ be the polar decomposition of $T \in \mathcal{B}(\mathcal{H})$. The generalized Aluthge transformation $T(s,t)$ with $0 < s,t$ is defined by $T(s,t) = |T|^sU|T|^t$. Recall that an operator $T \in \mathcal{B}(\mathcal{H})$ is said to be p-hyponormal if $(T^*T)^p \geq (TT^*)^p$, and class $wA(s,t)$ if $(|T^*|^t|T|^{2s}|T^*|^t)^{\frac{t}{s+t}} \geq |T^*|^{2t}$ and $|T|^{2s} \geq (|T|^s|T^*|^{2t}|T|^s)^{\frac{s}{s+t}}$ ([13]). In [18], the authors introduced class p-$wA(s,t)$ operators as follows,

Definition 1.1. [18] *Let $T = U|T|$ be the polar decomposition of T and let $0 < p \leq 1$ and $0 < s, t$. T is called class p-$wA(s,t)$ if*

$$(|T^*|^t|T|^{2s}|T^*|^t)^{\frac{tp}{s+t}} \geq |T^*|^{2tp},$$
$$(|T|^s|T^*|^{2t}|T|^s)^{\frac{sp}{s+t}} \leq |T|^{2sp}.$$

Class p-$wA(s,t)$ operators are extension of hyponormal operators and many interesting properties have been studied in [3, 4, 18–21].

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Let $\mathcal{G}$ denote the set of all compact subsets of $\mathbb{C}$ equipped with the Hausdorff metric. If $\mathcal{A}$ is a unital Banach algebra, then the spectrum can be viewed as a function $\sigma : \mathcal{A} \to \mathcal{G}$, mapping each $T \in \mathcal{A}$ to its spectrum $\sigma(T) \in \mathcal{G}$.

If $\{T_n\}$ is a sequence of elements of a unital Banach algebra $\mathcal{A}$, then

$$\liminf_n \sigma(T_n) = \{\lambda : \text{ there exists } \lambda_n \in \sigma(T_n) \text{ such that } \lambda = \lim_{n\to\infty} \lambda_n\},$$
$$\limsup_n \sigma(T_n) = \{\lambda : \text{ there exists } \lambda_{n_k} \in \sigma(T_{n_k}) \text{ such that } \lambda = \lim_{k\to\infty} \lambda_{n_k}\}.$$

If $\liminf_n \sigma(T_n) = \limsup_n \sigma(T_n)$, we say

$$\lim_n \sigma(T_n) = \liminf_n \sigma(T_n) = \limsup_n \sigma(T_n).$$

It is known that if T_n converges to T in $\mathcal{A}$, then

$$\liminf_n \sigma(T_n) \subset \limsup_n \sigma(T_n) \subset \sigma(T).$$

Hence the function σ is upper semicontinuous. We say σ is continuous if

$$\sigma(T) = \lim_n \sigma(T_n),$$

or equivalently

$$\sigma(T) \subset \lim_n \sigma(T_n).$$

It is known that σ does have points of discontinuity in noncommutative algebras. The work of J. Newburgh [17] contains fundamental results on spectral continuity in general Banach algebras. J. Conway and B. Morrel [6] have undertaken a detailed study of spectral continuity in the case where the Banach algebra is the C^*-algebra of all operators acting on a complex separable Hilbert space. Farenick and Lee [9] and Hwang and Lee [11] considered the spectral continuity when restricted to certain subsets of the entire manifold of Toeplitz operators. Hwang and Lee [12] proved that $\sigma(T)$ is continuous on the set of p-hyponormal operatrors. Jeon and Kim [15] proved that $\sigma(T)$ is continuous on the set of class $A(k, 1)$ operators by using the arguments of Chō and Yamazaki [5]. In this article we focus continuity of spectrum, Weyl spectrum and Browder spectrum on the set of class p-$wA(s, t)$ operators with $0 < p \le 1$ and $0 < s, t, s + t \le 1$. There are many papers in the topic of spectral continuity for non normal operators. We refer the reader to [7, 15, 16].

