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An Investigation of Incomplete $H-$Functions Associated with Some Fractional Integral Operators

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Abstract. Arbitrary-order integral operators find variety of implementations in different science disciplines as well as engineering fields. The study presented as part of this research paper derives motivation from the fact that applications of fractional operators and special functions demonstrate a huge potential in understanding many of physical phenomena. Study and investigation of a fractional integral operator containing an incomplete $H-$ functions (IHFs) as the kernel is the primary objective of the research work presented here. Specifically, few interesting relations involving the new fractional operator with IHFs in its kernel and classical Riemann Liouville(R-L) fractional integral and derivative operators, the Hilfer fractional derivative operator, the generalized composite fractonal derivate operaor are established. Results established by the authors in [1–3] follow as few interesting and significant special cases of our main results.

1. Introduction

Fractional calculus has been used in several scientific areas including fuzzy control, physics, automatic control, biology, signal processing and robotics because it can represent systems and physical phenomenon more accurately than classical integer-order calculus. Fractional derivatives are shown to give a more clear picture when applied to an engineering problem because of the broad spectrum they work in and their memory effects. See also [4–12]. The IHFs find applications in heat conduction, propbability theory and to study the Fourier and Laplace transforms. The solutions to a number of problems of applied mathematics, astrophysics, nuclear physics, ground water, statistics and engineering may be represented in the form of IHFs. Many researchers across the globe have established the fact that the investigations of analytic solutions of different problems occuring in various science and engineering fields are enhanced by the function $\Gamma(\mu)$ and its incomplete versions $\gamma(\mu, u)$ and $\Gamma(\mu, u)$ (see, e.g., [13–16]).
The dynamic integration of Fractional derivative operators and IHFs will contribute greatly to the literature of applied Mathematics for the researchers to use it in specific problems of science and engineering along with their interesting properties as shown in this research work.

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2. Preliminary tools

Not long ago, following definitions of IHFs $\gamma_{p,q}^{m,n}(z)$ and $\Gamma_{p,q}^{m,n}(z)$ was given by Srivastava et al. [17, Eqs.(2.1)-(2.4)]:

$$\gamma_{p,q}^{m,n}(z) = \gamma_{p,q}^{m,n}\left[z \;\middle|\; \begin{array}{c} (e_1, E_1, y), (e_i, E_i)_{2,p} \\ \\ (f_i, F_i)_{1,q} \end{array} \right] = \gamma_{p,q}^{m,n}\left[z \;\middle|\; \begin{array}{c} (e_1, E_1, y), (e_2, E_2), \cdots, (e_p, E_p) \\ \\ (f_1, F_1), (f_2, F_2), \cdots, (f_q, F_q) \end{array} \right]$$
$$:= \frac{1}{2\pi i} \int_{\mathfrak{C}} \mathbb{G}(\xi, y)\, z^{-\xi} d\xi,$$

where

$$\mathbb{G}(\xi, y) = \frac{\gamma(1 - e_1 - E_1\xi, y) \prod_{i=1}^{m} \Gamma(f_i + F_i\xi) \prod_{i=2}^{n} \Gamma(1 - e_i - E_i\xi)}{\prod_{i=m+1}^{q} \Gamma(1 - f_i - F_i\xi) \prod_{i=n+1}^{p} \Gamma(e_i + E_i\xi)}. \tag{1}$$

and

$$\Gamma_{p,q}^{m,n}(z) = \Gamma_{p,q}^{m,n}\left[z \;\middle|\; \begin{array}{c} (e_1, E_1, y), (e_i, E_i)_{2,p} \\ \\ (f_i, F_i)_{1,q} \end{array} \right] = \Gamma_{p,q}^{m,n}\left[z \;\middle|\; \begin{array}{c} (e_1, E_1, y), (e_2, E_2), \cdots, (e_p, E_p) \\ \\ (f_1, F_1), (f_2, F_2), \cdots, (f_q, F_q) \end{array} \right]$$
$$:= \frac{1}{2\pi i} \int_{\mathfrak{C}} \mathbb{F}(\xi, y)\, z^{-\xi} d\xi,$$

where

$$\mathbb{F}(\xi, y) = \frac{\Gamma(1 - e_1 - E_1\xi, y) \prod_{i=1}^{m} \Gamma(f_i + F_i\xi) \prod_{i=2}^{n} \Gamma(1 - e_i - E_i\xi)}{\prod_{i=m+1}^{q} \Gamma(1 - f_i - F_i\xi) \prod_{i=n+1}^{p} \Gamma(e_i + E_i\xi)}. \tag{2}$$

The IHFs $\gamma_{p,q}^{m,n}(z)$ and $\Gamma_{p,q}^{m,n}(z)$ given by Eqs. (1) and (2) survive for every $y \geqq 0$ inside identical contour $\mathfrak{C}$ and the similar group of conditions asserted in [17] (see, also, for details, [18, 19]). A complete description of (1) and (2) can be found in [17].
We recall here classical definition of R-L fractional integral operator $\mathbb{I}_{a+}^{\eta}$ given by [18, 20, 21]:

$$(\mathbb{I}_{a+}^{\eta} f)(x) = \frac{1}{\Gamma(\eta)} \int_a^t \frac{f(x)}{(t-x)^{1-\eta}}\, dx, \qquad \Big(\Re(\eta) > 0\Big). \tag{3}$$

The familiar definition of Riemann-Liouville fractional derivative operator $\mathbb{D}_{a+}^{\eta}$ defined by [18, 20, 21]:

$$(\mathbb{D}_{a+}^{\eta} f)(x) = \left(\frac{d}{dx}\right)^{\mathfrak{n}} \left(\mathbb{I}_{a+}^{\mathfrak{n}-\eta} f\right)(x), \qquad \Big(\Re(\eta) > 0;\ \mathfrak{n} = [\Re(\eta)] + 1\Big), \tag{4}$$

here $[\xi]$ is the greatest integer in the real number ξ.
The well known definition of Liouville-Caputo fractional derivative ${}_a^C D_x^{\alpha}$ defined by [18, 21]:

$${}_a^C D_x^{\alpha} f(x) = {}_a\mathbb{I}_x^{n-\alpha} \mathbb{D}_x^n f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} f^n(t) dt, \qquad (n-1 < \alpha \leq n, n \in \mathrm{N}). \tag{5}$$

The operators in (4) and (5) were generalized by Hilfer [22] and definition of fractional derivative operator $\mathfrak{D}_{a+}^{\mu,\nu}$ of order μ $(0<\mu<1)$ and type ν $(0 \leqq \nu \leqq 1)$ w.r.t to x was given by:

$$\left(\mathfrak{D}_{a+}^{\mu,\nu} f\right)(x) = \left(\mathbb{I}_{a+}^{\nu(1-\mu)} \frac{d}{dx}\left(\mathbb{I}_{a+}^{(1-\nu)(1-\mu)} f\right)\right)(x). \tag{6}$$

The fractional operator defined in (6) reduces to the R-L fractional derivative operator $\mathbb{D}_{a+}^{\mu}$ whenever $\nu=0$. Further, when $\nu=1$, (6) gives the Liouville-Caputo fractional derivative operator [1, 23, 24].
The generalized composite fractional derivative (GCFD) is given by Garg et al. [26] as follows:

$$\left({}_a\mathcal{D}_x^{\alpha,\beta;\nu} f\right)(x) = \left({}_a\mathbb{I}_x^{\nu(n-\beta)} D_x^n \left({}_a\mathbb{I}_x^{(1-\nu)(n-\alpha)} f\right)\right)(x), \tag{7}$$
$$(n-1<\alpha,\beta\le n, 0\le\nu\le 1, n\in\mathbb{N}).$$

If $\nu=0$ and $\nu=1$, then (7) reduces to the R-L operator (4) and Liouville-Caputo derivative (5), resp.
If $\alpha=\beta$, then GCFD reduces to the Hilfer fractional derivative (6). The details of various fractional integral operators with different kernels can be found in [25]