Let $\alpha(T)$ and $\beta(T)$ denote the nullity and the deficiency of $T \in \mathcal{B}(\mathcal{H})$, defined by $\alpha(T)$= dim(ker(T)) and $\beta(T)$=dim(ker(T^*)). An operator T is said to be upper semi-Fredholm (resp., lower semi- Fredholm) if the range $R(T)$ of $T \in B(\mathcal{H})$ is closed and $\alpha(T) < \infty$ (resp., $\beta(T) < \infty$). Let $SF_+(\mathcal{H})$ (resp., $SF_-(\mathcal{H})$) denote the semigroup of upper semi-Fredholm (resp., lower semi-Fredholm) operators on $\mathcal{H}$. An operator $T \in B(\mathcal{H})$ is said to be semi-Fredhom if $T \in SF_+(\mathcal{H}) \cup SF_-(\mathcal{H}) = SF(\mathcal{H})$, and Fredholm if $T \in SF_+(\mathcal{H}) \cap SF_-(\mathcal{H}) = F(\mathcal{H})$. The index of semi-Fredholm operator $T \in SF(\mathcal{H})$ is defined by ind $(T) = \alpha(T) - \beta(T)$. Recall, the ascent $a(T)$ of $T \in B(\mathcal{H})$ is the smallest non negative integer p such that $\ker(T^p) = \ker(T^{p+1})$. If such p does not exist, then $a(T) = \infty$. The descent $d(T)$ of T is the smallest non negative integer q such that $R(T^q) = R(T^{q+1})$. If such q does not exist, then $d(T) = \infty$. An operator $T \in B(\mathcal{H})$ is Weyl if T is Fredholm of index zero, and Browder if T is Fredholm of finite ascent and descent. The Weyl spectrum $\sigma_w(T)$ and the Browder spectrum $\sigma_b(T)$ of T are defined by

$$\sigma_w(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Weyl}\},$$
$$\sigma_b(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Browder}\}.$$

We say that $T \in B(\mathcal{H})$ satisfies Weyl's theorem if

$$\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T)$$

where $\pi_{00}(T)$ denote the set of all isolated points $\lambda \in \sigma(T)$ for which $0 < \dim\ker(T - \lambda) < \infty$.

2. Results

We prove the spectrum, Weyl spectrum and Browder spectrum are continious in the class of all p-$wA(s,t)$ operator with $0 < p \leq 1$ and $0 < s, t, s+t \leq 1$.

Proposition 2.1. [18] *Let $T \in B(\mathcal{H})$ and $0 < p \leq 1$ and $0 < s, t$. Let $T = U|T|$ be the polar decomposion of T and $T(s,t) = |T|^s U |T|^t$. Then T is class p-$wA(s,t)$ if and only if*

$$|T(s,t)|^{\frac{2tp}{s+t}} \geq |T|^{2tp}$$

and

$$|T|^{2sp} \geq |T(s,t)^*|^{\frac{2sp}{s+t}}.$$

Hence

$$|T(s,t)|^{\frac{2\rho p}{s+t}} \geq |T|^{2\rho p} \geq |T(s,t)^*|^{\frac{2\rho p}{s+t}} \tag{1}$$

and $T(s,t)$ is $\frac{\rho p}{s+t}$-hyponormal for any $\rho \in (0, \min\{s,t\}]$.

A complex number λ is said to be an approximate eigenvalue of T if there exists a sequence $\{x_n\}$ of unit vectors such that

$$(T - \lambda)x_n \to 0 \quad (n \to \infty).$$

We denote the set of all approximate eigenvalues of T by $\sigma_a(T)$. We say that $\lambda \in \sigma(T)$ belongs to the (Xia's) residual spectrum $\sigma_r^X(T)$ of T if $(T - \lambda)\mathcal{H} \neq \mathcal{H}$ and there exists a positive number $c > 0$ such that

$$\|(T - \lambda)x\| \geq c\|x\| \quad \text{for} \quad x \in \mathcal{H}.$$

By the definition, $\sigma(T)$ is a disjoint union of $\sigma_a(T)$ and $\sigma_r^X(T)$.