Theorem 2.1. *[26, p. 1072, Eq. (17)] If $\lambda>0$, $n-1<\alpha,\beta\le n$, $0\le\nu\le 1$, $n\in\mathbb{N}$, then the power function formula for the GCFD is defined as follows:*

$${}_0\mathcal{D}_x^{\alpha,\beta;\nu}(x-a)^{\lambda-1} = \frac{\Gamma(\lambda)}{\Gamma(\nu(\alpha-\beta)+\lambda-\alpha)}(x-a)^{\nu(\alpha-\beta)+\lambda-\alpha-1}. \tag{8}$$

3. Operator involving the incomplete *H*-function

This section presents the introduction of a fractional integral operator involving incomplete *H*-function defined by (2) and hence explore some of its interesting properties.

$$\left(\mathbb{E}_{a^+;p,q;\mu}^{w;m,n;\lambda}\varphi\right)(x) = \int_a^x (x-\rho)^{\mu-1}\Gamma_{p,q}^{m,n}\left[w(x-\rho)^{\lambda}\right]\varphi(\rho)d\rho \qquad x>b, \tag{9}$$
$$\left(\lambda,\mu,w\in\mathbb{C};\quad \Re(\mu)+\lambda\min_{1\le j\le m}\Re\left(\frac{f_j}{F_j}\right)>0\right).$$

Initially the following result is established:

Theorem 3.1. *Assuming already stated conditions given in the definition (9), The operator $\mathbb{E}_{a^+;p,q;\mu}^{w;m,n;\lambda}$ is bounded on $\mathfrak{L}(a,b)$ and*

$$\|\mathbb{E}_{a^+;p,q;\mu}^{w;m,n;\lambda}\Theta\|_1 \leqq m\cdot\|\Theta\|_1, \tag{10}$$

where the constant m $(0<m<\infty)$ *is given by*

$$m := \frac{(b-a)^{\Re(\mu)}}{2\pi i}\int_{\mathfrak{C}} \mathbb{F}(\xi,y)\frac{[w(b-a)^{\Re(\lambda)}]^{-\xi}}{[\Re(\mu)-\Re(\lambda)\xi]}d\xi \tag{11}$$

Given, $\mathfrak{L}(a,b)$ to be space of Lebesgue measurable functions on a finite interval $[a,b]$ $(b>a)$ of the real line $\mathbb{R}$ given by

$$\mathfrak{L}(a,b) = \left\{\mathfrak{g} : \|\mathfrak{g}\|_1 := \int_a^b |\mathfrak{g}(x)|dx < \infty\right\}. \tag{12}$$

Proof. Using the reasoning used by Srivastava et.al[3], it is sufficient to prove that

$$\|\mathbb{E}^{w;m,n;\lambda}_{a^+;p,q;\mu}\Theta\|_1 = \int_a^b \left| \int_a^x (x-\rho)^{\mu-1}\Gamma^{m,n}_{p,q}\left[w(x-\rho)^{\lambda}\right]\Theta(\rho)d\rho \right| dx < \infty \tag{13}$$

$$\left(\lambda, \mu, w \in C; \quad \Re(\mu) + \lambda \min_{1\le j\le m} \Re\left(\frac{f_j}{F_j}\right) > 0\right)$$

Applying the definitions (9) and (12) along with expression (2) for IHFs. Making use of Dirichlet formula [20, p. 56] , for interchanging the order of integration, Theorem 2 follows from previous results in [2] and [3]. □

Remark 3.2. *The results established by Kilbas et al. in [1] , by Srivastava and Tomovski [2] as well as the results obtained in [3] are the special cases of our Theorem 3.1.*

Theorem 3.3. *Let*

$$x > a \ \left(a \in R^+ = [0, \infty)\right), \quad m-1 < \alpha, \beta \le m, \quad 0 \leqq \nu \leqq 1, \qquad m \in N.$$

Suppose also that

$$\lambda, \mu, w \in C, \quad \Re(\lambda) > 0$$

Then

$$\begin{aligned} &\left({}_a\mathcal{D}_x^{\alpha,\beta;\nu}\left[(t-a)^{\lambda-1}\ \Gamma^{m,n}_{p,q}[w(t-a)^{\mu}]\right]\right)(x) \\ &= (x-a)^{\lambda+\nu(\alpha-\beta)-\alpha-1}\Gamma^{m,n+1}_{p+1,q+1}\left[w(x-a)^{\mu}\ \middle|\ \begin{matrix}(e_1,E_1,y),(1-\lambda,\mu),(e_j,E_j)_{2,p} \\ (f_j,F_j)_{1,q},(1-\lambda-\nu(\alpha-\beta)+\alpha,\mu)\end{matrix}\right] \end{aligned} \tag{14}$$

provided that every term of the claim (14) *exists.*

Proof. To prove the assertion (14), with the help of (8) and (5), we get

$$\begin{aligned} &\left({}_a\mathcal{D}_x^{\alpha,\beta;\nu}\left[(t-a)^{\lambda-1}\ \Gamma^{m,n}_{p,q}[w(t-a)^{\mu}]\right]\right)(x) \\ &= {}_a\mathcal{D}_x^{\alpha,\beta;\nu}\left[(t-a)^{\lambda-1}\frac{1}{2\pi i}\int_L \mathbb{F}(\xi,y)[w(t-a)^{\mu}]^{-\xi}d\xi\right](x) \\ &= \frac{1}{2\pi i}\int_L \mathbb{F}(\xi,y)w^{-\xi}\left({}_a\mathcal{D}_x^{\alpha,\beta;\nu}[(t-a)^{\lambda-\mu\xi-1}]\right)(x)d\xi \\ &= \frac{1}{2\pi i}\int_L \mathbb{F}(\xi,y)w^{-\xi}\frac{\Gamma(\lambda-\mu\xi)}{\Gamma(\lambda+\nu(\alpha-\beta)-\alpha-\mu\xi)}(x-a)^{\lambda+\nu(\alpha-\beta)-\alpha-\mu\xi-1}d\xi \\ &= (x-a)^{\lambda+\nu(\alpha-\beta)-\alpha-1}\frac{1}{2\pi i}\int_L \mathbb{F}(\xi,y)(w(x-a)^{\mu})^{-\xi}\frac{\Gamma(\lambda-\mu\xi)}{\Gamma(\lambda+\nu(\alpha-\beta)-\alpha-\mu\xi)}d\xi \\ &= (x-a)^{\lambda+\nu(\alpha-\beta)-\alpha-1}\Gamma^{m,n+1}_{p+1,q+1}\left[w(x-a)^{\mu}\ \middle|\ \begin{matrix}(e_1,E_1,y),(1-\lambda,\mu),(e_j,E_j)_{2,p} \\ (f_j,F_j)_{1,q},(1-\lambda-\nu(\alpha-\beta)+\alpha,\mu)\end{matrix}\right] \end{aligned}$$

Hence the proof. □

Corollary 3.4. *Let*

$$x > a \ \left(a \in \mathrm{R}^{+} = [0, \infty)\right), \quad 0 < \alpha < 1, \quad 0 \leqq \nu \leqq 1.$$

Suppose also that

$$\lambda, \mu, \mathrm{w}, \in \mathrm{C} \quad \Re(\lambda) > 0$$

Then

$$\left(\mathbb{I}_{a+}^{\alpha}\left[(t-a)^{\lambda-1}\ \Gamma_{\mathrm{p,q}}^{\mathrm{m,n}}[\mathrm{w}(t-a)^{\mu}]\right]\right)(x)$$
$$= (x-a)^{\lambda-\alpha-1}\Gamma_{\mathrm{p+1,q+1}}^{\mathrm{m,n+1}}\left[\mathrm{w}(x-a)^{\mu}\left|\begin{array}{c}(\mathrm{e}_1,\mathrm{E}_1,\mathrm{y}),(1-\lambda,\mu),(\mathrm{e}_j,\mathrm{E}_j)_{2,\mathrm{p}}\\ \\ (\mathrm{f}_j,\mathrm{F}_j)_{1,\mathrm{q}},(1-\lambda-\alpha,\mu)\end{array}\right.\right] \tag{15}$$