Proposition 2.2. [19] *If $T = U|T| \in B(\mathcal{H})$ is class p-$wA(s,t)$ with $0 < p \leq 1$ and $0 < s, t, s+t \leq 1$ and if $T_\alpha = U|T|^\alpha$ with $s + t \leq \alpha$, then*

$$\begin{aligned} \sigma_a(T_\alpha) &= \{r^\alpha e^{i\theta} \mid re^{i\theta} \in \sigma_a(T)\}, \\ \sigma_r^X(T_\alpha) &= \{r^\alpha e^{i\theta} \mid re^{i\theta} \in \sigma_r^X(T)\}, \\ \sigma(T_\alpha) &= \{r^\alpha e^{i\theta} \mid re^{i\theta} \in \sigma(T)\}. \end{aligned}$$

Theorem 2.3. *Let $T_n, T \in B(\mathcal{H})$ and $0 < p \leq 1, 0 < s, t, s+t \leq 1$. If T_n is class p-$wA(s,t)$ and $\|T_n - T\| \to 0$, then T is class p-$wA(s,t)$ and $\lim_n \sigma(T_n) = \sigma(T)$. Hence the spectrum σ is continuous on the set of p-$wA(s,t)$ operators with $0 < p \leq 1$ and $0 < s, t, s+t \leq 1$.*

Proof. Let $T_n = U_n|T_n|, T = U|T|$ be the polar decomposions of T_n, T and T_n be class p-$wA(s,t)$. Let $\|T_n - T\| \to 0$. Then $|T_n|^2 = T_n^* T_n \to T^*T = |T|^2$. Similarly, $|T_n|^{2k} \to |T|^{2k}$ for $k = 1, 2, \cdots$. We may assume $\sigma(|T_n|), \sigma(|T|) \subset [0, M]$. Then for any $\varepsilon > 0$ there exists a polynomial $p(t)$ such that $\sup\{|\sqrt{t} - p(t)| : 0 \leq t \leq M\} < \varepsilon$. Since $p(|T_n|^2) \to p(|T|^2)$, we have $|T_n| \to |T|$. Similarly, we have $|T_n|^s \to |T|^s$ and $|T_n|^t \to |T|^t$. Since

$$T_n - U_n|T| = U_n(|T_n| - |T|) \to 0,$$

we have

$$U_n|T| = T_n - U_n(|T_n| - |T|) \to T = U|T|.$$

Hence

$$(U_n - U)|T| \to 0.$$

Then $(U_n - U)|T|^2, (U_n - U)|T|^3 \to 0, \cdots$, and we have $(U_n - U)|T|^t \to 0$. Then

$$U_n|T_n|^t = U_n(|T_n|^t - |T|^t) + (U_n - U)|T|^t + U|T|^t \to U|T|^t$$

because $\|U_n\| \leq 1$. Hence

$$T_n(s,t) = |T_n|^s U_n |T_n|^t \to |T|^s U |T|^t = T(s,t).$$

Since

$$T_n^* - |T|U_n^* = (|T_n| - |T|)U_n^* \to 0,$$

we have

$$|T|U_n^* = T_n^* - (|T_n| - |T|)U_n^* \to T^* = |T|U^*,$$

and

$$|T|(U_n^* - U^*) \to 0.$$

Then $|T|^2(U_n^* - U^*) \to 0$, and $|T|^t(U_n^* - U^*) \to 0$. Hence

$$|T_n|^t U_n^* = \left(|T_n|^t - |T|^t\right) U_n^* + |T|^t(U_n^* - U^*) + |T|^t U^* \to |T|^t U^*,$$

and

$$(T_n(s,t))^* = |T_n|^t U_n^* |T_n|^s \to |T|^t U^* |T|^s = (T(s,t))^*.$$

Since T_n is class p-$wA(s,t)$, this implies that T is class p-$wA(s,t)$.

Then $T_n(s,t), T(s,t)$ are $\frac{\rho p}{s+t}$-hyponormal by Proposition 2.1. Since σ is continuous on the set of p-hyponormal operators by [12], we have

$$\sigma(T(s,t)) = \liminf_n \sigma(T_n(s,t)).$$

Since

$$\sigma(|T|^s U |T|^t) = \sigma(U|T|^{s+t}), \sigma(|T_n|^s U_n |T_n|^t) = \sigma(U_n|T_n|^{s+t})$$

by [23, Lemma 6] and

$$\sigma(U|T|^{s+t}) = \{r^{s+t}e^{i\theta} : re^{i\theta} \in \sigma(T)\}, \sigma(U_n|T_n|^{s+t}) = \{r^{s+t}e^{i\theta} : re^{i\theta} \in \sigma(T_n)\}$$

by Proposition 2.2, we have

$$\sigma(T) = \liminf_n \sigma(T_n).$$

This completes proof. □

Corollary 2.4. *The Weyl spectrum and Browder spectrum are continious on the set of p-$wA(s,t)$ operators with $0 < p \leq 1$ and $0 < s, t, s+t \leq 1$.*