$$\left(\mathbb{D}_{a+}^{\alpha}\left[(t-a)^{\lambda-1}\ \Gamma_{\mathrm{p,q}}^{\mathrm{m,n}}[\mathrm{w}(t-a)^{\mu}]\right]\right)(x)$$
$$= (x-a)^{\lambda-\alpha-1}\Gamma_{\mathrm{p+1,q+1}}^{\mathrm{m,n+1}}\left[\mathrm{w}(x-a)^{\mu}\left|\begin{array}{c}(\mathrm{e}_1,\mathrm{E}_1,\mathrm{y}),(1-\lambda,\mu),(\mathrm{e}_j,\mathrm{E}_j)_{2,\mathrm{p}}\\ \\ (\mathrm{f}_j,\mathrm{F}_j)_{1,\mathrm{q}},(1-\lambda+\alpha,\mu)\end{array}\right.\right] \tag{16}$$

and

$$\left({}_{a}\mathcal{D}_{x}^{\alpha,\nu}\left[(t-a)^{\lambda-1}\ \Gamma_{\mathrm{p,q}}^{\mathrm{m,n}}[\mathrm{w}(t-a)^{\mu}]\right]\right)(x)$$
$$= (x-a)^{\lambda-\alpha-1}\Gamma_{\mathrm{p+1,q+1}}^{\mathrm{m,n+1}}\left[\mathrm{w}(x-a)^{\mu}\left|\begin{array}{c}(\mathrm{e}_1,\mathrm{E}_1,\mathrm{y}),(1-\lambda,\mu),(\mathrm{e}_j,\mathrm{E}_j)_{2,\mathrm{p}}\\ \\ (\mathrm{f}_j,\mathrm{F}_j)_{1,\mathrm{q}},(1-\lambda+\alpha,\mu)\end{array}\right.\right] \tag{17}$$

provided that every term of the claims (15), (16) *and* (17) *exists.*

Proof. Assertions (15), (16) and (17) can be obtained directly by the specialising the parametere of generalised composite fractional derivative in assertion (14). □

Theorem 3.5. *Following composition relations hold for some Lebesgue measurable function $\varphi \in \mathfrak{L}(\mathfrak{a}, \mathfrak{b})$, assuming the conditions stated in the definition* (9), :

$$\mathbb{I}_{a+}^{\eta}\mathbb{E}_{a+;\mathrm{p,q};\mu}^{\mathrm{w;m,n;}\lambda}\ \varphi = \mathbb{E}_{a+;\mathrm{p,q};\mu}^{\mathrm{w;m,n;}\lambda}\mathbb{I}_{a+}^{\eta}\ \varphi \tag{18}$$

and

$$\mathbb{D}_{a+}^{\eta}\mathbb{E}_{a+;\mathrm{p,q};\mu}^{\mathrm{w;m,n;}\lambda}\ \varphi = \mathbb{E}_{a+;\mathrm{p,q};\mu}^{\mathrm{w;m,n;}\lambda}\mathbb{D}_{a+}^{\eta}\ \varphi \tag{19}$$

Proof. The left hand side of (18) is temporarily representated as Δ and thereafter, making use of (3) and (9), following is obtained:

$$\Delta = \frac{1}{\Gamma(\eta)}\int_a^x (x-v)^{\eta-1}\ \int_a^v (v-t)^{\mu-1}\ \Gamma_{\mathrm{p,q}}^{\mathrm{m,n}}[\mathrm{w}(v-t)^{\lambda}]\varphi(t)dtdv \tag{20}$$

Further, the order of t-integral and the v-integral is interchanged (allowed for the conditions already stated). Hence the following is obtained easily:

$$\Delta = \frac{1}{\Gamma(\eta)}\int_a^x \left[\int_t^x (x-v)^{\eta-1}\ (v-t)^{\mu-1}\ \Gamma_{\mathrm{p,n}}^{\mathrm{m,n}}[\mathrm{w}(v-t)^{\lambda}]dv\right]\varphi(t)dt. \tag{21}$$

Next, following equation is obtained by substituting $v - t = \tau$ in expression (21),

$$\Delta = \frac{1}{\Gamma(\eta)} \int_a^x \left[\int_0^{x-t} (x-t-\tau)^{\eta-1} \, \tau^{\mu-1} \, \Gamma_{p,q}^{m,n}[w\tau^{\lambda}] d\tau \right] \varphi(t) dt. \tag{22}$$

The equation that follows by using the claim (15) in the right hand side of (22),

$$\Delta = \int_a^x (x-t)^{\mu+\eta-1} \Gamma_{p+1,q+1}^{m,n+1} \left[w(x-t)^{\lambda} \left| \begin{matrix} (e_1, E_1, y), (1-\mu, \lambda), (e_j, E_j)_{2,p} \\ (f_j, F_j)_{1,q}, (1-\mu-\eta, \lambda) \end{matrix} \right. \right] \varphi(t) dt. \tag{23}$$

For simplicity, the right-hand side of (18) is temporarily denoted with Ω. Thereafter, making use of the definitions (9) and (3), is given as below:

$$\Omega = \int_a^y (y-t)^{\mu-1} \, \Gamma_{p,q}^{m,n}[w(y-t)^{\lambda}] \, \frac{1}{\Gamma(\eta)} \int_a^t (t-x)^{\eta-1} \, \varphi(x) dx \, dt. \tag{24}$$

Next step involves the interchanging x-integral and the t-integral initially and then substituting $y - t = \tau$, following equation is obtained:

$$\Omega = \frac{1}{\Gamma(\eta)} \int_a^y \left[\int_0^{y-x} (y-\tau-x)^{\eta-1} \, \tau^{\mu-1} \, \Gamma_{p,q}^{m,n}[w\tau^{\lambda}] d\tau \right] \varphi(x) dx. \tag{25}$$

Finally, with the help of assertion (15) in the right hand side of the equation (25), an equation familiar with expression (23) is obtained. It implies the following:

$$\begin{aligned} I_{a+}^{\eta} \left(\mathbb{E}_{a+;p,q;\mu}^{w;m,n;\lambda} \, \varphi \right) &= \int_a^y (y-x)^{\mu+\eta-1} \Gamma_{p+1,q+1}^{m,n+1} \left[w(y-x)^{\lambda} \left| \begin{matrix} (e_1, E_1, y), (1-\mu, \lambda), (e_j, E_j)_{2,p} \\ (f_j, F_j)_{1,q}, (1-\mu-\eta, \lambda) \end{matrix} \right. \right] \varphi(x) dx \\ &= \left(\mathbb{E}_{a+;p,q;\mu}^{w;m,n;\lambda} I_{a+}^{\eta} \, \varphi \right). \end{aligned} \tag{26}$$

This completes the proof of (18). For the proof of (19), one can refer to the already given proof of (18), and that's why the details are not covered in this manuscript. □

Theorem 3.6. *For any Lebesgue measurable function $\varphi \in \mathfrak{L}(\mathfrak{a}, \mathfrak{b})$ in the constraints given in definition* (9), *each of the following composition relationships*:

$$\mathfrak{D}_{a+}^{\mu_1,\nu_1} \left(\mathbb{E}_{a+;p,q;\mu}^{w;m,n;\lambda} \, \varphi \right) = \mathbb{I}_{a+}^{(1-\nu_1)(1-\mu_1)} \, \mathbb{E}_{a+;p,q;\mu}^{w;m,n;\lambda} \, \varphi \qquad (0 < \mu_1 < 1; \; 0 \leqq \nu_1 \leqq 1) \tag{27}$$

holds true.