Proof. By [20, Theorem 5.1], class p-$wA(s,t)$ operator T with $0 < p \leq 1$ and $0 < s, t, s+t \leq 1$ satisfies Weyl's theorem. From Theorem 2.3, spectrum σ is continuous on the set of class p-$wA(s,t)$ operators with $0 < p \leq 1$ and $0 < s, t, s+t \leq 1$. Applying Theorem 2.2 and Theorem 2.3 of [8], it follows that Weyl spectrum and Browder spectrum are continious on the set of class p-$wA(s,t)$ operators with $0 < p \leq 1$ and $0 < s, t, s+t \leq 1$. □

It is known that p-hyponormal operators with $0 < p \leq 1$ and log-hyponormal operators are class 1-$wA(s,t)$ operators for any $0 < s, t$. Class 1-$wA(1/2, 1/2)$ is called p-w-hyponormal ([2], [13]).

Corollary 2.5. *The Weyl spectrum and Browder spectrum are continious on the set of p-w-hyponormal operators with $0 < p \leq 1$.*

Definition 2.6. [23] *Let $T = U|T|$ be the polar decomposition of T and let $0 < s, t$. T is called class $A(s,t)$ if*

$$(|T|^s|T^*|^{2t}|T|^s)^{\frac{s}{s+t}} \leq |T|^{2s}.$$

Ito and Yamazaki [14] proved that if T is class $A(s,t)$, then

$$(|T^*|^t|T|^{2s}|T^*|^t)^{\frac{t}{s+t}} \geq |T^*|^{2t}.$$

Hence T is class 1-$wA(s,t)$. Hence we have the following Corollary.

Corollary 2.7. *The spectrum, Weyl spectrum and Browder spectrum are continious on the set of class $A(s,t)$ operators with $0 < s, t, s+t \leq 1$.*

However, in this case, we prove more general results.

Theorem 2.8. *The spectrum, Weyl spectrum and Browder spectrum are continious on the set of class $A(s,t)$ operators with $0 < s, t \leq 1$.*

Proof. We prove the case $1 < s+t$. Let $T_n, T \in B(\mathcal{H})$ and $T_n = U_n|T_n|, T = U|T|$ be the polar decomposions of T_n, T. Let T_n be class $A(s,t)$ and $\|T_n - T\| \to 0$. Then T is class $A(s,t)$,

$$T_n(s,t) = |T_n|^s U_n |T_n|^t \to |T|^s U |T|^t$$

and

$$\sigma(T(s,t)) = \lim\inf_n \sigma(T_n(s,t)).$$

by the same arguement of Theorem 2.3. Since

$$\sigma(|T|^s U |T|^t) = \sigma(U|T|^{s+t}), \sigma(|T_n|^s U_n |T_n|^t) = \sigma(U_n|T_n|^{s+t})$$

by [23, Lemma 6] and

$$\sigma(U|T|^{s+t}) = \{r^{s+t}e^{i\theta} : re^{i\theta} \in \sigma(T)\}, \sigma(U_n|T_n|^{s+t}) = \{r^{s+t}e^{i\theta} : re^{i\theta} \in \sigma(T_n)\}$$

by [23, Theorem 5], we have

$$\sigma(T) = \lim\inf_n \sigma(T_n).$$

Hence the spectrum is continious on the set of class $A(s,t)$ operators with $0 < s, t \leq 1$.

It is known that $A(s,t)$ operator T with $0 < s, t \leq 1$ is class $A(1,1)$ by [13, Theorem 3.5] and class $A(1,1)$ operator is paranormal by [10, Theorem 1]. Class $A(1,1)$ operator is called class A and Uchiyama [22] proved class A operators satisfy Weyl's theorem. The rest of the proof is similar to the proof of Corollary 2.4. □

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