Proof. Initially applying definition (6) in the left-hand side of (27), following is obtained:

$$\mathfrak{D}_{a+}^{\mu_1,\nu_1} \left(\mathbb{E}_{a+;p,q;\mu}^{w;m,n;\lambda} \, \varphi \right) = \mathbb{I}_{a+}^{\nu_1(1-\mu_1)} \frac{d}{dx} \left(\mathbb{I}_{a+}^{(1-\nu_1)(1-\mu_1)} \mathbb{E}_{a+;p,q;\mu}^{w;m,n;\lambda} \, \varphi \right). \tag{28}$$

The equation obtained after applying assertion (26) is as follows:

$$\begin{aligned} &\left(\mathbb{I}_{a+}^{(1-\nu_1)(1-\mu_1)} \mathbb{E}_{a+;p,q;\mu}^{w;m,n;\lambda} \, \varphi \right)(x) \\ &= \int_a^x (x-t)^{\mu+(1-\nu_1)(1-\mu_1)-1} \, \Gamma_{p+1,q+1}^{m,n+1} \left[w(x-t)^{\lambda} \left| \begin{matrix} (e_1, E_1, y), (1-\mu, \lambda), (e_j, E_j)_{2,p} \\ (f_j, F_j)_{1,q}, (1-\mu-(1-\mu_1)(1-\nu_1), \lambda) \end{matrix} \right. \right] \varphi(t) dt. \end{aligned} \tag{29}$$

Further obtain the first derivative w.r.t x, of every term in the equation (29), to obtain the following:

$$\frac{d}{dx}\left(\mathbb{I}_{a+}^{(1-\nu_1)(1-\mu_1)}\mathbb{E}_{a+;\mathrm{p,q};\mu}^{\mathrm{w;m,n};\lambda}\,\varphi\right)(x)=\int_a^x (x-t)^{\mu+(1-\nu_1)(1-\mu_1)-2} \cdot \Gamma_{\mathrm{p+1,q+1}}^{\mathrm{m,n+1}}\left[\mathrm{w}(x-t)^{\lambda}\,\middle|\,\begin{matrix}(e_1,E_1,y),(1-\mu,\lambda),(e_j,E_j)_{2,p}\\ (f_j,F_j)_{1,q},(2-\mu-(1-\mu_1)(1-\nu_1),\lambda)\end{matrix}\right]\varphi(t)dt. \tag{30}$$

Next, the operator $\mathbb{I}_{a^+}^{\nu(1-\mu)}$ is applied to equation (30) (use the definition (3)):

$$\begin{aligned}&\left(\mathbb{I}_{a+}^{\nu_1(1-\mu_1)}\frac{d}{dx}\left(\mathbb{I}_{a+}^{(1-\nu_1)(1-\mu_1)}\mathbb{E}_{a+;\mathrm{p,q};\mu}^{\mathrm{w;m,n};\lambda}\,\varphi\right)(x)\right)(v)\\ &=\frac{1}{\Gamma\big(\nu_1(1-\mu_1)\big)}\int_a^v (v-s)^{\nu_1(1-\mu_1)-1}\,ds\int_a^s (s-t)^{\mu+(1-\nu_1)(1-\mu_1)-2}\\ &\qquad\cdot\Gamma_{\mathrm{p+1,q+1}}^{\mathrm{m,n+1}}\left[\mathrm{w}(s-t)^{\lambda}\,\middle|\,\begin{matrix}(e_1,e_1,y),(1-\mu,\lambda),(e_j,E_j)_{2,p}\\ (f_j,F_j)_{1,q},(2-\mu-(1-\mu_1)(1-\nu_1),\lambda)\end{matrix}\right]\varphi(t)dt.\end{aligned} \tag{31}$$

Next step involves the interchange of t-integral and the s-integral (allowed for the given conditions) followed by the substitution $s-t=\tau$ in the equation (31), following is obtained:

$$\begin{aligned}&\left(\mathbb{I}_{a+}^{\nu_1(1-\mu_1)}\frac{d}{dx}\left(\mathbb{I}_{a+}^{(1-\nu_1)(1-\mu_1)}\mathbb{E}_{a+;\mathrm{p,q};\mu}^{\mathrm{w;m,n};\lambda}\,\varphi\right)(x)\right)(v)\\ &=\frac{1}{\Gamma\big(\nu_1(1-\mu_1)\big)}\int_a^v \varphi(t)dt\int_0^{v-t}(v-\tau-t)^{\nu_1(1-\mu_1)-1}\tau^{\mu+(1-\nu_1)(1-\mu_1)-2}\\ &\qquad\cdot\Gamma_{\mathrm{p+1,q+1}}^{\mathrm{m,n+1}}\left[\mathrm{w}\tau^{\lambda}\,\middle|\,\begin{matrix}(e_1,E_1,y),(1-\mu,\lambda),(e_j,E_j)_{2,p}\\ (f_j,F_j)_{1,q},(2-\mu-(1-\mu_1)(1-\nu_1),\lambda)\end{matrix}\right]d\tau\\ &=\int_a^v \mathbb{I}_{0+}^{\nu_1(1-\mu_1)}\Bigg(\tau^{\mu+(1-\nu_1)(1-\mu_1)-2}\\ &\qquad\cdot\Gamma_{\mathrm{p+1,q+1}}^{\mathrm{m,n+1}}\left[\mathrm{w}\tau^{\lambda}\,\middle|\,\begin{matrix}(e_1,E_1,y),(1-\mu,\lambda),(e_j,E_j)_{2,p}\\ (f_j,F_j)_{1,q},(2-\mu-(1-\mu_1)(1-\nu_1),\lambda)\end{matrix}\right]\Bigg)\varphi(t)dt.\end{aligned} \tag{32}$$

Finally, the application of the result (26) to the righthand side of the equation (32), gives the following equation:

$$\begin{aligned}&\left(\mathbb{I}_{a+}^{\nu_1(1-\mu_1)}\frac{d}{dx}\left(\mathbb{I}_{a+}^{(1-\nu_1)(1-\mu_1)}\,\mathbb{E}_{a+;\mathrm{p,q};\mu}^{\mathrm{w;m,n};\lambda}\,\varphi\right)(x)\right)(v)\\ &=\frac{1}{\Gamma\big(\nu_1(1-\mu_1)\big)}\int_a^v (v-t)^{\mu+(1-\nu_1)(1-\mu_1)-1}\\ &\qquad\Gamma_{\mathrm{p+1,q+1}}^{\mathrm{m,n+1}}\left[\mathrm{w}(v-t)^{\lambda}\,\middle|\,\begin{matrix}(e_1,E_1,y),(1-\mu,\lambda),(e_j,E_j)_{2,p}\\ (f_j,F_j)_{1,q},(1-\mu-(1-\mu_1)(1-\nu_1),\lambda)\end{matrix}\right]\varphi(t)dt\\ &=\left(\mathbb{I}_{a+}^{(1-\nu_1)(1-\mu_1)}\,\mathbb{E}_{a+;\mathrm{p,q};\mu}^{\mathrm{w;m,n};\lambda}\,\varphi\right)(v),\end{aligned} \tag{33}$$

that is the entire proof of Theorem 3.6. □

Remark 3.7. *The results established by Srivastava and Tomovski et al.[2] are obtained as special cases of our findings if the incomplete H−function reduces to the Mittag-Leffler type function $E^{\gamma,\kappa}_{\alpha,\beta}(z)$. Similarly, if the incomplete H−function is reduced to the multi-index Mittag-Lefflerfunction $E^{\gamma,\kappa}_{(\alpha_j,\beta_j)_m}(z)$, the results obtained by Srivastava et al. [3] follow as special cases.*

4. Conclusion

In this research work some interesting results have been established by integrating fractional derivative operators with that of IHFs. It is already specified in the introduction the significance of both fractional operators and IHFs. As noted in Remark 1 and Remark 2, our main findings viz. Theorem 2, Theorem 3, Theorem 4 and Theorem 5 generalize the interesting and significant results obtained by several authors in the past [1], [2] and [3]. Therefore, we conclude by stating that this research work definitely contributes to the vast and ever changing Mathematical literature of fractional calculus and special functions.

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